Mechanical behavior of materials

Dr. Niraj Mohan Chawake

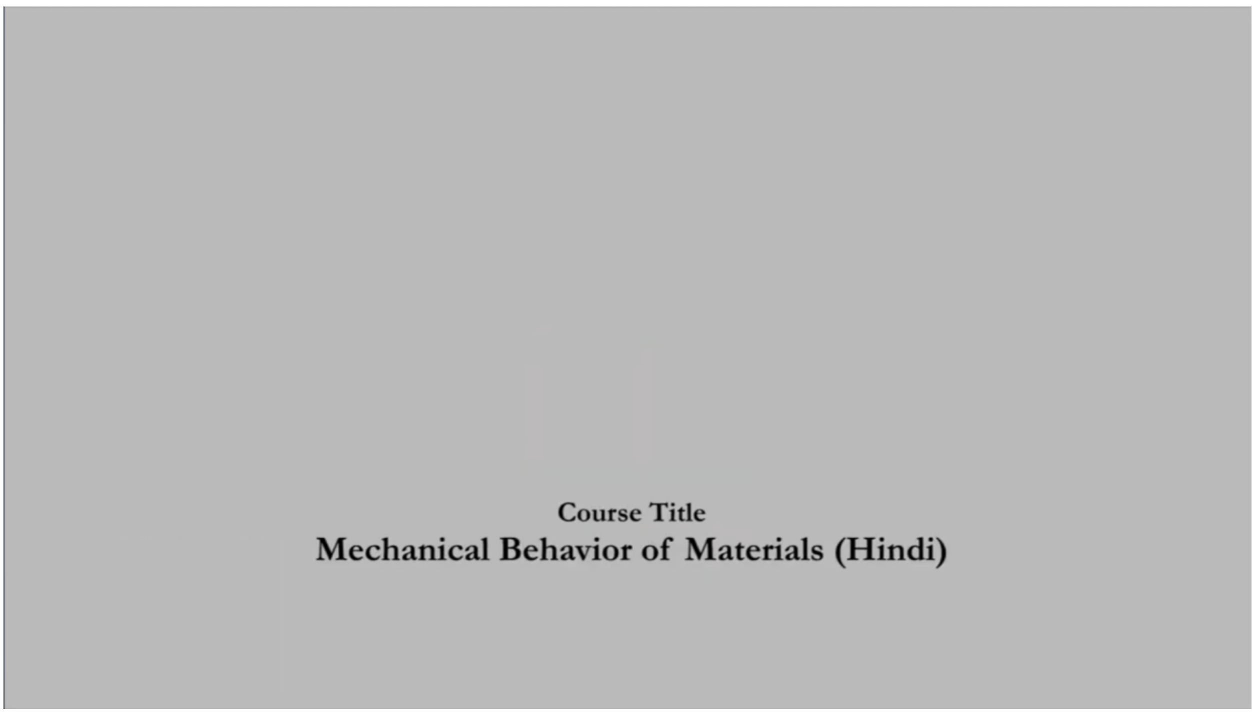
Department of Materials Science and Engineering

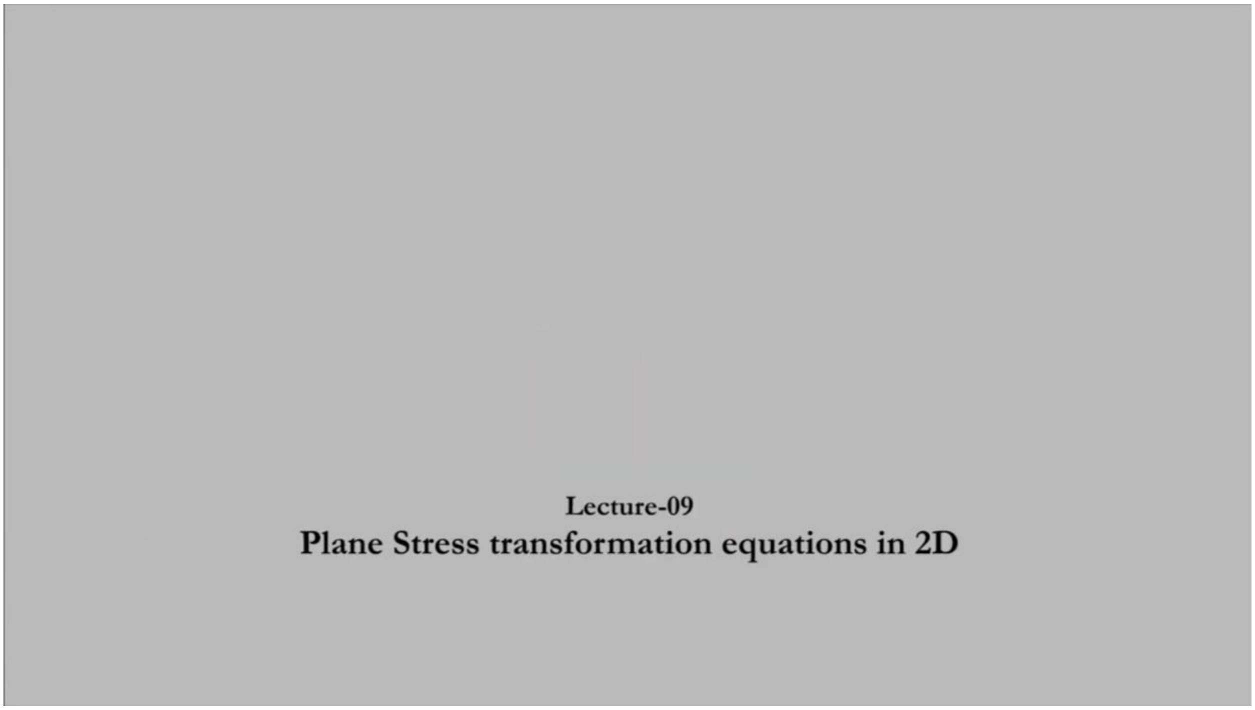
Indian Institute of Technology, Kanpur

Week-2

Lecture-9

Plane Stress transformation equations in 2D



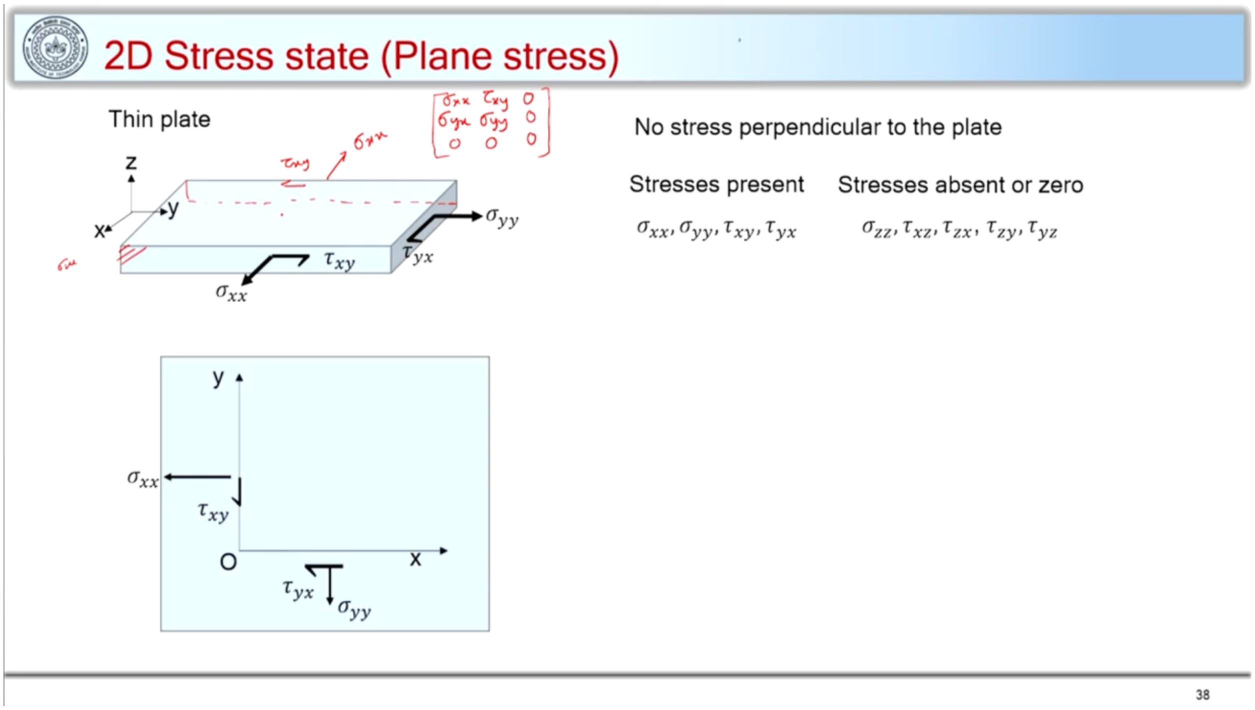


So – humne last part mein dekha tha ki humare paas nine stress components hain – aur in nine stress components mein chhah independent stress components hote hain.
To – is case ya is part mein hum dekhenge ki plane stress kya hota hai – ya 2D stress state kya hota hai. 2D stress state arise hota hai – yaani tayyar hota hai thin plates mein.

Thin plates – yaane agar main maan loon ki mere dimensions aise hain: x, y aur z – to z–direction mein, yaani jo thickness direction hai wo bahut kam hai x aur y ki tulna mein.
To – is tarah se jo components hote hain – unmein mujhe 2D stress state ya plane stress condition milti hai. Ismein kuch stress components hum mark kar lete hain – jaise $\boldsymbol{\sigma_{xx}}$, $\boldsymbol{\tau_{xy}}$, $\boldsymbol{\tau_{yx}}$, aur $\boldsymbol{\sigma_{yy}}$. $\boldsymbol{\sigma_{xx}}$ – aap jaanenge ki ye σ hai jo x–direction mein hai aur x–plane pe act karta hai – yaani mera y–x plane ho jaayega. Humne dekha hai ki ye mera x–plane hai aur y–along x–direction hai to ye $\boldsymbol{\sigma_{xx}}$ ho jaayega. Similarly baaki ke stress components hum likh sakte hain.
Aur last part mein humne dekha tha ki $\boldsymbol{\tau_{xy}}$ aur $\boldsymbol{\tau_{yx}}$ saman hone chahiye equal hone chahiye tab jaake body static equilibrium maintain karegi. Is part mein hum dekhenge – ki ye 2D stress state hoti kya hai. 2D stress state – yaane stress perpendicular to the plate zero hota hai yaani z–direction par jo bhi stresses hain, ya z–plane par jo bhi stresses act karte hain – wo sab zero hain. Agar hum dekhenge – humare paas stresses kaun se hain – yahan par humare paas chaar stresses hain: $\boldsymbol{\sigma_{xx}}$, $\boldsymbol{\sigma_{yy}}$, $\boldsymbol{\tau_{xy}}$, aur $\boldsymbol{\tau_{yx}}$. Stresses absent kahaan pe hain – to humare paas ye jo stress components hain:

σ_{zz}, **τ_{xz}**, **τ_{zx}**, **τ_{yz}**, aur **τ_{zy}** – ye sab z–plane par act ho rahe hain ya along z–direction act kar rahe hain – to in sabki value 2D stress state mein zero ho jaayegi. Is tarah se main mark kar loonga agar 2D stress state mein dekhun to – main x aur y coordinates yahan pe mark kar deta hoon – aur ye point O jahaan maine stress define kiya hai. **σ_{xx}** – main is direction mein mark karunga – jo plane mera x–plane hai, jo ki perpendicular to x–direction hai. To **σ_{xx}** main is direction par mark karunga. Agar main dekhu to yahan par bhi is direction mein **σ_{xx}** hoga – aur jo negative x–plane hai us direction pe bhi stresses act ho rahe hain. Similarly – **τ_{xy}** is direction mein hoga.Ye jo stress components hain – yahan par mark kar raha hoon. Main iska hi representation 2D mein mark kar raha hoon – ye **σ_{xx}** ho gaya mera, ye **σ_{yy}** ho gaya jo ki mera y–plane hai aur along y–direction hai. Similarly – **τ_{xy}** aur **τ_{yx}** – to ye mera y–plane hai aur **τ_{yx}** along x–direction act kar raha hai. Main isko is tarah se mark karunga – **τ_{xy}** x–plane hai jo along y–direction act kar raha hai – to **τ_{xy}** yahan is tarah se act hoga. Agar hum 2D stress state dekhein – to humare paas stress components kaun se hain? Humne bola hai – ki z–direction mein stress components nahi hain – to z–direction ke saare stress components zero ho jaayenge. To humare paas ye do hi stress components hain – aur isi ko hum **state of stress** kehte hain. Ab hum dekhenge – agar ye mere paas 2D stress state hai – to agar main koi element yahan likh loon – aisa kuch element yahan mark kar diya. Maan lijiye – ye along y–direction hai aur ye along x–direction hai – ye mera ek stress state element hai. To yehi main yahan par mark kar raha hoon.

Maan lijiye – mujhe ek plane QP par stress nikalni hai – to main kaise nikaaloon? Yehi hum solve karenge is part mein.



Jaise maine ek element maan liya hai – agar koi aisa element hai jo kuch angle bana raha hai x–direction aur y–direction ke hisaab se – ye mera x–direction hai aur ye meri y–direction hai – to kuch angle θ bana raha hai. Ye mera ek element ho gaya. Is element par kya state of stress hogi – agar mujhe ye state of stress is element ki pata hai – to yehi hum aaj is part mein dekhenge. Do cheezein ho sakti hain – ek to main element change kar raha hoon – ya main apna coordinate axis hi change kar raha hoon. Maan lijiye – plane QP hai jahan par mujhe stress nikalni hai – aur iske along jo coordinate axes hain – main mark kar loonga x′ aur y′. Humne yahan dekha tha – is plane par ye jo direction hai – ye meri x′ hai – aur ye jo direction hai meri y′ hai. To humein stress x′ aur y′ direction mein nikalni hai – jab humein state of stress x aur y coordinate reference mein pata hai.

Maan lijiye plane QP ka area A hai aur ye jo x' direction se angle bana raha hai wo theta hai. To agar ye angle θ hai to doosra angle 90 – θ ho jaayega yeh bhi 90 minus θ hoga, aur yeh θ ho jaayega angle, aur similarly yeh bhi angle θ ho jaayega. To abhi hum dekhenge is area jo mark karenge, area PQ jo mark kiya, maan lijiye iska area A hai PQ ka. To area of plane OP kya hoga OP hoga m * A. A kya hai? Yahaan par m sin θ ho jaayega. To agar yeh Q-P hai, iska area A hai, to QOP yahaan par is direction me hai jo angle 90 − θ bana raha hai X-axis se, to agar hum iska projection dekhenge, so OP kya hai yeh area ka projection hai X-axis pe. To yeh m × A likh sakte hain, jahaan pe $m = \sin \theta$ hoga. Aur OQ yeh jo OP hai, yeh ho jaayega mera X-plane, aur jiska area ho jaayega A, aur l yahaan pe = $\cos \theta$ hai. To is tarah se, agar mujhe abhi hamein kya karna hai hamare paas yeh stresses hain jo is plane par act kar rahe hain, to abhi hum is plane par jo stresses act kar rahe, inko hum convert karenge forces along X aur forces along Y direction. To kaise karenge? Hamein stresses pata hain stresses X direction mein kaun se act ho rahe hain? To agar main dekhoon, stresses mere X-direction par act ho rahe σ_{xx} aur $\tau_x y$.

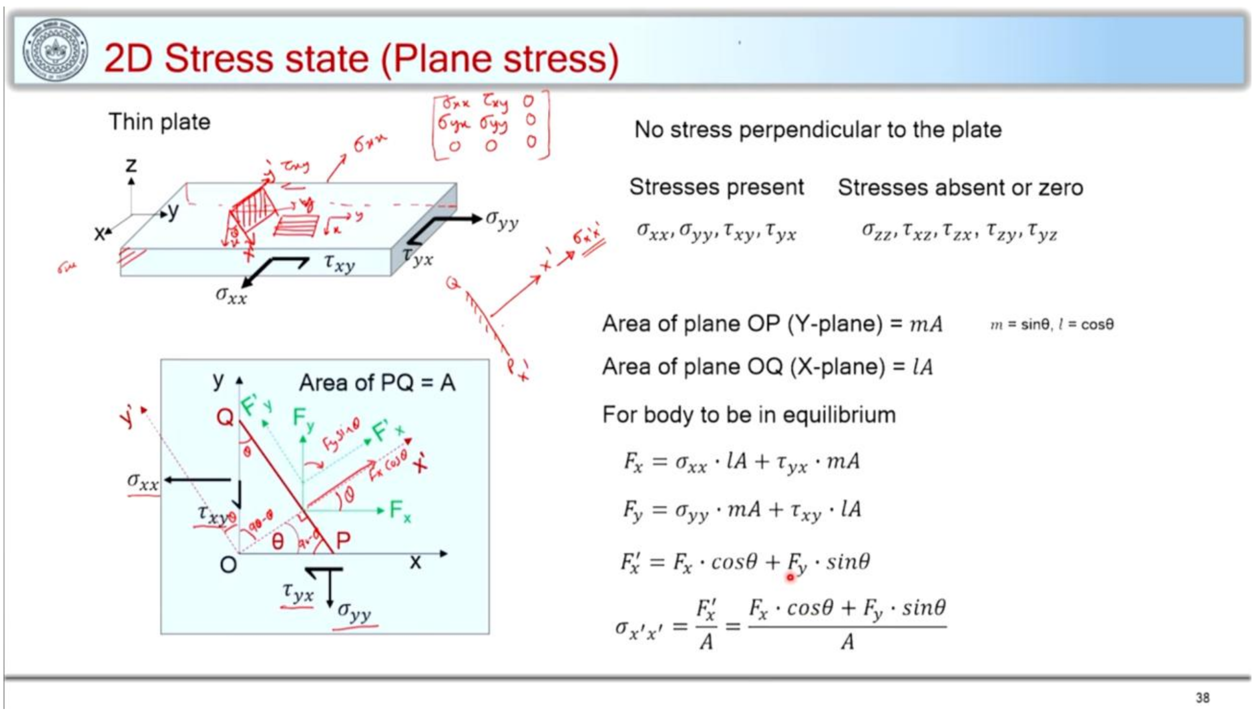
Aur Y-direction par kaun se stresses act honge τxy aur σyy. Yeh mere Y-direction ke liye act honge. To agar mujhe force nikaalna hai X-direction ke liye, to mujhe is stress ko convert karna padega force mein. To, for body equilibrium ke liye hamein chahiye yeh forces.

Agar main F_x yaani is direction mein forces nikaaloon to $\sigma_{xx} \times lA + \tau_x y \times mA$. σ_{xx} kis plane par act ho raha hai? Is OQ par act ho raha hai, to OQ ka area hai mere paas lA. Aur τyx τyx act ho raha mera OP plane par, to uska area hai mA. To yeh ho jaayega mera force along X-direction. Similarly, force along Y-direction hoga σyy jo ki O-P par act ho raha hai, uska area hai mA, aur τxy yeh plane along Y-direction act kar raha hai aur yeh on-plane OQ act kar raha hai, to uska area hai lA. To yeh hum likhenge $Fy = \sigma yy\ mA + \tau xy\ lA$. To yeh forces ho gaye mere is coordinate axis mein jo ki along X aur Y direction pe hain. Abhi mujhe forces nikaalne hain is naye coordinate axis mein, jo humne choose kiye $F'x$ aur $F'y$ direction mein, kyunki mujhe stress nikaalna hai is plane par PQ par. To agar mujhe stress nikaalna hai, to pehle mujhe is direction par forces pata hone chahiye yaani F_x' aur Fy' pe. To iske liye hum karenge jaise F_x' agar hum nikaalna chahte hain, to hum agar hamein forces pata hain Fx aur Fy direction mein, to hum Fx' ke along forces nikaal sakte hain.

To agar mujhe yeh Fx pata hai, aur yeh angle pata hai yeh angle mera ho jaayega θ, to Fx ka component is direction mein Fx' direction mein aayega $Fx \cos \theta$, aur Fy ka jo component aayega is direction mein, wo aayega $Fy \sin \theta$.

Jo ki humne yahaan par likha hai. So, mujhe mere paas $Fx' = Fx \cos \theta + Fy \sin \theta$ mil gaya. Yeh force mujhe pata chal gaya Fx' direction par. Agar mujhe iska stress nikaalna hai, to mujhe kya karna padega? Mujhe sirf agar stress nikaalna hai $\sigma x'x'$ direction pe

$X'X'$ direction yaani kya hoga? Yeh mera X'-plane ho gaya. Agar main is direction mein dekhoon, to yeh mera X'-direction hai to yeh agar X' direction hai, to yeh jo plane hoga mera jo ki Q-P hai yahaan pe yeh mera X'-plane hoga. To agar stress agar is direction mein nikaalna hai mujhe, to yeh stress kya hoga? Jo X'-plane pe act ho raha hai along X' direction. To yeh mera ho jaayega $\sigma x'x'$. To $\sigma x'x'$ hum nikaalenge Fx' upon area of this plane, yaani Q-P ka area. To yeh hum isko saral tarah se likhenge $(Fx \cos\theta + Fy \sin\theta) / A$.



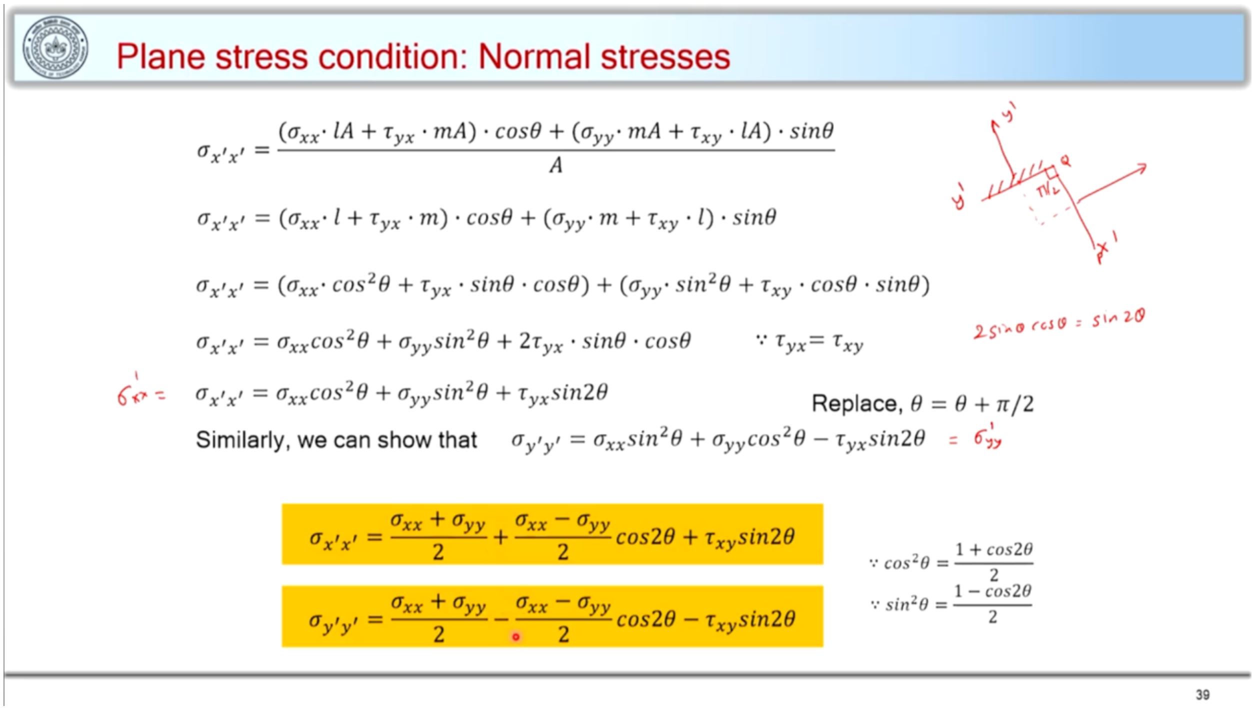
To ismein $\sigma x'x'$ ko hum is tarah se likh paayenge $\sigma x'x' = \sigma_{xx} l^2 + \tau yx\, m \cos\theta + \sigma yy\, m^2 + \tau xy\, l \sin\theta$.

Aur l aur m ki value hum yahan pe likhenge, aur yeh jo area hai yahaan pe dekhenge saare terms mein A area aaya hai, to hum area agar baahar nikaalenge saare brackets se, to yeh numerator aur denominator ka area cancel ho jaayega. To hamare paas ek simple term rahegi yahaan pe $(\sigma_{xx}\, l \cos\theta + \tau yx\, m \cos\theta) + (\sigma yy\, m + \tau xy\, l) \sin\theta$. Aur l aur m ki value hum rakhenge yahaan par $l = \cos\theta$ aur $m = \sin\theta$. to hamare paas agar ye isko ham multiply karenge is term ko to hamare paas ye relation aayega. Aur agar aap dekhenge, agar isko main thoda simplify karun to ye $\sigma_{xx} \cos^2\theta$ ho jaayega. Ye term $\sigma yy \sin^2\theta$ yahan pe aa gayi aur in dono ka addition ham karenge $\tau xy \sin\theta \cos\theta$. To ham τxy aur τyx ye similar hain. Hamne pichhle part mein dekha tha ki ye jo stress components hain shear stress components hain τxy aur τyx samaan hone chahiye static equilibrium maintain karne ke liye. To yahan par ham τyx ki jagah, is τxy ki jagah τyx likhenge, to ye 2 times ho jaayega. Uske baad ham isko simplify karke is tarah se likhenge. Aur agar hamein $\sigma y'y'$ nikaalna hai, to hamein kya karna tha hamne stress nikaala tha is plane par, ye mera x' plane tha. To mera y'd plane dekhenge, ye mera QP tha to ye mera y'plane hoga aur ye mera y' direction. To hamein is plane par agar stress nikaalna hai to hamein sirf is relation mein θ ki jagah $(\theta + \pi/2)$ rakhna hai, kyunki x' jo plane hai, ye y' se $\pi/2$ se change hai. To agar ham yahan par $\pi/2$ replace kar denge $\theta + \pi/2$, to hamein stress mil jaayega y'-plane par. To ham $\sigma y'y$ is tarah se nikaalenge: $\sigma_{xx} \sin^2 \theta + \sigma yy \cos^2 \theta - \tau yx \sin 2\theta$. To yahan par ek ye identity bhi istamal ki gayi hai jo ki likh dete hain: $2 \sin \theta \cos \theta = \sin 2\theta$. To ye ek identity bhi yahan par use hui hai. To hamare paas $\sigma x'd$ x'd aur $\sigma y'd$ y'd aa gaye. Kisi books mein ye aise bhi likha rehta hai isko ham σ'_{xx} likhte hain aur kuch books isko $\sigma' yy$ bhi likhte hain. Confuse nahi hona hai, sirf ye dekhna hai ki ye state of stress kisi aur plane ya kisi aur coordinate axis mein nikaalna hai. Abhi ham $\cos^2 \theta$ ko is tarah se likhenge: $(1 + \cos 2\theta)/2$, aur $\sin^2 \theta$ ko likhenge: $(1 - \cos 2\theta)/2$. Ye trigonometric identity hai. Agar ham ismein replace karenge to hamein $\sigma x'd$ x'd aise milega:

$\sigma x' x' = (\sigma_{xx} + \sigma yy)/2 + (\sigma_{xx} - \sigma yy)/2 \cos 2\theta + \tau xy \sin 2\theta$.

Aur $\sigma y'd$ y'd yahan pe milega: $\sigma y' y' = (\sigma_{xx} + \sigma yy)/2 - (\sigma_{xx} - \sigma yy)/2 \cos 2\theta - \tau xy \sin 2\theta$.

To yahan par sirf convenience ke hisaab se, agar mujhe $\sigma x'd$ nikaalna hai, to yahan par signs change ho jate hain yahan plus tha to dusre mein minus hai, yahan plus tha to dusre mein minus hai.



Ab hamein shear stress nikaalna hai. Shear stress nikaalne mein ham ye plane dekhenge QP. Yahan par hamne normal stress nikaala tha σx'x'. Shear stress kya hoga? Shear stress hoga hamara is plane par, is direction mein jo parallel hai plane ke. Hamare shear stress ko summation nikaalna hai along y' direction. To τx'y' kya hoga? To total force parallel to y' direction. Kyunki yahan par hamne force act kiya tha x' direction par parallel, yahan par ham total force y'd direction par parallel karenge, to hamein shear stress milega. To shear f y' kya milega hamein? To yahan ham summation of forces along x'y' direction karenge. Ye jo fx hai, ye θ hai, ye 90° − θ hoga. To fx is direction mein, y' ke direction mein, fx ka aayega fx sin θ. Ye component fx ka is direction mein aayega. To ye negative y' direction mein hoga, kyunki upar side positive hai to neeche side negative hoga. Isliye ham yahan par −fx sin θ likh rahe hain.

Aur ye jo fy hai, iska component aayega fy cos θ. Aur ham divide karenge area se. Ab jab ham fx nikaale the to fx ki value thi (σ_{xx} A + τxy mA) aur fy ki value thi (σyy A + τyx LA). L aur m ham put karenge to ham isko simplify karenge aur ye relation milega:

τx'd y'd = (σyy − σ_{xx})/2 sin 2θ + τxy cos 2θ.

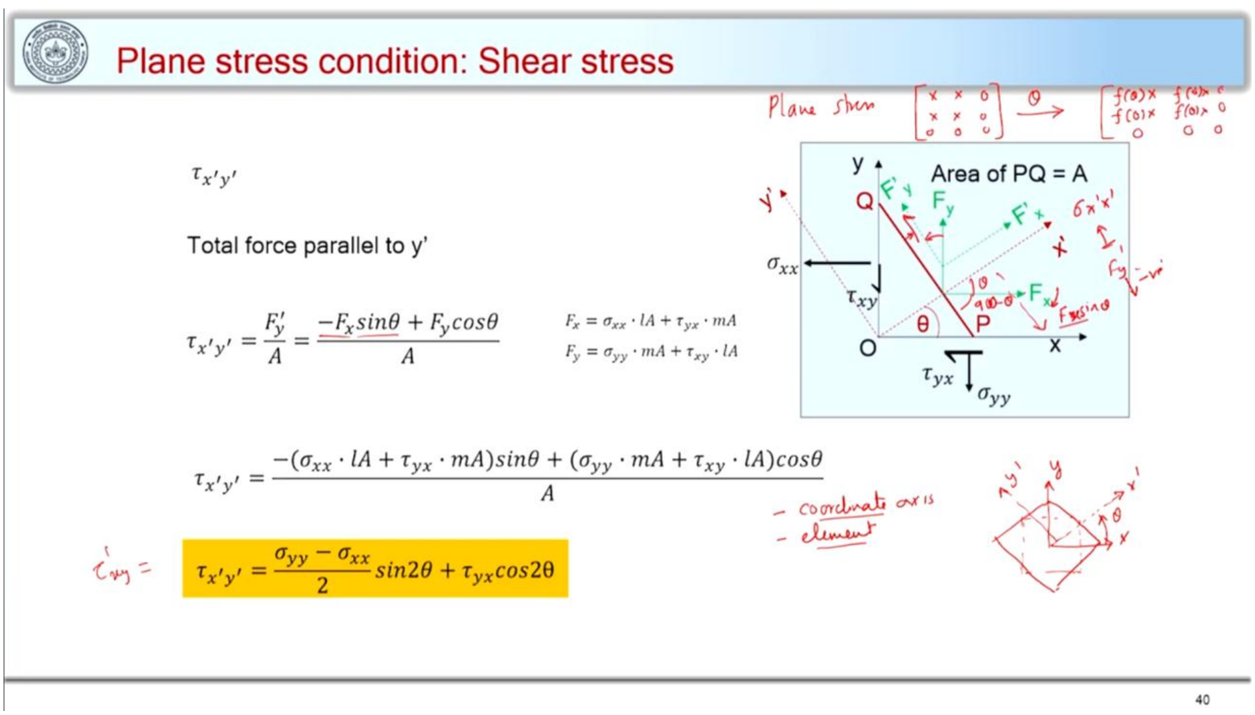
Isko bhi kuch books mein aise likha jata hai: τd x'y'.

To hamare paas do relations hain σx’d x’d aur τx’d y’d. Yaani hamne kya kiya hamare paas state of stress tha kisi point O ke around jo defined tha ek particular coordinate axis mein x aur y ke direction par.

Agar hamein state of stress nikaalna hai jaise ye hamara element hai, ye mera original element hai, aur mujhe koi doosra element diya gaya hai jo is direction mein kuch angle bana raha hai x aur y se jaise ye element ho gaya x’d aur ye y’d ho gaya, jo kuch angle θ bana raha hai x axis se.

To jab coordinate axis change karni hai ya reference axis change karni hai, ya element change karna hai dusre element mein, to hamein sirf ek relation pata hona chahiye ki ye kya angle bana raha hai original axis se.

Agar mujhe θ angle pata hai aur ye stresses pata hain is element mein σ_{xx}, σ_{yy}, aur τ_{xy} to main state of stress nikaal sakta hoon kisi bhi doosre element mein by this relation.



To hamne dekha tha ki hamare paas plane stress condition hai. Yaani hamare paas ek stress tensor hai jahan z direction mein value zero hai. To isko main convert kar sakta hoon agar mujhe θ pata hai. Ye chaaron values pata hain to ek function hoga θ ka, jo hamne yahan determine kiya hai.

Aur ye hi transformation aaj hamne is part mein dekha hai. To yahan par main abhi rukta hoon. Next step mein ham dekhenge ki ye jo transformed state of stress hai, isse ham Mohr circle kaise nikaalenge. Is equation jo hamne derive ki hai, iska importance kya hai, ya agar aapko ek state of stress kisi point par pata hai to aap doosre plane par ya geometry ke liye kaise nikaal sakte hain ye ham dekhenge Mohr circle ke construction se. To yahin par main abhi rukta hoon.

Thank you.