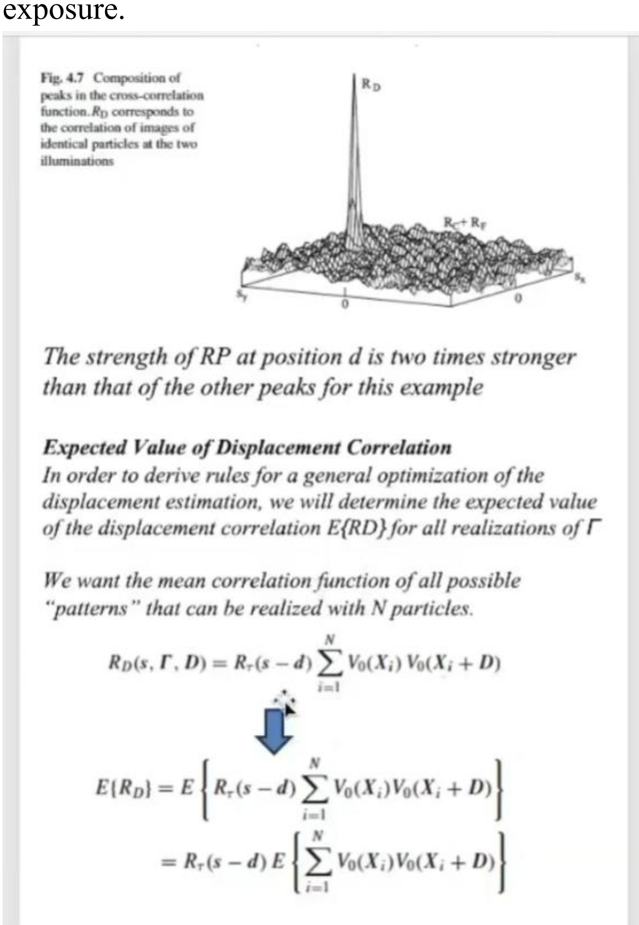
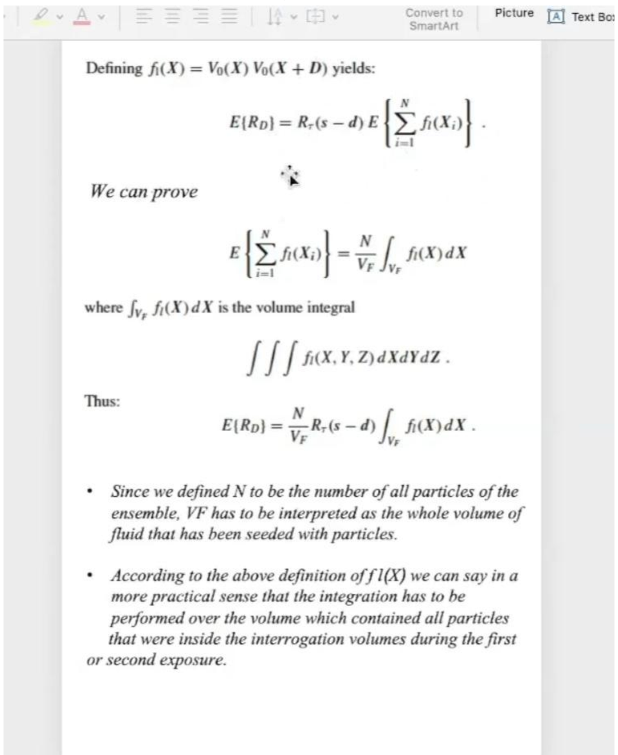
Advanced Measurement Techniques in Fluid Mechanics and Heat Transfer
Prof. Saptarshi Basu
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru
Week – 08
Lecture - 36
Particle Image Velocimetry – 6

All right, so in this particular instance, we are going to continue with our PIV lectures, and it will probably be the last lecture of this series. So this, we already, if you recall, did what a cross-correlation will yield. So it is nearly the same correlation peaks as those that happen in autocorrelation, but there is a displacement of D. So, in order to derive rules for a general optimization of the displacement estimation, we use the expected value of the displacement correlations for all realizations of γ. So E actually means the expected value, okay? So, an R_d is basically the displacement correlation. If you recall, that represents the cross-correlation function when I = J, right? But the first exposure images of identical particles are the same; I mean, for the same set of particles as in the second exposure.



Got it? So that is what we want. So we want the mean correlation function for all possible patterns. And this can be realized with n particles, N. So if this is R_d for all n, this is the R_d that you get.

Now, the expected function will be something like this. So these are the math courses that you might want to take a look at. And this is the expected value. Okay, so defining F1 as Vox multiplied by, you know, V_o x +d; d is the actual displacement in the object plane. Okay, you have the expected R_d equal to this.



Defining $f_I(X) = V_0(X)\,V_0(X+D)$ yields:

$$E\{R_D\} = R_\tau(s-d)\,E\left\{\sum_{i=1}^{N} f_I(X_i)\right\}.$$

We can prove

$$E\left\{\sum_{i=1}^{N} f_I(X_i)\right\} = \frac{N}{V_F}\int_{V_F} f_I(X)\,dX$$

where $\int_{V_F} f_I(X)\,dX$ is the volume integral

$$\int\int\int f_I(X,Y,Z)\,dX\,dY\,dZ\,.$$

Thus:

$$E\{R_D\} = \frac{N}{V_F}R_\tau(s-d)\int_{V_F} f_I(X)\,dX\,.$$

- *Since we defined N to be the number of all particles of the ensemble, VF has to be interpreted as the whole volume of fluid that has been seeded with particles.*
- *According to the above definition of f1(X) we can say in a more practical sense that the integration has to be performed over the volume which contained all particles that were inside the interrogation volumes during the first or second exposure.*

where we can prove that the expected value of this f function, which is basically the multiplication of v and v, is given by this. Okay, so where this part, the one in the integral, is the volume integral, which basically means it is integrated thrice over x, y, and z. So, f_l or f_i, okay, so thus the expected function becomes something like this: e of $R_d = n/v_f\, R\tau$ s - d into the integral, into the volume integral. So we have defined n to be the number of all particles in the ensemble; therefore, your v_f is the whole volume of the fluid that has been seeded with the particles, because these are all particles in the ensemble. According to this definition, your f f_l x.

In a more practical sense, it has to be performed over the whole volume that contained all the particles inside the interrogation volumes during the first or second exposure. So the expected value is this; therefore. Okay, so how the volume integral is to be performed and what the extent of the volume is, that is what is being covered here. Then we can rewrite the integral now; this f dx, as you know, the integral of the intensities as well as the window functions, which brings something like this. So where F_l X, this F_i rather, dx is given by this, and $F_o$dz is given by this.

We can rewrite the integral over fl(X) as

$$\int_{V_F} f_I(X)\,dX = \int I_0(Z)\,I_0(Z+D_Z)\,dZ \times \int\int W_0(X,Y)\,W_0(X+D_X,Y+D_Y)\,dX\,dY = \int_{V_F} V_0^2(X)\,dX \cdot F_O(D_Z)\,F_I(D_X,D_Y)$$

with

$$F_I(D_X,D_Y) = \frac{\int\int W_0(X,Y)\,W_0(X+D_X,Y+D_Y)\,dX\,dY}{\int\int W_0^2(X,Y)\,dX\,dY}$$

and

$$F_O(D_Z) = \frac{\int I_0(Z)\,I_0(Z+D_Z)\,dZ}{\int I_0^2(Z)\,dZ}.$$

FI as a factor expressing the in-plane loss-ofpairs, and FO as a factor expressing the out-of-plane loss-of-pairs. When no in-planeor out-of-plane loss-of-pairs are present the latter two are unity.
Finally

$$E\{R_D(s,D)\} = C_R\,R_\tau(s-d)\,F_O(D_Z)\,F_I(D_X,D_Y)$$

constant C_R is defined as:

$$C_R = \frac{N}{V_F}\int_{V_F} V_0^2(X)\,dX\,.$$

So F_i, as a factor, expresses the in-plane loss of pairs, and F_o is a factor which expresses the out-of-plane loss of pairs. When in-plane or out-of-plane loss of pairs is present, the latter, when there is nothing, these two factors are actually unity. Therefore, your EDE is equal to something like this, where this constant is nothing but this integral. So it's a lot of math, but you can follow it. You can work out the math in your free time, and this is what you are going to get.

So now that we have done in the previous one, we have calculated what the expected value is and stuff like that. We can now move on to the image evaluation methods in PIV. So the main objective here is the statistical evaluation of PIV recordings at medium image density. So basically, you cannot track the individual particles, but you are supposed to determine the displacement between two patterns of randomly distributed particles. which are stored as a 2D distribution of gray levels.

Image Evaluation Methods for PIV

The main objective of the statistical evaluation of PIV recordings at medium image density is to determine the displacement between two patterns of randomly distributed particle images, which are stored as a 2D distribution of gray levels.

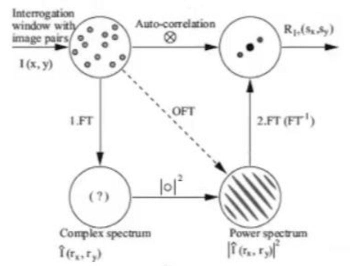


Fig. 5.1 Sketch of relation between 2D correlation function and spatial spectrum by means of the Wiener–Khinchin theorem. FT – Fourier transform, FT^{-1} – inverse Fourier transform, OFT – optical Fourier transform

Figure 5.1 illustrates that the auto-correlation function can either be determined directly in the spatial domain (upper half of the figure) or indirectly by Fourier transform FT (left hand side), multiplication, that is the calculation of the squared modulus, in the frequency plane (lower half of the figure), and by inverse Fourier transform FT−1 (right hand side).

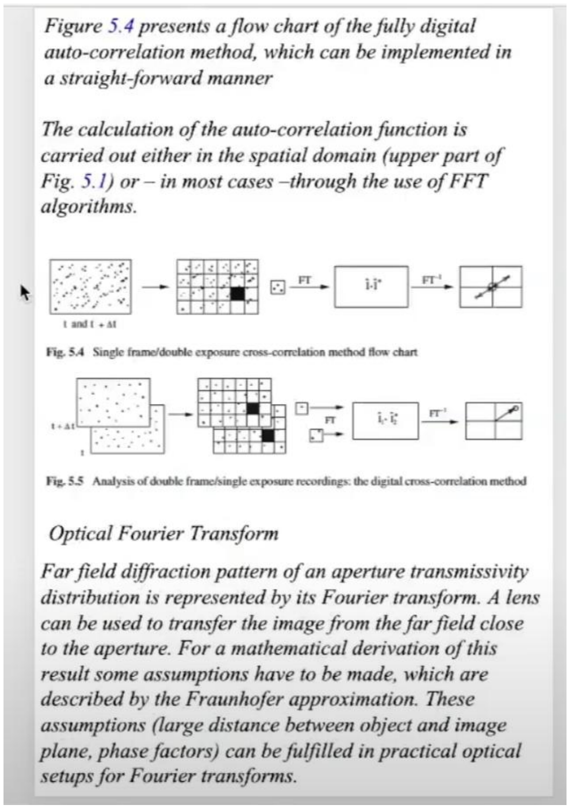
Wiener-Khinchin theorem [8, 10]. This theorem states that the Fourier transform of the auto-correlation function R_I and the power spectrum $|\hat{I}(r_x, r_y)|^2$ of an intensity field $I(x, y)$ are Fourier transforms of each other.

So that is what it is. So that is what we are supposed to find out: how these patterns relate and what the displacement is between these two patterns. So, this is an interrogation window with an image pair. Okay, so there are two ways to do this. You can do it directly, then perform autocorrelation.

Okay, or you can do what we call an optical Fourier transform, and you get the power spectrum. Okay, and from the power spectrum, you perform an inverse Fourier transform and get the same autocorrelation function. Or you can do an FFT first or a Fourier transform first, then calculate the power spectrum, and then using the power spectrum, you can do an inverse Fourier transform to get the same autocorrelation function. So this is the figure that illustrates that the autocorrelation function can be determined directly in the spatial domain. That means you can directly determine this part in the spatial domain, or you can determine it indirectly from the Fourier transform.

Multiplication, and that is how you calculate the square of the modulus in the frequency plane, followed by the inverse Fourier transform. So here the important theorem is Wiener's theorem. So this theorem states that the Fourier transform of an autocorrelation function, which is R(i), and the power spectral density, which is i^2, are Fourier transforms of each other, so you understood. One is that you do it in the spatial domain, and one is that you do it in the frequency domain. You can do it in two ways.

One is the optical Fourier transform, and the other is a normal digital Fourier transform. So in this case, what is the important theorem? The theorem states that the Fourier transform of the autocorrelation function, this function, and the power spectral density, which is this function, are Fourier transforms of each other. So you can perform an inverse Fourier transform, and you can obtain that particular autocorrelation. So this basically uses a mathematical tool to do the calculations in what we call the frequency domain and not in the spatial domain. So here you have a flow chart of a fully digital autocorrelation method that can be implemented in a very straightforward manner.



So, this is what happens. You get the images at t + Δ t. You take those interrogation windows. You perform a Fourier transform on the images in that. You multiply it by its complex conjugate, and you get the i.i'.

And then you perform an inverse Fourier transform to get the velocity. So the calculation of the autocorrelation function is carried out either in the spatial domain, which is this, and in most cases, we do it through the use of the, so not this particular figure, but the figure before that. Or, in most cases, it is done through the FFT algorithms. So this is a single-frame double exposure, and it's an analysis of the double-frame single exposure. Both utilize this kind of flow that is t + Δ t, and this is t; then you take the two interrogation volumes, do the FFT from them, multiply one by the conjugate of the other, and then you do an inverse Fourier transform.

You can also do, if you look at the previous one, something that is called OFT, which stands for optical Fourier transform. So the optical Fourier transform is an aperture transmittance distribution. The far-field diffraction pattern of an aperture transmissivity distribution is represented by a Fourier transform. So a lens can be used to transfer the image from the far field to the aperture. So, from a mathematical derivation of this, some assumptions have to be made; for example, we have the Fraunhofer approximation.

These assumptions, such as large distances, as we know, allow Fraunhofer diffraction over large distances between the object and the image, which can be fulfilled in practical optical systems for Fourier transform. So what it does is provide two configurations for the optical Fourier transform. In this arrangement, the left-hand side of the object, which would consist of a transparency to be Fourier transformed, similar to the PIV recording, can be placed in front of the so-called Fourier lens. At one f-stop and at the other focal length, in the second setup it is placed behind the lens. The result of this optical Fourier transform is a power spectrum of great value in the distribution of the transparency.

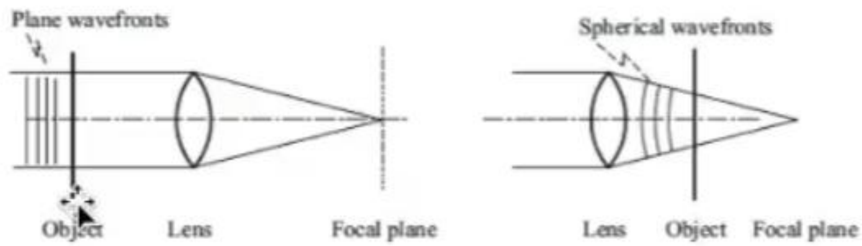


Fig. 5.2 Optical Fourier processor, different positions of object and Fourier lens

- *Two different configurations for such optical Fourier processors.*
- *In the arrangement on the left hand side the object, which would consist of a transparency to be Fourier transformed (e.g. the photographic PIV recording), is placed in front of the so-called Fourier lens (at -f usually). In the second setup (right hand side) the object is placed behind the lens.*

The result of the optical Fourier transform (OFT) is the power spectrum of the gray value distribution of the transparency.

So basically, the same thing is the power spectrum that we get over here. So just to recap, the far-field diffraction pattern of an aperture transmissivity distribution is represented by its Fourier transform. So a lens can be used to transfer the image from the far field to the aperture. So, from a mathematical point of view, we say that these are represented by the Von Hopper approximation. These assumptions, such as the large distance between the object and image, phase factors, etc.

, can be addressed by these setups for optical Fourier transforms. So, let us look at these two images. One is at sample one and sample two. One is i, and one is i’. This is image one; this is image two.

These are separated by a Δ t between them. Now, from a signal image processing point of view, the first image may be considered; this particular image is an input image to a system whose output produces the second image of the pair. Okay, if you think about it like this, is that okay? So how is this done? This is the input image, and then you have an image transfer function, which is essentially a spatial shift. Then you add some noise, additive noise, and then you get the output, which is I', which is the second image. So the system's transfer function from this to this converts the input image i to an output image; this is i’.

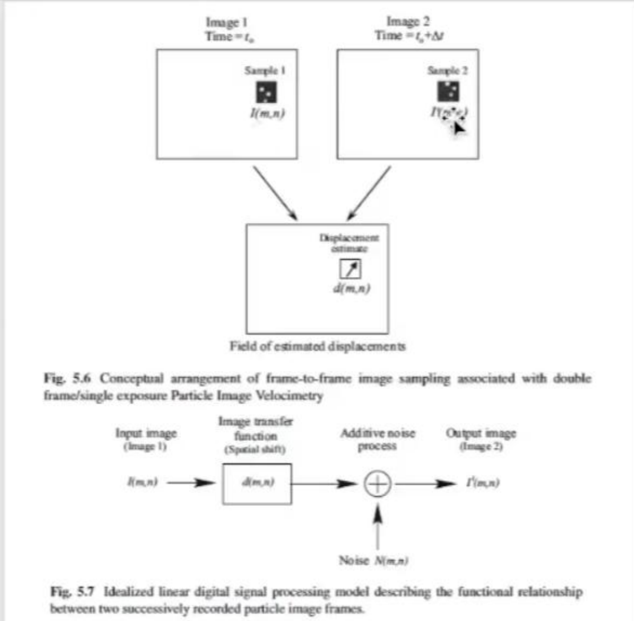


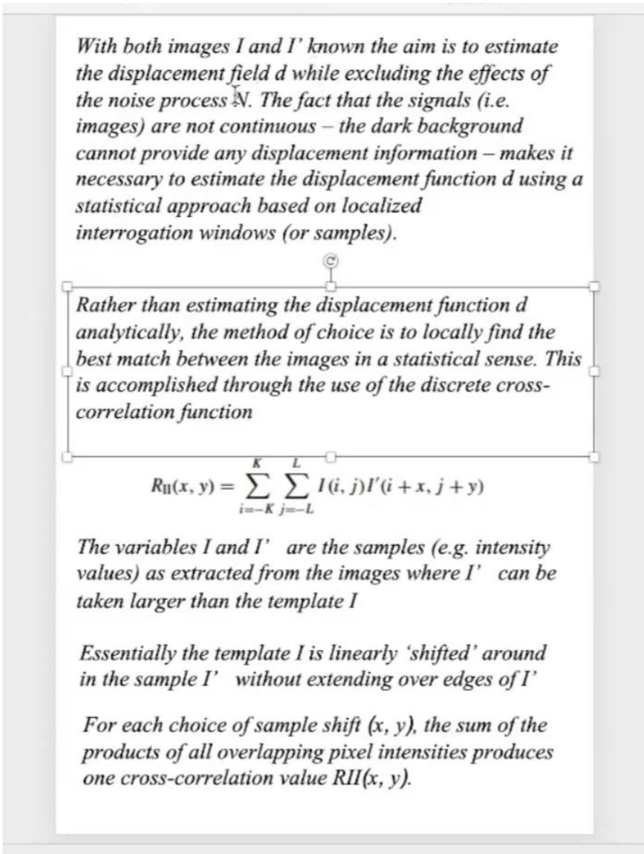
Fig. 5.6 Conceptual arrangement of frame-to-frame image sampling associated with double frame/single exposure Particle Image Velocimetry

Fig. 5.7 Idealized linear digital signal processing model describing the functional relationship between two successively recorded particle image frames.

- *From a signal (image) processing point of view, the first image may be considered the input to a system whose output produces the second image of the pair (Fig. 5.7).*
- *The system’s transfer function, H, converts the input image I to the output image I and is comprised of the displacement function d and an additive noise process, N.*
- *The function of interest is a shift by the vector d as it is responsible for displacing the particle images from one image to the next.*

I'm sorry, this should be I’ and is comprised of the displacement function. There is a displacement, and then there is an additive noise process, okay, n. The function of interest is a shift of this vector d, as it is responsible for displacing the particle images from one image to the next. Okay, so there is a shift. The function of interest is this vector shift, which is the displacement that we are most concerned about.

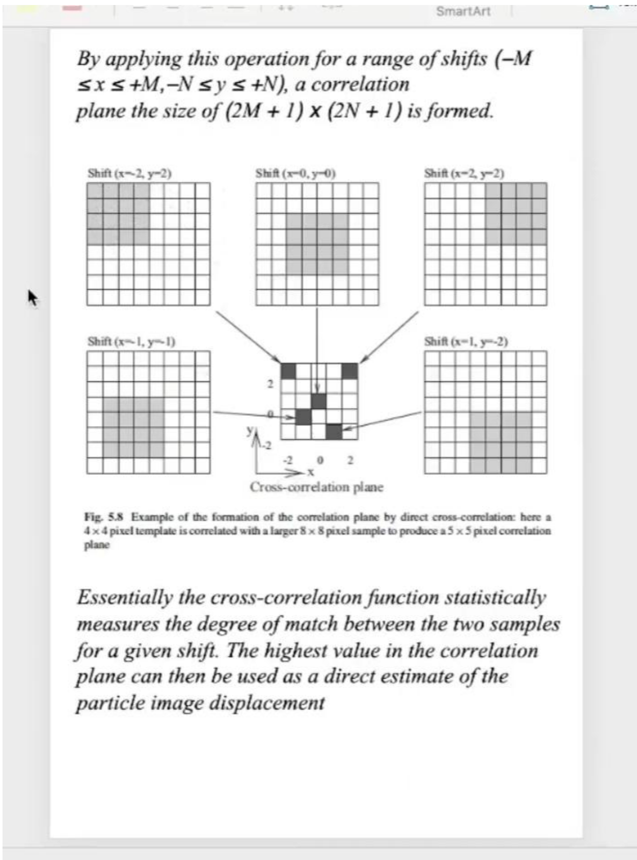
Right, so these are the two images, and this is ultimately the displacement estimate that we get for image evaluation. So when both images, i and i’, are known, this is i’. Okay, that is known; the aim is to estimate the displacement d. While excluding the effects of the additive noise, which is n, the fact that the signals are not continuous—meaning there is a dark background—cannot provide displacement information; it is necessary to estimate the displacement function d using a statistical approach on these interrogation windows, with which we are already well versed. So rather than estimating the displacement function d analytically, the method of choice is to find the best match locally between the two images in a statistical sense.



So this is done using the discrete cross-correlation function. So this is an image, and this is the image that is shifted. These are the samples, example intensity values extracted from the image. Where I prime can be taken larger than the template I. So I', the second output image, can be larger than the input image.

So essentially, the template I is linearly shifted around the sample I prime without extending over the edge. So you can see that this one may be smaller than this, so you are shifting this all over this image I'. And we will actually see it in the next slide. So, for each shift, the sum of the products of all overlapping pixel intensities produces one cross-correlation value. So, that is what we are going to see here.

So, this is what happens. So you see that there is a larger I prime, and this is the image. This is the I. So we are shifting it across different spatial dimensions; you know, the shift can be X equal to two, Y = 2,0. Zero means there is no shift.



By applying this operation for a range of shifts (−M ≤ x ≤ +M, −N ≤ y ≤ +N), a correlation plane the size of (2M + 1) x (2N + 1) is formed.

Fig. 5.8 Example of the formation of the correlation plane by direct cross-correlation: here a 4 × 4 pixel template is correlated with a larger 8 × 8 pixel sample to produce a 5 × 5 pixel correlation plane

Essentially the cross-correlation function statistically measures the degree of match between the two samples for a given shift. The highest value in the correlation plane can then be used as a direct estimate of the particle image displacement

It is in the center. It should not cross the edge. That is what it is trying to say. And then you can shift it in all possible directions. So there is a range of shifts which is given as - m to + m and - n to + n. This produces a correlation plane which is of the size two m + one multiplied by two n + 1.

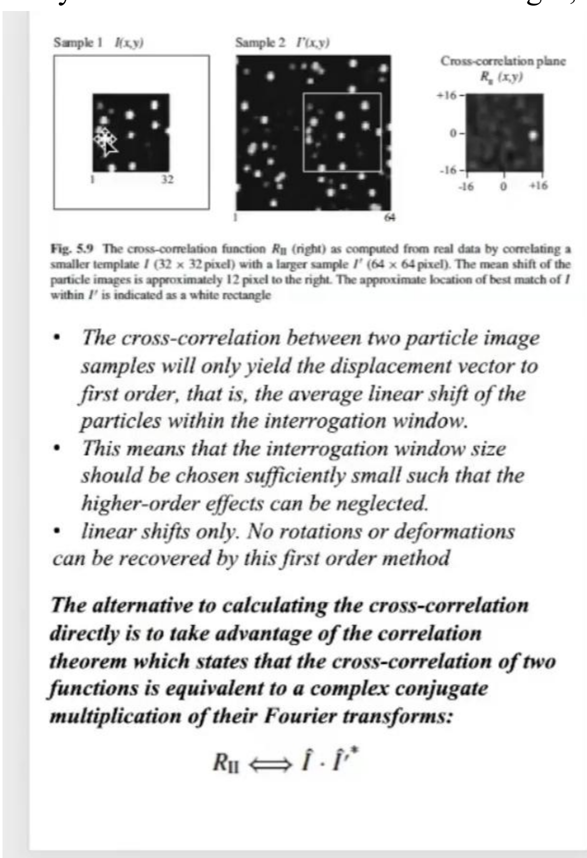
So, for example, in this case, if you look at it, this is a 4x4 pixel template, and this is on a much larger template of eight by eight. Okay, so eight by eight and the shift; therefore, you can do + 2 and you can do - 2 on both sides. So, 2x2 is 4, +1, which is 5, and it's similar with N because it's a square. So , 2x2 is 4, +1, so this is also 5. So you produce a 5x5 pixel correlation plane that you ultimately get out of it.

So essentially, what the cross-correlation function is statistically doing is trying to measure the degree of match between two samples for a given shift. The highest value of the correlation can then be used as a direct estimate of the particle displacement. So you understand this very carefully. So the variables are sample images, essentially the template, and where I can take larger images than the template.

And, uh... and you know, uh... for each choice, without extending over the edges of, uh... edges of i prime, we do not extend beyond i prime. So, for each choice, the sum of the products of all the overlapping pixel intensities produces one cross-correlation value, and therefore, by shifting this, we are trying to see the degree of match between the two samples.

The highest value in the correlation plane is therefore a direct estimate of the particle distribution, and we also know now how to calculate this: 2m + 1 into 2n + 1. Where does it come from? Okay, and this is the first one; this is a much larger second pass. All right, so the Cross-correlation between, for example, this is the first image and this is the second image. So this is a much larger second image. So the cross-correlation between two particle images will yield the displacement vector only to the first order.

That means it will calculate the average linear shift of the particles within that interrogation window. So this means that the interrogation window should be chosen sufficiently small so that higher-order effects can be neglected. That means these are linear shifts only; no rotations or deformations can be derived from this. So, for example, if you look at these two images, this is the cross-correlation plane.



You see, there is a very bright spot. So, the mean shift of the particles is 12 pixels. For example, in this case, it is 32 x 32 with a larger sample of 64 x 64. The mean shift is approximately 12 pixels to the right. So the approximate location of the best match of i within I' is indicated by that white triangle, the white spot, and the white rectangle, actually. So that is the shift that you are actually getting, all right? So the alternative to calculating the cross-correlation directly is to take advantage of the correlation theorem, which states that the cross-correlation of two functions is equal to the complex conjugate multiplication of their Fourier transforms.

So this is, once again, you go to the spatial domain. So the cross-correlation function is nothing but the multiplication of the Fourier transform, the intensity, and its complex conjugate. That means i‘ , and then the i and I’ conjugate.

So this is the complex conjugate. All right? So... Implementation of the cross-correlation using the Fourier transform involves having image i and then image 2; then you perform a real-to-complex FFT and again a real-to-complex FFT. After that, you do a complex conjugate multiplication to obtain i from 1, which is the FFT of the first image, and then the FFT of the second image, followed by a complex conjugate multiplication. Then you do a complex-to-real inverse Fourier transform, and this will give you the correlation data. Okay, so computing two-dimensional FFTs on equal size samples is followed by the complex conjugate multiplication of the resulting Fourier coefficients. So these are then inversely Fourier transformed to the actual cross-correlation plane, which has the same spatial dimensions.

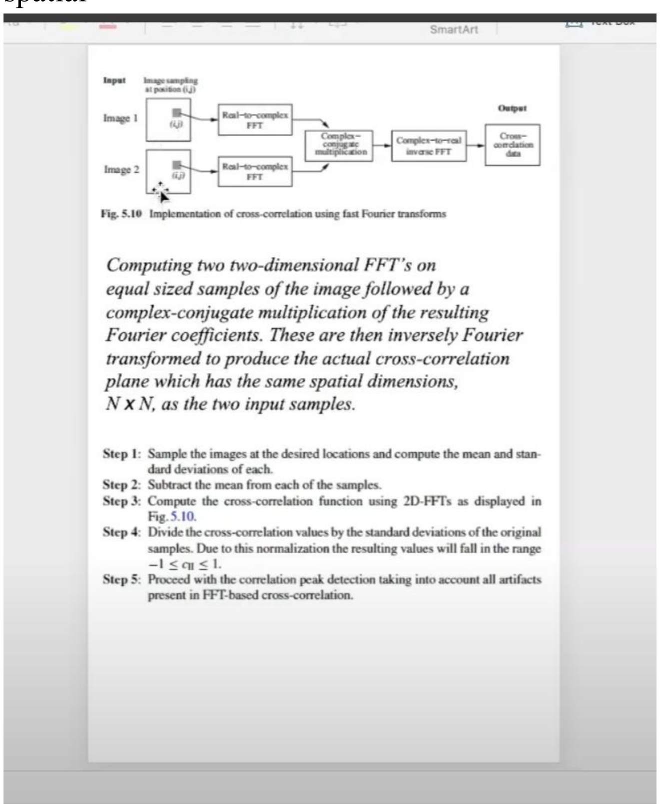


Fig. 5.10 Implementation of cross-correlation using fast Fourier transforms

Computing two two-dimensional FFT's on equal sized samples of the image followed by a complex-conjugate multiplication of the resulting Fourier coefficients. These are then inversely Fourier transformed to produce the actual cross-correlation plane which has the same spatial dimensions, N x N, as the two input samples.

Step 1: Sample the images at the desired locations and compute the mean and standard deviations of each.
Step 2: Subtract the mean from each of the samples.
Step 3: Compute the cross-correlation function using 2D-FFTs as displayed in Fig. 5.10.
Step 4: Divide the cross-correlation values by the standard deviations of the original samples. Due to this normalization the resulting values will fall in the range $-1 \leq c_{II} \leq 1$.
Step 5: Proceed with the correlation peak detection taking into account all artifacts present in FFT-based cross-correlation.

So these are the steps that sample the images at the desired location, compute the mean and the standard deviation, subtract the mean from each sample, then calculate the cross-correlation using 2D FFTs, and then divide the cross-correlation by the standard deviations just for normalization purposes. Due to this normalization, all values will fall within the range of - 1 to + 1, and then proceed with correlation peak detection, taking into account all these artifacts by using this FFT-based correlation. So, these are the steps. So, in some cases, you also need to do image pre-processing. So the image pre-processing occurs because of the correlation signal; you know, why do we need that? It's strongly affected by the variations in image intensity.

Image Pre-processing

- *The correlation signal is strongly affected by variations of the image intensity.*
- *The correlation peak is dominated by brighter particle images with weaker particle images having a reduced influence.*
- *Also the non-uniform illumination of particle image intensity, due to light-sheet non-uniformities or pulse-to-pulse variations, as well as irregular particle shape, out-of-plane motion, etc. introduce noise in the correlation plane*

Background subtraction from the PIV recordings reduces the effects of laser flare and other stationary image features

A filter-based approach to image enhancement is to high-pass filter the images such that the background variations with low spatial frequency are removed leaving the particle images unaffected.

Thresholding or image binarization, possibly in combination with prior highpass filtering, results in images where all particles have the same intensity and thus have equal contribution to the correlation function.

The application of a narrow-width, low-pass filter may be suitable to remove high frequency noise (e.g. camera shot noise, pixel anomalies, digitization artifacts, etc.)

Normalization by the time-average image intensity is also an effective technique to remove stationary background with time-varying intensity.

And since the correlation peak is dominated by the brighter particle images, the weaker particle images have a reduced influence. So non-uniform illumination can occur due to light sheet problems, as well as out-of-plane motion, noise, and many other factors. So the steps that we do normally is to do a background subtraction from the PIV recordings, which reduces the effects of laser flare. And then you use a filter-based approach to image enhancement, like a high-pass filter, to remove the background variations, and a low-pass spatial filter removes the background variations that have low spatial frequency, which is removed by the high-pass filter.

Then you do thresholding or image minimization. This is in combination with prior high-pass filtering that results in images that have the same intensity. Okay, then you apply a narrow width or low pass filter to remove the high-frequency noise. So this may be camera shot noise; you know, pixel anomalies, digitization artifacts, etc. And then you perform a normalization by the time-averaged image intensity, which can also remove this stationary background to quite some extent. In some cases, you use what we call ensemble correlation techniques.

So, in these cases, the previous technique was used within a given PIV image pair. Now it can be applied to a sequence of images. So, this approach is called ensemble correlation. So rather than obtaining a displacement peak for an individual image pair, you take the average of the correlation planes obtained from a sequence of events and then calculate the mean displacement of the flow. But in this case, as you can imagine, all the unsteady information is lost because of this averaging and this ensemble averaging.

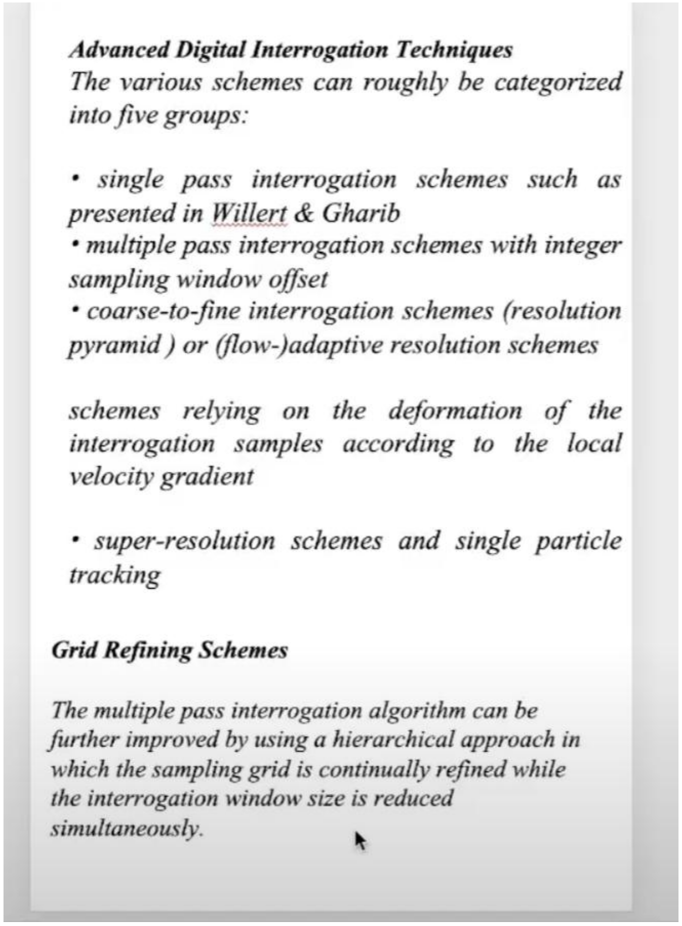
Ensemble Correlation Techniques

While the previous method is applied within a given PIV image pair, it can also be applied to a sequence of images. This PIV processing approach, also known as ensemble correlation or correlation averaging

Rather than obtaining displacement data for each individual image pair, the technique relies on averaging the correlation planes obtained from a sequence of images. With increasing frame counts a single correlation peak will accumulate for each correlation plane reflecting the mean displacement of the flow

All Unsteady Information is lost

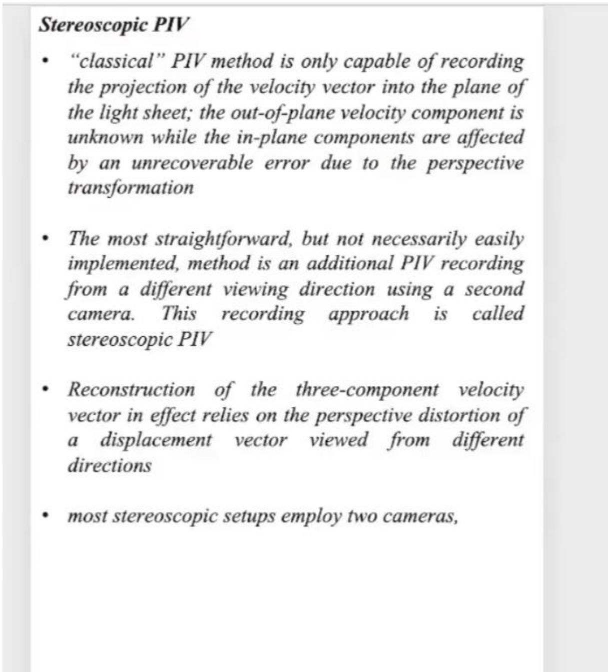
Maybe it's good enough for, you know, low-speed flow as well as for time-periodic flows. So there are advanced digital interrogation techniques, as well. For example, there are various schemes that can be roughly categorized. For example, there may be single-pass interrogation schemes such as those presented in Willert and Karib. Then multi-pass interrogation schemes, courses to find interrogation schemes, and schemes relying on the deformation of the interrogation windows.



Super-resolution schemes. These are not within the scope of this, but you can read the book by Marcus Raphael to learn more about it. There are also grid refining schemes that use a multi-pass interrogation algorithm by decreasing the window size hierarchically. So one result from one pass actually gets fed into the next one with grid refinement. So this enables us to get an even finer distribution.

The last thing that we will do very quickly is stereoscopic PIV. Classical PIV is capable of recording the projection of the velocity vectors on the plane. Out-of-plane components actually lead to the deterioration of the in-plane components. As we already know, we have covered this topic in detail because of what we call perspective transformation. These are systemic or unrecoverable errors. So in order to image this third velocity component, you use an additional PIV recording from a different viewing angle.

So this approach is called stereoscopic PIV, which kind of looks at our eye. You are looking at the same thing from two different perspectives, essentially. So, the reconstruction of the velocity depends on the perspective distortion of the displacement vector, which we observe from two different directions. So most of these setups will utilize two cameras.



One camera can also be used with relevant optics. So there are two ways to do this. You can have a translation mode and a translation method, which basically means that this is the lens and you shift the lens. Basically, this is like that. So first, the magnification across the field of view is constant.

There is no image deformation. And second, all particles are in focus. As a depth of field, this one is parallel to the main plane of the lens. But this has a lot of disadvantages as well. The alternative arrangement is called the angular method, or the angular displacement method. This is what you see over here. So this angular method aligns the lens with respect to the principal viewing direction.

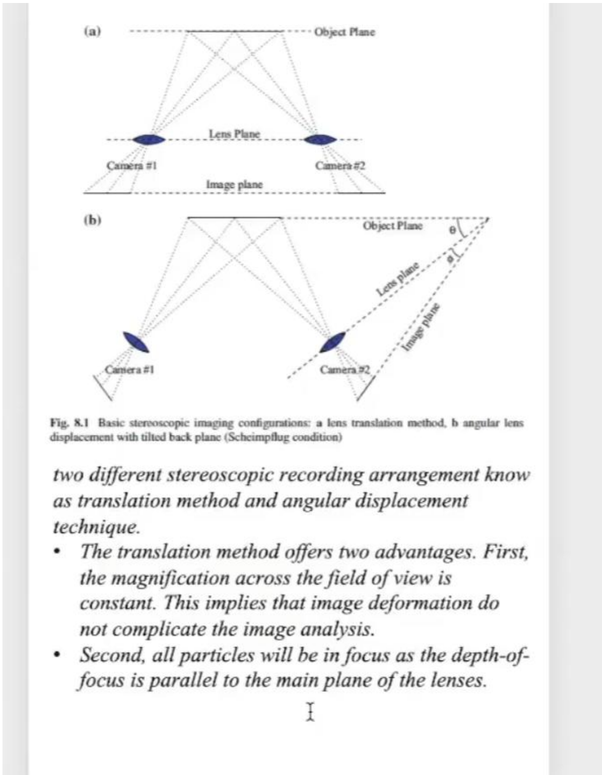


Fig. 8.1 Basic stereoscopic imaging configurations: a lens translation method, b angular lens displacement with tilted back plane (Scheimpflug condition)

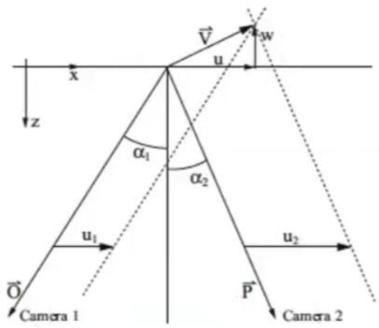
two different stereoscopic recording arrangement know as translation method and angular displacement technique.

- *The translation method offers two advantages. First, the magnification across the field of view is constant. This implies that image deformation do not complicate the image analysis.*
- *Second, all particles will be in focus as the depth-of-focus is parallel to the main plane of the lenses.*

So, if this is the viewing direction, this is how we have angled it. The Scheimpflug criteria is used in which the camera, the image plane, the lens plane, and the object plane all intersect in a common line. Remember? Okay, so here the perspective distortion results in a magnification factor that is no longer constant across the field of view and requires additional calibration means. So these are the different techniques by which you can look at the flow. So this involves what we call reconstruction geometry. So, how do we reconstruct the geometry? So if you look at it, this is the XZ plane, and the Y plane is inside the paper.

Reconstruction Geometry

This section describes the geometry necessary to reconstruct the three-dimensional displacement field from the two projected, planar displacement fields.



For particle image displacement assuming geometric imaging:

$$x'_i - x_i = -M\left(D_X + D_Z \frac{x'_i}{z_0}\right)$$

$$y'_i - y_i = -M\left(D_Y + D_Z \frac{y'_i}{z_0}\right)$$

$$\tan\alpha = \frac{x'_i}{z_0}$$

$$\tan\beta = \frac{y'_i}{z_0}$$

We will use the angle α *in the XZ plane between the Z axis and the ray from the tracer particle through the lens center O to the recording plane Correspondingly,* β *defines the angle within the Y Z plane*

So if you look at this, there are two cameras: camera one and camera two. Okay, so if you look at it carefully for a particle image displacement, assuming geometric imaging, this is something that you have already seen: that $x'_i - x'$ is given as the magnification into the actual movement in the object plane, and then you have these other factors due to the out-of-plane motion. So, there is an angle that it creates. You can see that this angle is α. So this angle in the x-z plane between the z-axis and the ray from the tracer, which is this, is passing through the lens center.

The lens center is somewhere around here. Okay, this angle is α. And therefore, there is also an angle β in the YZ plane, which is the plane that is perpendicular to this. Okay, so you can define tan α. There will actually be 2tan αs. Tan α is x'_i / z_o.

And $\tan\beta = Y'_i / Z_o$. Okay, so this is the arrangement. These are all from the geometry, so you can see this is the arrangement that we have now if the velocity components, which are measured by the left camera, which is this camera, are given by this. Okay u_1 / v_1 so you can see what u_1 and v_1 are. Okay, they are given as x'_i - xi divided by the magnification factor into Δ t, which is the time differential. So for the right camera, it will be u_2 and v_2, and it will be exactly the same type of expressions.

Now, if you use this for, you have u_1, u_2, and v_1, v_2. Using the above equations, the three velocity components, now you can cast it and write it in this form. Look at the thing that this is $(u_1 \tan \alpha_2 + u_2 \tan \alpha_1)/ \tan \alpha_1$ and $\tan \alpha_2$, where this is the angle that it makes in the xz plane, okay, α. And similarly, for v, there is the β. Which is basically the angle that this is the same, actually, but it is in the yz plane, which is perpendicular to the plane of the paper. Now you can recover the w component in two ways: you can recover it from $u_1 - u_2$ or from $v_1 - v_2$

The velocity components measured by the left camera are given by

$$U_1 = -\frac{x_i' - x_i}{M\,\Delta t}$$

$$V_1 = -\frac{y_i' - y_i}{M\,\Delta t}$$

The velocity components for the right camera U2 and V2 can be determined accordingly. Using the above equations, the three velocity components (U, V,W) can be reconstructed from the four measured values.

$$U = \frac{U_1 \tan\alpha_2 + U_2 \tan\alpha_1}{\tan\alpha_1 + \tan\alpha_2}$$

$$V = \frac{V_1 \tan\beta_2 + V_2 \tan\beta_1}{\tan\beta_1 + \tan\beta_2}$$

$$W = \frac{U_1 - U_2}{\tan\alpha_1 + \tan\alpha_2} = \frac{V_1 - V_2}{\tan\beta_1 + \tan\beta_2}$$

Note, that there are three unknowns and four known measured values, which results in an overdetermined system that can be solved in a least-squares sense

To use the above reconstruction, the displacement data set must first be converted from the image plane to true displacements in the global coordinate system while taking into account all the magnification issues.

So you can see that there are three unknowns, u, v, w, and four known measurements, which are u_1, u_2, v_1, and v_2. So it results in an over-determined system that needs to be solved in a least squares sense. So to use this reconstruction, the displacement data set must first be converted from the image plane to the true displacement while taking into account all the magnification issues. So, it's a simple enough thing. So, just by looking at it and using the angles, you can actually determine the three velocity components.

Of course, there are many errors and many things that are uncertain about it. Those things you can read in the book, but the time is too short to go into the details of all of them.