Advanced Measurement Techniques in Fluid Mechanics and Heat Transfer
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Week – 07
Lecture - 35
Particle Image Velocimetry – 5

All right, so in today's lecture, we are going to look at the mathematical background of statistical PIV evaluation. So this part is rather mathematical, so we'll go a little faster. You can look up the math in the book. There is Ron Adrian's book, and there is Marcus Raffel's book. These notes are taken from Marcus Raffel's book. So the first detailed mathematical description of statistical PIV evaluation was given by Ron Adrian.

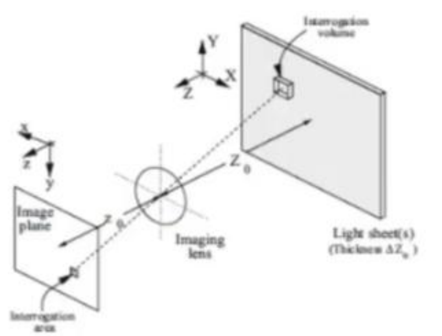
Mathematical Background of Statistical PIV Evaluation

The first detailed mathematical description of statistical PIV evaluation has been given by Adrian. This early work from 1988 concentrated on auto-correlation methods and was later expanded to cross-correlation analysis. A complete and careful mathematical description of digital PIV has been given by Westerweel. Adrian & Westerweel published the most complete book on Particle Image Velocimetry in 2011

Particle Image Locations

- Typically, PIV recordings are subdivided into interrogation areas during evaluation. These areas are called interrogation windows.[1] Due to reasons stated afterwards, for cross-correlation analysis those interrogation areas need not necessarily be located at the same position of the PIV recording. Their geometrical back-projection into the light sheet will be referred to as interrogation volumes
- The local sample of a PIV image from which a velocity vector is determined is referred to as the interrogation window. Its size determines to what degree the recovered velocity field is spatially smoothed.

Fig. 4.1 Schematic representation of geometric imaging



Two interrogation volumes used for statistical evaluation together define measurement volume

And later on, a kind of beautiful and complete mathematical description of digital PIV was provided by Westerwill and Adrian. in their seminal book. So, remember, this is a handbook, okay? So this is more practical. So typically, when you look at a PIV image, let's examine this particular image.

So this is the light sheet, the interrogation window, the object distance, and the image plane from the imaging lens. And the light sheet has a thickness of Δ Z. So the interrogation window is right over there, and XYZ is the object plane while xyz is the image plane. So, typically, the PIV recordings are subdivided, as we know, into interrogation windows during evaluation. These areas are basically called the interrogation windows.

So we will show later that for cross-correlation analysis, these windows need not be located at the same position. And their geometric back projection will be referred to as the interrogation volumes. Now this is a local sample of a PIV image from which a velocity vector is determined and referred to in the velocity vector of that interrogation window. Its size determines the degree to which the recovered velocity value is spatially smoothed. Okay, so these two interrogation volumes are used for statistical evaluation, which, when measured together, constitutes the measurement volume, these two interrogation volumes, okay? Okay, so this is the interrogation volume, and this is the interrogation area.

All right, so let's look at the next one. So a single exposure is considered. It consists of a random distribution of particles, okay? So, which corresponds to the following pattern of N tracer particles inside the flow. So this γ that you see over here describes the state of the ensemble at a time T. X_i, X_i, is a position vector of particle I at time T.

A single exposure recording is considered. It consists of a random distribution of particle images, which correspond to the following pattern of N tracer particles inside the flow:

$$\Gamma = \begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N \end{pmatrix} \quad \text{with} \quad X_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

Γ describes the state of the ensemble at a given time t. Xi is the position vector of the particle i at time t.

The lower case letters refer to the coordinates in the image plane

$$x = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$X_i = x_i / M \quad \text{and} \quad Y_i = y_i / M$$

Particle position and the image position are related by a constant magnification factor M

All right, so if you go back once again, you will see that, okay, these are the capital XYs, and you know, this is basically an ensemble, a state of the ensemble at a particular given time t, where X_i is basically the position vector, okay? Position vector xy of the particle at time t: the lowercase refers to the coordinates in the image plane now, okay? So this x, again, going back to the figure, you will see that this is basically the small xy. So xy, the lowercase, refers to the coordinate in the image plane. So, how are they related? XY = xy / m, and $Y_i = y_i$ / m. So the particle position and image position are related by a constant magnification factor, which is given as m. All right, so this is given as m.

The particle position and the image position. So this should not be very difficult to comprehend. Now let us look at the image intensity fields. This is where things become a little complicated. So you know that the mathematical representation of the intensity distribution on the image plane is what is to be determined, which is given.

Image Intensity Field

- Mathematical representation of the intensity distribution in the image plane is given. It is assumed that the image can best be described by a convolution of the geometric image and the impulse response of the imaging system, the point spread function [3]. For infinite small particles and perfectly aberration-free, well focused lenses the amplitude of the point spread function can mathematically be described by the square of the first order Bessel function also known as Airy function

- In the following analysis we assume the point spread function of the imaging lens $\tau(x)$ to be Gaussian versus x and y. The convolution product of $\tau(x)$ with the geometric image of the tracer particle at the position x_i therefore describes the image of a single particle located at position X_i. Furthermore, we restrict the analysis to infinitely small geometric particle images which would be the case for small particles imaged at small magnifications. Therefore, we use the Dirac delta-function shifted to position x_i to describe the geometric part of the particle image. As schematically illustrated in Fig. 4.2, the image intensity field of a single exposure may be expressed by:

$$I = I(x, \Gamma) = \tau(x) * \sum_{i=1}^{N} V_0(X_i)\,\delta(x - x_i)$$

where $V0(Xi)$ is the transfer function giving the light energy of the image of an individual particle i inside the interrogation volume VI and its conversion into an electronic signal. τ (x) is considered to be identical for every particle position.

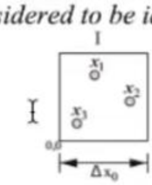


Fig. 4.2 Example of an intensity field I (single exposure)

So, what is the image? It is assumed that the image can be described by the convolution of two things. One is the geometric image, and the other is a point spread function, which is the impulse response of the imaging system. So for infinitely small particles, and if we assume that our lenses are aberration-free and well-focused, et cetera, et cetera, the amplitude of the point spread function, which is the impulse response function, is basically described by the first-order Bessel function, which is also called the Airy function. We already did this a little earlier. So, what is it? The image is a convolution of the geometric image and the impulse response, which is the point spread function.

So for infinite small particles and for perfect lens assembly, this is described by a first order Bessel function. So the Airy function is called the first-order Bessel function. So in the analysis, we are going to assume that the point spread function of the imaging lens, which is $\tau(x)$, is Gaussian in terms of x and y. Okay, so we are going to say that the point spread function of the imaging lens is Gaussian. So the convolution product of $\tau(x)$, with the geometric image of the tracer particle at a position x_i, therefore, describes the image of the single particle located at X_i.

The X_i is the actual particle that is located at XY in the object plane. So here we are assuming that if we convolute the product, if we $\tau(x)$, which is the point spread function that is Gaussian, with the geometric image of the tracer particle at the image plane, which is xy, together this convolution will describe the image of the single particle located at XY. Therefore, we restrict the analysis to infinitely small geometrical particles, which would be the case for all small particles imaged at small magnifications. Therefore, what we can do is use the Dirac Δ function shifted to a position xi to describe the geometric part of the particle image. The geometric part of the image, so you already know that the spread function is Gaussian.

Now the geometric part of the image for small particles and at small magnifications is described by the Dirac Δ function shifted to that particular particle position $\tau(x)$.So the schematic shown over here, if you look at the schematic, is an example of an intensity field with x_i. Therefore, we use the Dirac Δ function shifted to small xi to describe the geometric part. So the image intensity field therefore becomes i, which is a function of x; γ is equal to the point spread function, which is $\tau(x)$ multiplied by this term, where Δ is nothing but the Dirac Δ function shifted to the position of x_i. So this v_0 is basically a transfer function that gives the light energy of the image of an individual object i inside the interrogation volume v_i and its conversion into an electronic signal.

So $\tau(x)$ is considered to be identical for every particle's position. That means the area of the disc is the same for all particles. All right? Only the Δ x – x_i is basically shifted to the corresponding particle locations, x_i. And what is V_o x_i? It is just a transfer function, which is basically the light energy of the image of an individual particle and its conversion into an electronic signal, and that's about it. So in many situations, what happens is that we assign different weights to different locations inside the interrogation window by multiplying the recorded image with a kernel, a weighted kernel, basically.

- In many situations different weight is assigned to different locations inside the interrogation area by a multiplication of the recorded image intensity with weight kernels.
- Further on, we presume that Z is the viewing direction, the light intensity inside the interrogation volume is only a function of Z and the image intensity finally analyzed depends on X and Y only due to the weight function.
- Therefore, $V_0(X)$ just describes the shape, extension and location of the actual interrogation volume:

$$V_0(X) = W_0(X, Y)\, I_0(Z)$$

where I0(Z) is the intensity profile of the laser light sheet in the Z direction and W0(X, Y) is the interrogation window function geometrically back-projected into the light sheet.

$$I_0(Z) = I_Z \exp\left(-8\frac{(Z-Z_0)^2}{\Delta Z_0^2}\right)$$

Gaussian intensity profile of the laser light sheet, where Z0 is the thickness of the light sheet measured at the e−2 points and IZ is the maximum intensity of the light sheet.

$W_0(X, Y)$ can be described in a similar way if a Gaussian window function with a maximum weighting W_{XY} at position X_0, Y_0 has to be considered:

$$W_0(X, Y) = W_{XY} \exp\left(-8\frac{(X-X_0)^2}{\Delta X_0^2} - 8\frac{(Y-Y_0)^2}{\Delta Y_0^2}\right)$$

Intensity distribution which is closer to a top-hat function than to a Gaussian function and since digitized recordings are commonly interrogated with rectangular windows, V0(X) can also be defined as a rectangular box:

$$I_0(Z) = \begin{cases} I_Z \text{ if } & |Z - Z_0| \le \Delta Z_0/2 \\ 0 \text{ elsewhere} & \end{cases}$$

So we assign different weights at different points in the interrogation window; further, we assume that z is the viewing direction. Okay, the light intensity inside the interrogation volume is only a function of Z, and the image intensity finally analyzed depends on x and y only due to the weight function. Okay, so if Z is the viewing direction, the light intensity inside the interrogation volume is only a function of Z. Therefore, V_o X describes the shape, extension, and location of the actual interrogation window. So what is V_o X? It's a multiplication of W_o XY into I_o Z.

So I_o Z is the intensity profile of the laser light sheet in the z-direction. And W_o XY is the interrogation window function geometrically back-projected into the light sheet. This is also assumed to be so the Gaussian intensity profile is assumed; the laser light sheet intensity profile is assumed to be Gaussian, where Z_o is the thickness of the light sheet, and IZ is basically the maximum intensity of the light sheet. This we have already covered a little bit earlier, which is why the laser intensity is Gaussian. And W_o XY, which is the interrogation window function, geometrically back-projected, is described in a similar way, by a Gaussian window function, with a maximum weighting, which is WXY, at a position (x0, y0).

All right. So this is also provided by this particular relationship. OK, so this is a weighting function. And because of the viewing angle, we have this weighting function as well as the laser intensity sheet profile. This is also to remember that the interrogation window, and the different locations of the interrogation window, are assigned different weights. All right, so the V_o, which describes the shape, extension, and location of the actual interrogation window, is a multiplicative of two terms.

One is the laser intensity profile, which is Gaussian and basically spreads, okay? The other one is a Gaussian window function, which has its peak location at X_o, Y_o. So this is the thing that we actually see here. All right? Okay, and $\tau(x)$, remember once again, is a point spread function or the Airy function. Okay? So the intensity distribution, if the intensity distribution is like a top hat function rather than a Gaussian function, then the digitized recordings are commonly interrogated with rectangular windows. In that case, first and foremost, instead of these complications, we write that I_o z, which is the laser intensity profile, is equal to I z, which is the maximum intensity, because it is now a top hat profile.

$$W_0(X, Y) = \begin{cases} W_{XY} & \text{for } |X - X_0| \leq \Delta X_0/2 \text{ and } |Y - Y_0| \leq \Delta Y_0/2 \\ 0 & \text{elsewhere.} \end{cases}$$

The factor $I_0(Z_i)$ represents the amount of light received from the particle i inside the flow, and located at distance $|Z_i - Z_0|$ from the center plane of the laser light sheet. ΔZ_0 is the light sheet thickness and therefore the extension of the interrogation volume in the Z direction. $\Delta X_0 = \Delta x_0/M$ and $\Delta Y_0 = \Delta y_0/M$ are the extension of the interrogation volume in the X- and Y-direction respectively. With $\tau(x - x_i) = \tau(x) * \delta(x - x_i)$ and the assumption that the particle images under consideration do not overlap, Eq. (4.1) can alternatively be written as:

$$I(x, \Gamma) = \sum_{i=1}^{N} V_0(X_i)\,\tau(x - x_i)\,. \quad \text{(see Appendix B)} \qquad (4.5)$$

$$I = I(x, \Gamma) = \tau(x) * \sum_{i=1}^{N} V_0(X_i)\,\delta(x - x_i)$$

This expression for the image intensity field will intensively be used

Remember, it's not Gaussian anymore. It's a top hat, which means it's a constant profile. So if $Z - Z_o$ is less than or equal to $\Delta Z_o/2$, this is IZ, and it is zero everywhere. So this is the intensity distribution that is closer to a top hat function, okay? Similarly, the window function can also be written in the same way, where it is WXY within a certain band and zero everywhere else. So now, therefore, what will happen? The factor I_o z represents the amount of light that is received by the particle inside the flow, and $z_i - z_o$ is from the center plane of the laser sheet.

Δz_o is the laser sheet thickness, and $\Delta X_o = x_o/m$, so all these terms are actually present over here, okay? Now what we see is that if you take a look at this particular equation, this is what we already discussed. You can transform this into this particular form, which you see over here. So there are many things that we'll come across a little later, but this is the form that is to be used. So now we go to the extent of spatial estimators for mean value and the variance of the image intensity field because this is the one that is utilized for the normalization of the cross-correlation. So, the spatial average is defined as this.

So these bracketed terms basically denote an average, and AI is the interrogation window. So basically, you integrate the image intensity across the interrogation window and divide it by the interrogation window size. Okay, so if you look at this particular expression, this is what we derived. Okay, and if you take the mean of this, this is what you are going to arrive at. Okay, so you can do these calculations again.

Spatial estimators for the mean value and the variance of the image intensity field, because these quantities will be used for the normalization of the cross-correlation.

The spatial average is defined as

$$\langle I(x,\Gamma)\rangle = \frac{1}{a_I}\int_{a_I} I(x,\Gamma)\,dx$$

where aI is the interrogation area.

$$I(x,\Gamma) = \sum_{i=1}^{N} V_0(X_i)\,\tau(x - x_i)$$

$$\Downarrow$$

$$\langle I(x,\Gamma)\rangle = \frac{1}{a_I}\int_{a_I}\sum_{i=1}^{N} V_0(X_i)\,\tau(x - x_i)\,dx$$

The mean value of the intensity field can be approximated by:

$$\mu_I = \langle I(x,\Gamma)\rangle = \frac{1}{a_I}\sum_{i=1}^{N} V_0(X_i)\int_{a_I}\tau(x - x_i)\,dx$$

The auto-correlation of the single exposure intensity field

$$R_I(s,\Gamma) = \langle I(x,\Gamma)I(x+s,\Gamma)\rangle$$
$$= \frac{1}{a_I}\int_{a_I}\sum_{i=1}^{N} V_0(X_i)\,\tau(x - x_i)\sum_{j=1}^{N} V_0(X_j)\,\tau(x - x_j + s)\,dx$$

where s is the separation vector

This is a spread function, and v_o is basically the transfer function. Right? So the mean value of the intensity field, which is, say, μ_I, is basically this average of the I field, and if you take the $v_o x$ out, it is an interrogation of the τ over the entire area. This is an approximation. Okay. Now, if we look at the autocorrelation of the single exposure intensity field, that means it is a correlation of the intensity with itself, with a little bit of a separation.

So you take the intensity fields and you basically multiply them by a separation vector. I mean, you evaluate the intensity field with a different separation vector, and then you calculate the total intensity. This is called the autocorrelation of a single exposure intensity field. Okay, so what happens is that if you take a look at this, then you multiply these two terms, and this is the integral because the integral arises because of the bracket, which is basically the mean averaging. So you can see one in this case is $\tau(x) - x_i$; the other one is $x - x_j$, plus there is a factor s, which is the separation vector.

This is i over one to n, and this is j over one to n. So just remember, this is the mean value of the intensity field, and this is the autocorrelation of a single exposure. Intensity field. So these are the spatial estimators. So where do they come from? They come from these places.

By distinguishing the i not equal to j terms which represent the correlation of different particle images and therefore randomly distributed noise in the correlation plane, and the i = j terms which represent the correlation of each particle image with itself, we come to the following representation:

$$R_1(s,\Gamma) = \frac{1}{a_I}\sum_{i\neq j}^{N} V_0(X_i)\,V_0(X_j)\int_{a_I} \tau(x - x_i)\,\tau(x - x_j + s)\,dx$$
$$+\frac{1}{a_I}\sum_{i=j}^{N} V_0^2(X_i)\int_{a_I} \tau(x - x_i)\,\tau(x - x_j + s)\,dx\,.$$

Following the decomposition proposed by ADRIAN, we can write:

$$R_1(s,\Gamma) = R_C(s,\Gamma) + R_F(s,\Gamma) + R_P(s,\Gamma)$$

where Rc(s, Γ) is the convolution of the mean intensities of I and RF(s, Γ) is the fluctuating noise component both resulting from the i = j terms.

RP(s, Γ) finally is the self-correlation peak located at position (0, 0) in the correlation plane. It results from the components that correspond to the correlation of each particle image with itself (i = j terms).

And for the V_0, we have already estimated what the V_0 will be by looking at these terms. And by assuming top hat profiles and whatnot. The math is a little cumbersome, which you can go through. By distinguishing the I not equal to J terms, which represent the correlation of different particles, let's just denote that not equal to J terms, which basically represent the correlation of the different particle images and therefore randomly distributed particles and randomly distributed noise in the correlation plane. And i = j terms now represent the correlation of each particle image with itself.

So, therefore, if we look at this particular situation, which is r_i, s,τ, okay, we write it; this is once again the expression where you are separating the two terms. This is i not equal to j, and this is i = j. Alright, so x_i and x_j are equal, and their x_i and x_j are not equal. So, i not equal to j are the terms which represent the correlations of different particle images and therefore randomly distributed noise in the correlation plane, and i equal to j represents the correlation of each particle image with itself.

Okay. Therefore, if we look at this and Adrian did this and Ron did this, and then we actually follow this particular decomposition. So, this is the total autocorrelation function. So this autocorrelation function now has three parts. One part, this RC part, is the convolution of mean intensities of I.

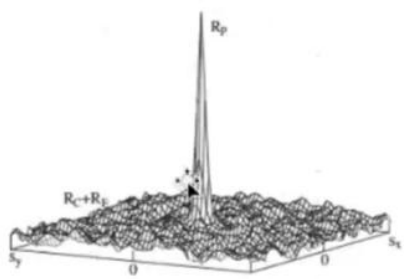
Okay. Mean intensities of I, and RF, which is the second term, is the fluctuating noise component. Both of these arise from the terms I not equal to J. Again, not equal to. Sorry, it came out like this.

Not equal to J terms. Then the RP is, finally, the self-correlation peak. which is located at position 0, 0 in the correlation plane. It results from the component that corresponds to the correlation with itself, or the I equal to J terms. So the first two terms are RC and RF. RC is a convolution of mean intensities, and RF is the convolution of fluctuating noise, both when I is not equal to J.

RP is the only term, which is the self-correlation peak where I is equal to J. Okay, so these are the two things that one should remember. So this is how the autocorrelation peak actually looks. It shows a very central RP, and then you can see that the RC plus RF terms are all there. It shows that this is basically what we call a noisy floor.

The auto-correlation of actual particle image data is provided in Fig. 4.3 and clearly shows a strong central self-correlation peak surrounded by a noise floor.

Fig. 4.3 Composition of peaks in the auto-correlation function. R_P depicts the self-correlation peak



For a Gaussian particle image intensity distribution

$$\tau(x) = K \exp\left(-\frac{8|x|^2}{d_\tau^2}\right)$$

$R\tau$ (s) is again a Gaussian function with a width that is √2 dτ

$$R_P(s, \Gamma) = \sum_{i=1}^{N} V_0^2(X_i) \exp\left(\frac{-8|s|^2}{(\sqrt{2}d_\tau)^2}\right) \frac{1}{a_1} \int_{a_1} \tau^2\left(x - x_i + \frac{s}{2}\right) dx$$

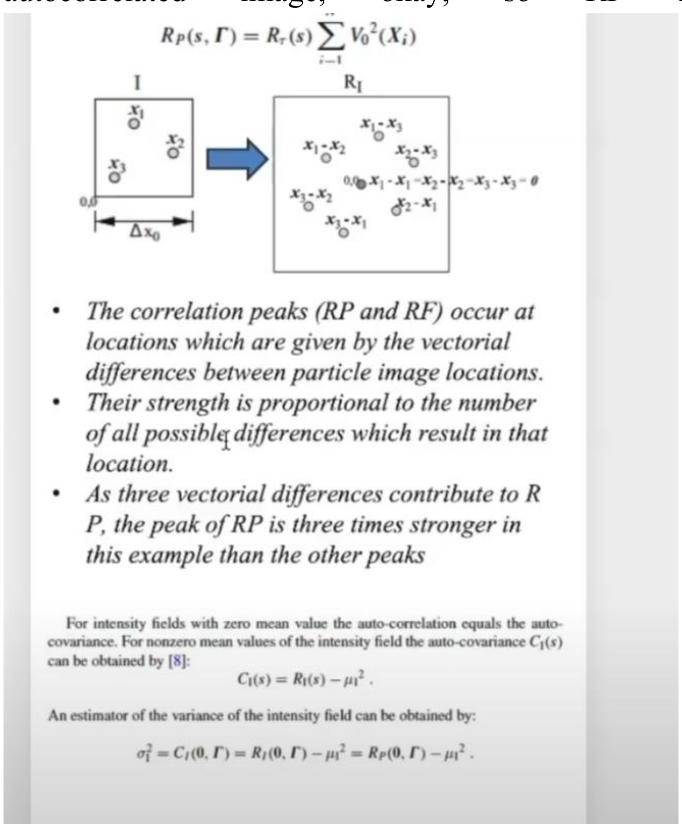
$$R_\tau(s) = \exp\left(\frac{-8|s|^2}{(\sqrt{2}d_\tau)^2}\right) \frac{1}{a_1} \int_{a_1} \tau^2\left(x - x_i + \frac{s}{2}\right) dx$$

Maximum of $R\tau$ (s) is located at |s| = 0 and the characteristics of its shape is given by the particle images shape.

So R and RP; RP is a peak where everything else is kind of distributed. Now for a Gaussian particle image velocity and particle image intensity distribution, this τ(x), which is basically the Airy function or the spread function, is given as this. Okay, R(τ) is again a Gaussian function. Okay, let's assume R(τ) is a Gaussian function with a width that is the √dτ; this is the same as this. Okay, so r p is written as this because it's a multiplication with a point spread function, which is essentially what we have over here, this s bar square.

Okay, and r τ is given as this, so the maximum r τ is located at Δ s, which is the separation vector equal to zero, and the characteristics of its shape are given by the particle image shapes. So the R(τ), okay, so the R(τ) that you see over here, let's move a step back, okay, so the R(τ) that you see over there, okay. It is basically located at the central peak. This is the RP. And the characteristics of its shape are given by the particle shapes.

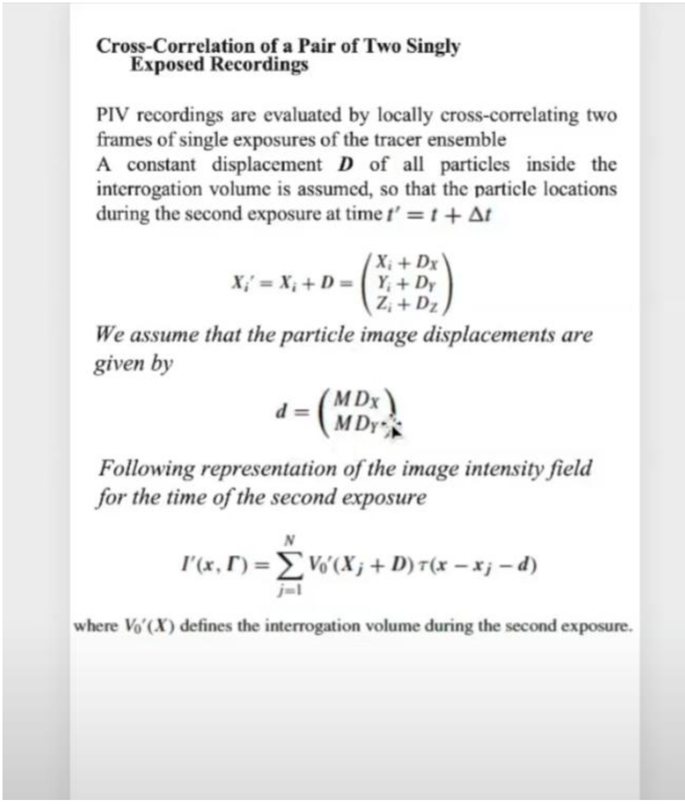
So this is how the autocorrelation function actually looks. Here, everything is kind of taken to be Gaussian, as you can see, because this is also considered to be Gaussian, the airy function. And then you have the Gaussian function with the width of Δ τ, Δ 2, or √dτ, not Δ τ. All right, so what do you see? So when you are talking about the correlation peaks, RP and RF, okay, so RP, for example, is the correlation peak, and you know RF, which is from the fluctuating noise. Okay, so if you look at this image and then the autocorrelated image, okay, so RP is basically R(τ) V_o^2 x_i.



The correlation peaks R_p and R_f occur at locations that are given by the vectorial differences between the particle image locations. Now you can see that their strengths are proportional to the number of all possible differences that result in that location. So now that there are three vectorial differences, okay, which contribute to R, the peak of R_p is three times stronger in this example than the other peaks, okay? So that is what we are, you know, what we get out of this, okay? So for intensity fields with zero mean value, the autocorrelation equals the autocovariance. So, autocovariance CIS is equal to this. An estimator of the variance of the intensity field can be derived using this particular method.

So you understand that the correlation peaks R_p and R_f occur at locations given by the vectorial differences between the image locations, and their strength is proportional to the number of all such possible differences. Okay, now three vectorial differences contribute to R_p, and therefore the peak is three times stronger in this example than in any other peak. So this is for a three-particle kind of system, okay, that you see. So the cross-correlation of a pair of singly exposed recordings, the PIV recordings, is evaluated by locally cross-correlating two frames of single exposure of the tracer ensemble. So again, what you have is a constant displacement d of all the particles inside the interrogation volume; remember this is the object plane.

So the particles during the second exposure lie at a distance x_i + d, which essentially has three components. We have seen this already. Now we assume that the particle displacements are in the image displacement; we have basically, you know, d, which is the magnification, into the D. Now, here we have neglected the out-of-field velocity components, as you can see. So the following representation of the image intensity field for the second exposure, remember, has an added term, which is D over here.



Cross-Correlation of a Pair of Two Singly Exposed Recordings

PIV recordings are evaluated by locally cross-correlating two frames of single exposures of the tracer ensemble
A constant displacement $\boldsymbol{D}$ of all particles inside the interrogation volume is assumed, so that the particle locations during the second exposure at time $t' = t + \Delta t$

$$\boldsymbol{X}_i' = \boldsymbol{X}_i + \boldsymbol{D} = \begin{pmatrix} X_i + D_X \\ Y_i + D_Y \\ Z_i + D_Z \end{pmatrix}$$

We assume that the particle image displacements are given by

$$\boldsymbol{d} = \begin{pmatrix} M D_X \\ M D_Y \end{pmatrix}$$

Following representation of the image intensity field for the time of the second exposure

$$I'(\boldsymbol{x}, \boldsymbol{\Gamma}) = \sum_{j=1}^{N} V_0'(\boldsymbol{X}_j + \boldsymbol{D})\,\tau(\boldsymbol{x} - \boldsymbol{x}_j - \boldsymbol{d})$$

where $V_0'(\boldsymbol{X})$ defines the interrogation volume during the second exposure.

So, this D is the image displacement. d is the image displacement. And it is related to the actual displacement, which is the constant displacement within the interrogation volume. Remember, all particles in the interrogation volume are assumed to have a constant displacement, which is D.

Okay. And so, this is what you get. V_0' defines the interrogation volume during the second exposure. Right? So you can see that this is x_j + D, which comes from the transfer function. And then τ is basically the point spread function. Here, the d is actually the image displacement in the second exposure. So we first consider identical light sheet and windowing characteristics.

we first consider identical light sheet and windowing characteristics, the crosscorrelation function of the two interrogation areas can be written as

$$R_{II}(s, \Gamma, D) = \frac{1}{a_I} \sum_{i,j} V_0(X_i)\, V_0(X_j + D) \int_{a_I} \tau(x - x_i)\tau(x - x_j + s - d)\, dx$$

where s is the separation vector in the correlation plane. Analogous to the procedure used

$$R_{II}(s, \Gamma, D) = \sum_{i,j} V_0(X_i)\, V_0(X_j + D)\, R_\tau(x_i - x_j + s - d)$$

By distinguishing the i not equal to *j terms which represent the correlation of different randomly distributed particles and therefore mainly noise in the correlation plane and the i = j terms, which contain the displacement information desired, we obtain*

$$R_{II}(s, \Gamma, D) = \sum_{i \neq j} V_0(X_i)\, V_0(X_j + D)\, R_\tau(x_i - x_j + s - d) + R_\tau(s - d) \sum_{i=1}^{N} V_0(X_i)\, V_0(X_i + D)\,.$$

Again, we can decompose the correlation into three parts:

$$R_{II}(s, \Gamma, D) = R_C(s, \Gamma, D) + R_F(s, \Gamma, D) + R_D(s, \Gamma, D)$$

where R D(s, Γ , D) represents the component of the cross-correlation function that corresponds to the correlation of images of particles obtained from the first exposure with images of identical particles obtained from the second exposure (i = j terms):

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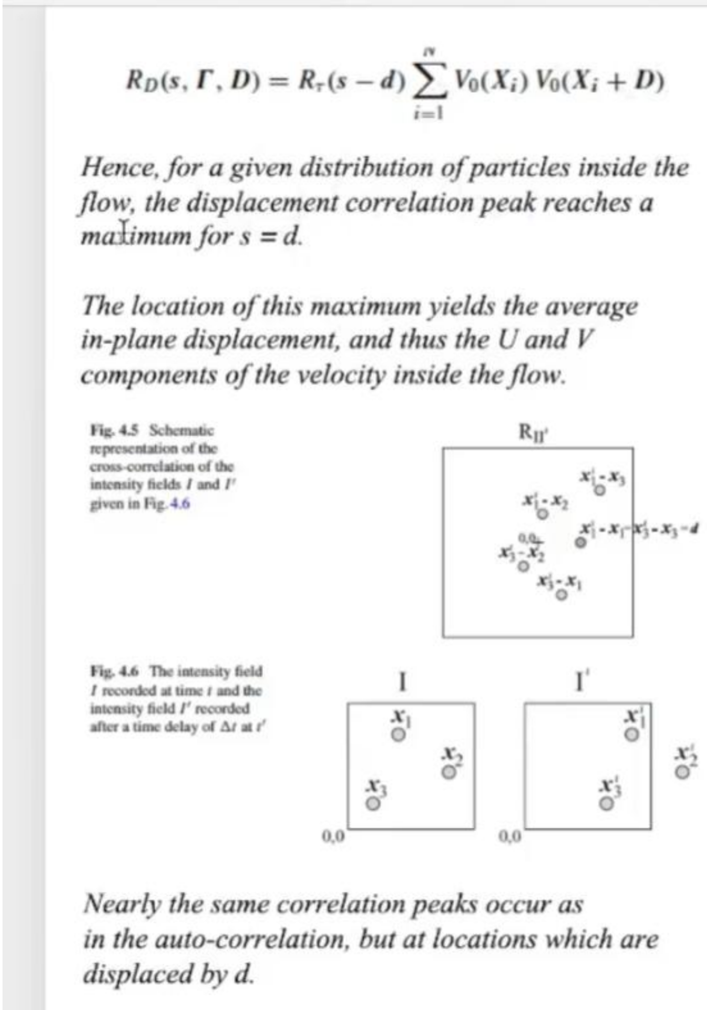
So it's the same; the cross-correlation function of these two interrogation windows is given as this. Okay. So, this is the interrogation. S is a separation vector that is used in the correlation plane, and then, analogous to the previous procedure we used, we get this. By distinguishing that i is not equal to j, terms which represent the correlations of differently randomly distributed particles, and therefore mainly noise in the correlation plane, and then the i equal to j terms, which contain the displacement information, we get these two values.

One is that i is not equal to j, which is basically the correlation of differently randomly distributed particles and mainly noise. The other one is basically, so this is i not equal to j, and this is i equal to j terms. Again, we decompose them into three parts, where Rd represents the component of the cross-correlation function that corresponds to the correlation of images of particles obtained from the first exposure with images of identical particles obtained from the second exposure, which are the i equal to j terms. The rest of the terms, as we know, one is the mean and one is the random fluctuations, both arising from the terms where i is not equal to j. So R_d is the term that represents the component of the cross-correlation function that corresponds to the correlation of the images obtained from the first exposure with identical images of the particle obtained from the second exposure with i equal to j terms.

So, this is the cross-correlation method. So this R_d is basically given as this. Hence, for a given distribution of particles inside the flow, the displacement correlation peak reaches a maximum when the separation vector is equal to D. So this location of the maximum yields the average in-plane displacement. And thus the u and v components of the flow inside.



This is the most important thing that we will take away. As you can see from these images, nearly the same correlation peaks occur as in the autocorrelation, but the locations are displaced by d. And when s = d occurs, we get the maximum displacement peak for a given distribution of particles. You get the displacement correlation peak that reaches a maximum for s = d. The location of this maximum actually yields the average in-plane displacement.

And in this case, this is exactly what happens. This is how the cross-correlation technique actually works. So the peaks are nearly the same as the autocorrelation for this three-particle exposure, but the locations are displaced by d.