Advanced Gas Dynamics
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Lecture – 38
Supersonic Flow past a 3D Cone at an angle of attack: Governing Equations

So, let us begin with a numerical part right. So, we sort of try to understand the 3D picture of the chronicle fluoride. And we saw that is a you know how some of the phenomena if we have a cone at an angle of attack and so on and so forth, right. So, let us see how in order to solve for the properties.

You know, around such a surface which is a cone, right. And then you have a shockwave. So, to solve for these in numerically say you know, how do the how do. We take account of that we have for example, vertical singularity how would that be taking care of by the governing equations, the governing equations are again the continuity momentum and energy. So, how do we make sure that these governing equations are actually giving us what we try to see physically in the couple of last lectures.

So, let us see what will first try to do is exactly find out how the governing equations, right you know, give us the vertical singularity show that we have each streamline has a single value of entropy and so on and so forth. Let us let us do that, right. Now so, along a streamline therefore, each streamline basically has a single value of entropy right. So, therefore, there is no change in the entropy across along a streamline.

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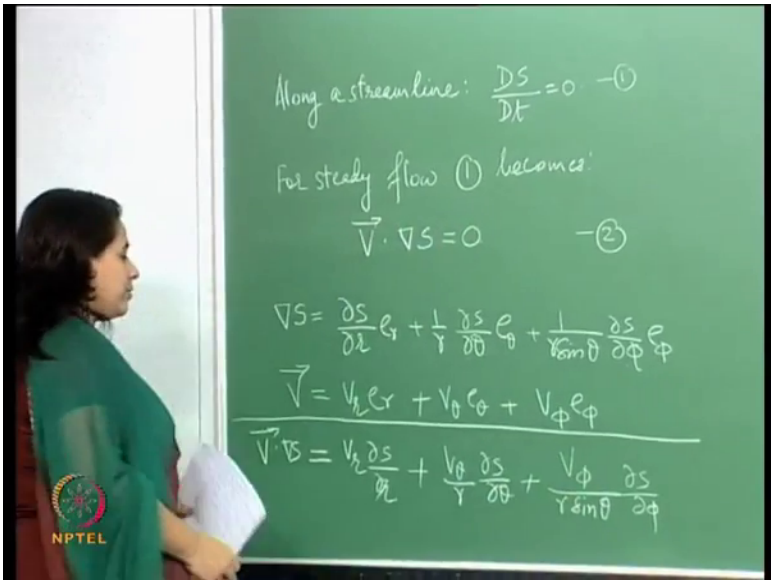

So, along a streamline I can write, right I write that as a total derivative. So, therefore, for the for let us and let us call this as a 1. And therefore, for steady case steady flow we get from 1, right. We just have the convective part. So, for steady case you know, it boils down to this. Now what we will do is write this whole thing you know in spherical coordinates. Because gradient s therefore, becomes and ok.

So, if I write 2 in terms of you know spherical coordinates. So, gradient s becomes this and V becomes this. Let us just make sure we understand what this means. So, what we are seeing here V basically in here has these 3 components. Now hopefully you aware and this e r e theta and e phi are unit vectors, right. Unit vectors normal score whatever. So, so these let me just, I am going to write it like this e r is a synonymous to I j k in it is analogous to the I j k in cartesian coordinates.

So, basically, we have r theta and phi directions. In which direction we have taken? It is unit vectors and the velocity components in those directions is r theta and phi. So, again here. So now, gradient of s is basically the change in s is a convective change in s. So, you see del s del r in the r direction 1 by r del s del theta and theta direction, 1 by r sine theta del s del phi in the phi direction. So, let us just sort of make sure we understand what this means.

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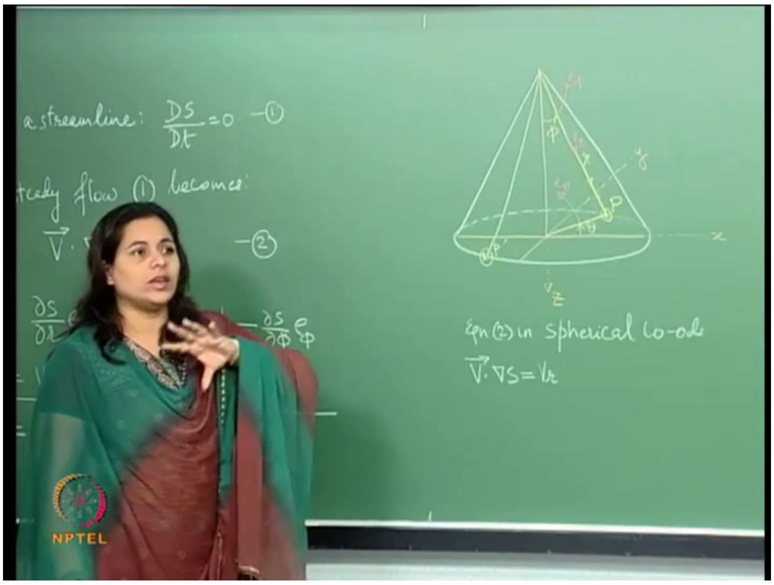

So, let us say this is what we have. And this is x, and this is y. This is x, and this is y. And in this particular case. So, x y and z. This is the 3D picture I am trying to sort of draw this, right. And on so, therefore, my if I do this from this vertex, if I draw this line here, now this is my cone. So, essentially so, this becomes my cone, right cone surface. Now let us take any point say here. Let us take say any point over here.

This point here and we will drop a radial line from here this, which makes an angle theta, right. With the x axis, and will also drop radial line from this vertex still here, this is r. And this angle is phi, this angle is phi, right. Something we know from before right. So, then in that case now along this direction, right; we take a unit vector and call that as e r, then along this direction we take an unit vector and call that as e theta. And in this direction, we call a unit vector and call that as e phi, right. Make sure so, this make sense.

So, this is something that I think hopefully if you look at those 3 diagrams, it will you know it will register something it will restore in your mind. So, essentially so, this is what I am I am talking about. So, in this particular case. So, in here if I take the total velocity at this particular point. So, I have a velocity component. So, I instead of I could have here for example, if I was looking just at this particular plane x, say let me say. So, at this particular point, let us call that as P, right.

At this particular point the now the velocity could also be V a like, u V and w which is in the x y and z coordinate system; which is the same, but if I instead of the if instead of taking the components of that in the x y and z direction. If I take the components in the r theta and phi direction, that is one I get V theta V phi and V r. So, that is what it means, and this is the change this is the change in r theta and phi directions. Essentially so, if I have a point a point goes from one point to the other.

So, if I have a point say this p at was at some location here, was originally at some location here, and it moves from p dash to p. So, you can see that here the theta the phi and the r are all different, is not it? These theta r and if I were let us say do this here. So, you can see that the theta r and phi are all different. So, said p it moves from p dash to p, this is the convective change in the entropy, say it moves from p dash to p. If it does that, then there is a change in the entropy, because there is a change in the radial direction, there is a change in the theta, there is a change in the phi, which is what is represented here, which is what is represented here. Now why is it not just del del s del theta? And this is the math part of it geometry part of it is easy to find out. I would request you to do that or consult a very basic standard text. And this should make sense.

You do that once and then just take these equations as they come. But, if you have not understood this by now; maybe it is psi time just go and take a very standard text look at polar coordinates and spherical coordinates and cylindrical coordinates. And just make sure you understand why you know these you know these terms come in here. You have to do that once as long as you know that is fine you understand this we need to memorize this. So, for a steady flow we can write this, and in the spherical coordinates this is how we would write. So, therefore, I will write this equation 2 so, they we will write it like this here. Let me write equation 2 as so, we have equation 2 in spherical coordinates right. So, basically, we have so, if you if you if we look over here V dot it is a dot product V dot delta s gradient s. So, we all we need to we have these 2 vectors here, these are 2 vectors we need to take a dot product.

So, that is all I mean e r dot e r is 1. So, that is what we get. So, we get V r why do not we write it here itself. Why do not we write it here itself? So, if you look at these 2 terms over here. So, what we get is V r del s del r plus V theta by r del s del theta. This then plus V phi r sin theta del s del phi, right. This is what we get now from here what we know is that along this is still it the chronicle nature of the flow is still maintained, and the property along a radial direction is constant, right.

We had you know talked about this earlier. So, in like in the cone at zero degree angle of attack for the cone at a certain angle of attack as well, the property along the radial direction is still constant. So, that does not change. So, therefore, the entropy along the radial direction does not change. So, if I am moving from point p dash to p, the entropy will change with respect to the change in theta and phi only it will not change with respect to r.

So, let me call this equation as 3. And then taking that out. So, basically what we will have is that this term will go. So, we will have this as only this term and this term, right. This 3 3 is basically left-hand side of equation 2, right. This is the left-hand side of 2.

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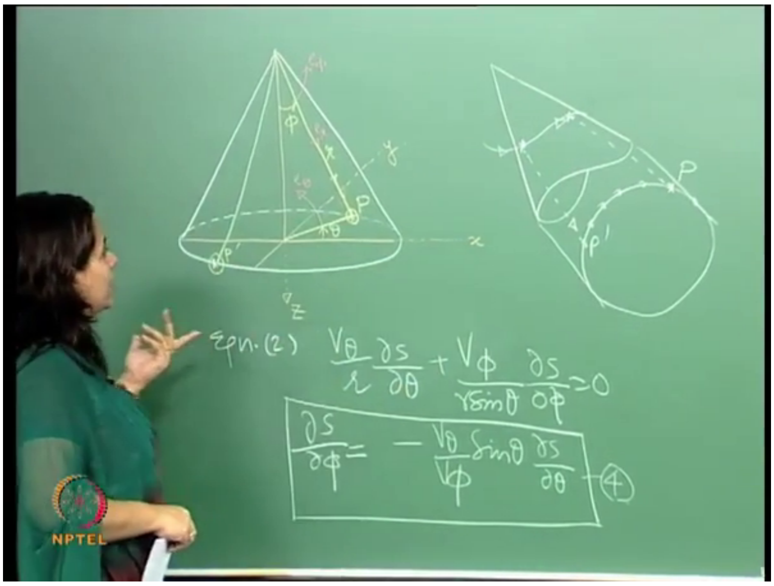

So, therefore, equation 2 becomes basically V theta by r del s del theta plus V phi by r sin theta del s by del phi equals to 0, right. Or what we can write here is del s del phi, I can write that as minus V theta by V phi sin theta del s del theta. So, this equation this equation as we see here. So, this is basically the an expression for the entropy change an expression for the entropy change on a streamline, right. This is and what we can see here that it is connecting the entropy with the physical parameters theta and phi. This theta and phi is what the theta and phi.

Now so, therefore, if I you know if you sort of recall you know the way we were looking at the streamlines if you remember this. So, let me just quickly sort of remind you what I am talking about. So, we have a say cone like that, right. We have a cone like that, and if I look here, right. So, this is a, super this is a if I you know if I project this onto here. So, I will basically look at a circle right. So, if you remember, right that we had we had you know streamlines like this we had streamlines like this coming over right. So, then now these streamlines if I projected here, there were nothing but points on the points on this projected plane right. So, therefore, in here now this is if they if say the point moves from p dash to p there is a change in r theta and phi, but what we can see here is; that my change is happening only with respect to theta and phi. Because if the flow is still the pro property still are constant along r directions. So, which means that in here.

So, here what we have written the entropy change in terms of the velocity parameters velocity V theta and plus V phi. So, let us see how this if I were to look at this change just in terms of changing the physical parameters theta and phi just the geometry. So, this here for is a relationship between the change in entropy with respect to the velocity components.

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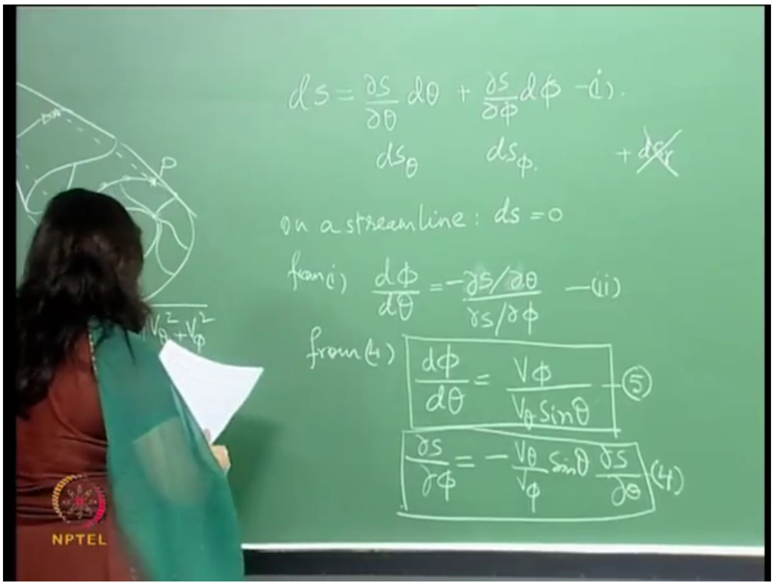

So, if I do this. So, say that total say d s, change in the entropy. You see what I am trying to write over here which is different from here. Now the way we got equation 4 is from here. So, we use the equation of the not exactly the physical equation of a straight line, but the fact that the entropy is single valued on a particular streamline, which means which means this right. So, therefore, when I have this particular streamline. So, you can think of this you can think of this, when I say it is moving like this going from p dash to p right.

So, if you are looking at say this is say p and this is p dash. So, basically, I am going from this point to say this point here. You know, which are the same streamlines. So, therefore, there is a change in theta and phi when I go from here to there, but the there is no change in the radial direction the property still remains same. So, and we use this equation and we wrote out the change in the property in the entropy with respect in in terms of the velocity and velocity in the theta and phi direction. Now what we try top say is that you know if I would have look at it look at it another way that the entropy change will goes from say p dash to p you know. And I look at it say as a change in theta and phi. So, what I am writing it as that the change in total change in entropy is a change in entropy in the theta direction, I am changing entropy in the phi direction. That is what it s amount to, isn't it?

So, this is this you can write as d s in the theta direction, and this is d s in the phi direction. So, these are the 2 directions in which it is changing it is not changing. So, I do not have a d s in r direction. We do not have that because it is constant. But we do have this. Now on a streamline d s is what; there is no change in the entropy, right. Because it is single valued on one particular streamline it is single valued. So, it is equal to this therefore, from this expression what we can write I will write this as say 1. So, therefore, from one what we get? So, what I will write here is the phi by d theta, which is equal to; let me call this as 2. So, all I do from here is write it. So, what I look here is the relationship you know it is a make the change in the theta direction, and the change in the phi direction of entropy is given in terms of change in the theta and phi itself.

Equation 4 was giving us change in the you can write it here also actually del. The change in the theta and phi direction if you look at this, in terms of the velocity components in terms of the velocity components. Whereas, here we have the change in the theta and phi directions of entropy in terms of the change in phi and theta itself. So, therefore, using again so, if I were to use here if I were to use equation 3, sorry 4. So, therefore, what we will get is that. So now, what I have here is the d phi d theta in terms of V phi with the velocity components. So, what is this giving us? If you were to think of this as an x y coordinate system, x and y coordinate system this would be like d y by d x which is the slope. So, this equation is essentially giving us the slope the slope of the streamline in the theta phi plane; which is the projection. Which is the projection here. The theta phi projection here.

So, what this is giving us is the shape of the streamline on this projected plane. And I do not know if I remember if I mentioned this, right. The this plane is called the cross-flow plane. This plane is called the cross-flow plane because if you look at this you have all sorts of you know you if you remember, right. We had this was for the angel for the 0-angle cone, right. 0 angle of attack cone, and for something like this for something like the at certain angle of attack we had something like this so on and so forth, right.

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We had also so, this was this plane is called the cross-flow plane, and the velocity that we had talked about, right. Which was this V theta V phi V phi in this plane is total velocity this is the cross-flow velocity. And when this goes to 0, then you have a stagnation point, right.

So now, therefore, let us so, therefore, equation 4, an equation 5; now what equation 5 is giving us is the shape is the shape of this streamline in the theta phi direction in the theta phi plane. So, in the theta phi plane 5 gives us the shape of the streamline and 4 gives us; that for a change in that theta and phi what should be the change in corresponding entropy. So, armed or equipped with these 2 equations.

Let me write this equation one more time. Just let me write it here for to I want to make a statement here. So, this is d s or yeah let me write it like this d s by d phi is equal to minus let us by del theta. This is our equation 4. Now if you look at both these equations so, here I have V phi by V theta and V theta by V phi. The question to ask here is which is fine, which is always fine. So, this gives us a shape it in terms of these velocities, and this gives us a change in entropy in terms of these velocities.

So, numerically that is it should have no problem, right. Is there any issues here with this? The hint is that in here in this space we had a stagnation point if you remember, these are the 2 stagnation points and this was also a point of vertical singularity, right. If you remember right. So, what was this stagnation point? The stagnation point was when this is equal to 0, it means it is a stagnation point, right. Then this becomes a right so, this velocity this is a cross flow velocity. When this becomes 0 what we get is a stagnation point, right. Makes sense. So, therefore, if in that particular therefore, now 5 and 4 are fine 5 and 4 are fine as long we have a finite V phi by V theta.

But what if V theta and V phi are 0 right. So, the exception is that when that if we have a stagnation point right what we have is a stagnation point and this become indeterminate right.

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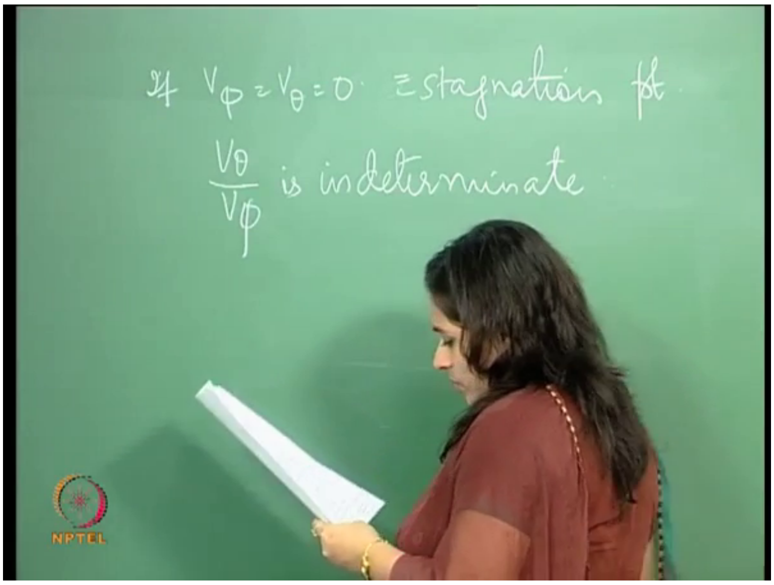

So, this becomes so, basically V theta by V phi or whatever is indeterminate. This equations therefore, if you look at these equations 4 and 5 therefore, they are allowing us you know more than one solution or for the entropy value, for the entropy at a particular point. So, therefore, this is going to allow us, for multiple values of the entropy at a particular point. Which means that we have we have what we have vertical singularity. If you remember, right. Vertical singularity meaning at a particular point we have more than one value of the yeah more than one value of the entropy. So, this is now numerically being shown.

So, if I look at this equation here for time being for just for a second forget that this V phi and V theta are velocities. Just consider them as some in our arbitrary random value something is just a number just think of it as a number. So, if I look at that if I see that this then the ratio of these numbers V phi or V theta something. They are not velocity. They just some arbitrary you know value that I am taking if this. If each of these goes to 0, then I have an indeterminate equation over here. I have an indeterminate equation over here. So, and in this particular case; however, this V theta and V phi corresponds to the velocities in the theta phi directions. So, what we will therefore, get is a stagnation point, and hence at a particular point we will have more than one value of the entropy.

Which means we have multiple values given by 4 and 5 and hence which is the vertical singularity. So, the therefore, the vertical singularity is being shown by the equations that we are using. Hence this is a correct representation, right. Of the flow around a cone at an angle of attack. That is the whole sort of gist of it. So now, what we will do is do this task of you can of course, look this up you can a look this up at from any standard book, but I can of want to write this down write this down and give you a feel for this. So, how these will look.

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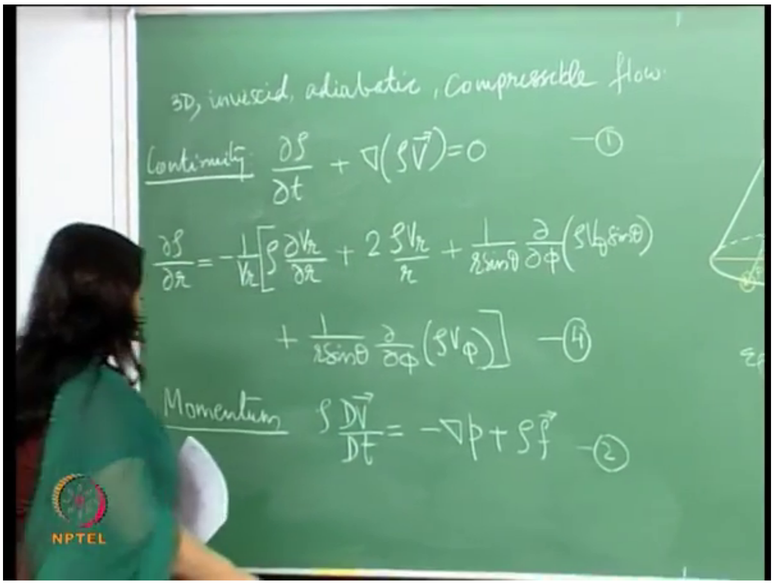

Now, let us go ahead in do that the governing equations in our usual. So, these are a governing equations for 3D inviscid, adiabatic, compressible right. So, this is our flow and what is our this is our continuity equation right.

So, this is the continuity equation as we know, it in our usual coordinate system if I write this in spherical coordinates. What does this look like? So, the way I will write it is in the way I will write this is you know all the changes here, in terms of the change in the radial direction. I will write all properties like that. So, this is it and if you look at this, and I am going to call this equation as 4. I am going to call this equation as just call this as 4. So, this is my continuity equation and this is in the spherical coordinates.

So, then we have the momentum equation I will write the momentum equation here. So, momentum equation, you recall that is 2. Now this is something that I will now write in the r theta, and phi directions. R theta and phi directions. And let us see this momentum equation. So, then the momentum this is r direction, in the r direction so, that looks like this which is so, this is the equation in the r direction.

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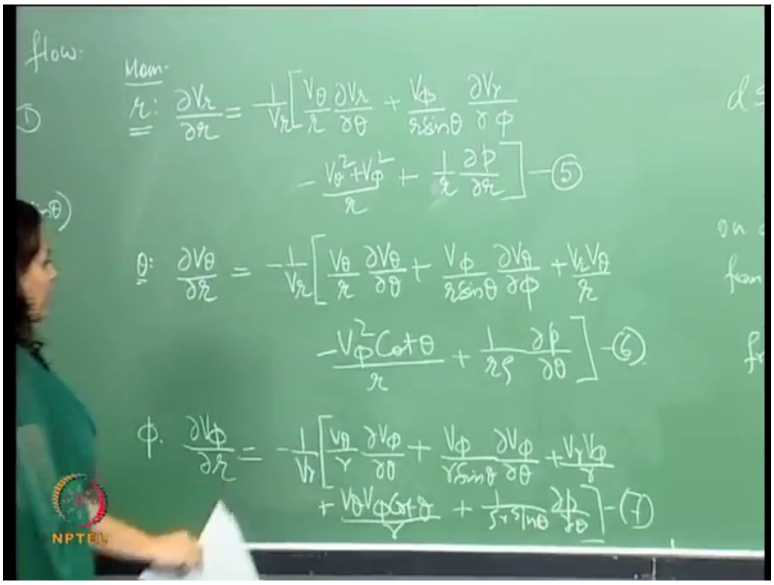

So, I am going to just cross check that V theta by r V phi r sin theta that yeah, looks all right. So, this is in the r direction. So, then we have the theta direction. Then we have the theta direction which looks like this. So, like I said I am writing all the parameters with respect you know with respect to r on the left-hand side. So, I do that; that is a 0 direction and finally, in the phi direction finally, in the phi direction; which is plus that running is really bad that is it. So, that is how the momentum equations look like in the r theta and phi directions. And along with this so, along with this we also need the energy equation, right and the energy equation.

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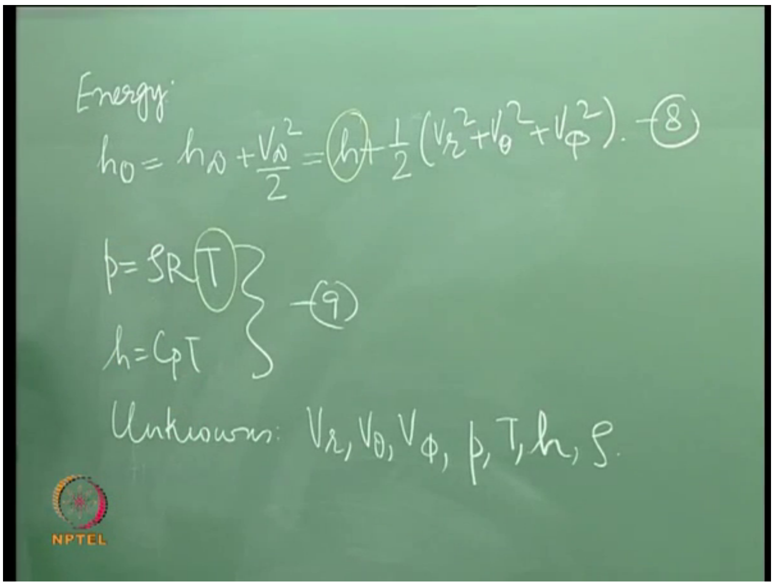

So, the energy equation is; so, total enthalpy. So, which is equal to total velocity at one point, is not it? And we call this and along with this we will equally along with this of course, we have the equation of state and the calorically perfect gas relationship. So, which is, right this is the equation of state right. So, these are the basically our equations. So, we wrote down you can see here.

So, this continuity equation, then equation the continuity equation this is in the spherical coordinates. Then the momentum equations, momentum equations in r theta and phi directions. Energy equation and the equation of state and perfectly the equation. We have this. And what I think at this point I will kind of briefly go over a method; which was I guess some of right at the beginning when people started doing this, and I will just illustrate briefly as to how this method will proceed, how we can do this how we can solve for the properties here.

So, let us look at these equations, now if you look at these equations here. So, let us see what are the unknowns that we have in all the equations that we have written. So, one is this V r, then we have V theta, we have V theta and we have V phi. So, that makes it 3 unknowns. What else do we have? We have the pressure. So, that is one and what else do we have over here? So, then again, we have the enthalpy and we have the temperature, isn't it? So, if you look at this if I were to write that down. So, our total unknowns are our V r, V theta, V phi then pressure, temperature, enthalpy and what else? And the density, forget about that one. So, let us look at the density yeah sorry.

So, there we go so, the density. So, therefore so, we have this 7 unknowns, right. We have the 7 unknowns which need to be calculated using these equations that we have here. So, basically so, this is a 1 2 3 4 5 6 7. So, these are the relationships etcetera, we would use to calculate the 4 2 8 we will use equations to solve for our unknowns. And this is a numerical procedure a sort of it is not possible to sort of clearly explained you know you know in a board you have to write a computer program.

And if you have a brief bit of idea as to how to progress I think you can do that; buts unless and until you write a computer program is difficult to explain you know the whole thing, but I will explain I will try to go through go over the gist of it. And it is important that you know some of these earlier methods are also you know given in these references

I would like to sort of write those down for you. So, you can take a look at it. So, one reference if I were to sort of briefly write this some of the references.

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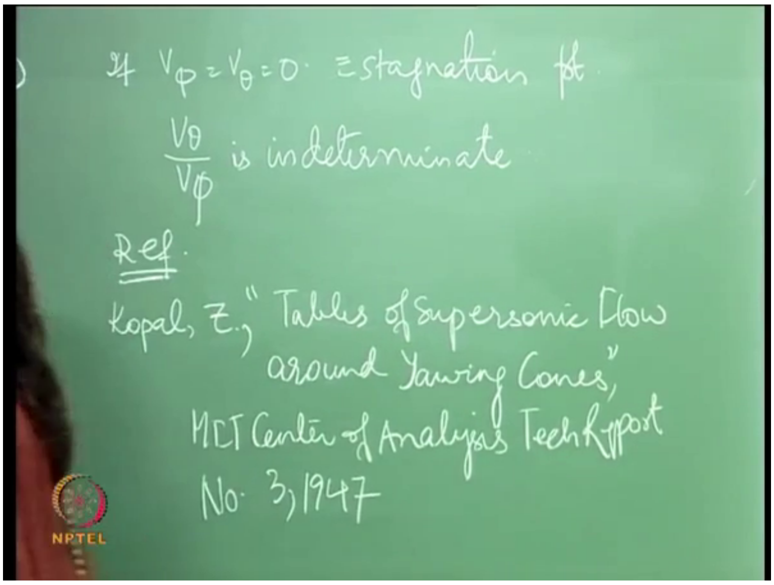

If you look at the references for example, by Kopal this, and this is stable the supersonic flow around Younkins this is tables of supersonic flow Yauring.

So, this was the you know this was basically analysis tech it is a tech report. These some of the like really early methods. This is one reference. And other reference is Joseph Sims actually.

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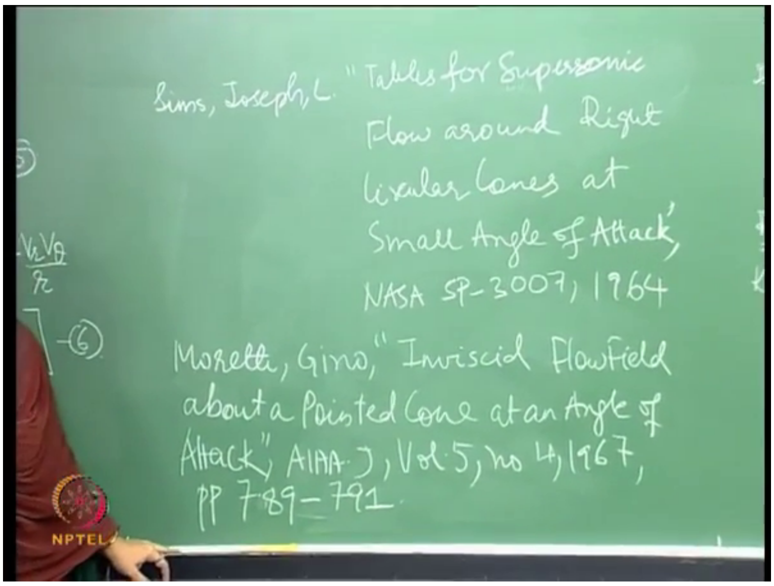

So, tables of supersonic flow around, right. Circular cone that is a small angle of attack. So, again this is tables for supersonic flow around, right circular cones a small angle of attack. So, you can write that; this is say so, you know this were these were these are all sort of you know before I was born. So, yeah so, it is like the old, but it is nice to know from where you know things have started. And the method that I will talk about the method that is I will sort of explain is basically you know towards, right. At the beginning is given by Moretti. So, you can look at the paper yourself, and go through the details of it I will just make a brief gist of it and the sky.

So, Moretti so, which is inviscid flow field, about a pointed cone at an angle of attack. And this one (Refer Time: 49:50) which you know. So, I think what I will do is basically stop here. We will pick it up from here to in the next lecture actually. And so, all of these I mean you know all of these are kind of old, because this I wrote before my you know earlier before I was born.

So however, this is a method that I will show you, and also picked up a paper which is more recent; which is more recent and it is got some nice pictures numerical results basically. And you know I will give you the reference and you can take a look at it and of course, you yourself also can look at other references and more sort of recent work in this if you are interested in do some recent work yourself. So, I will stop here, and we will sort of go over this you know briefly as to what this; how we will be able to calculate you know 7 of these properties. So, we will stop here.

Thanks.