Muffler Acoustics-Application to Automotive Exhaust Noise Control
Prof. Akhilesh Mimani
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 34 and 35
Two–interacting Duct Configurations: Development of Equations and Concentric Tube Resonators

Welcome to lectures 4 and 5 of week 7. The last two lectures of this particular week will be combined because now, we are trying to we are trying to get into the really the business end of the perforated things which will also continue for the entire week 8 and some parts of week 9. This is probably one of the very important muffler components.

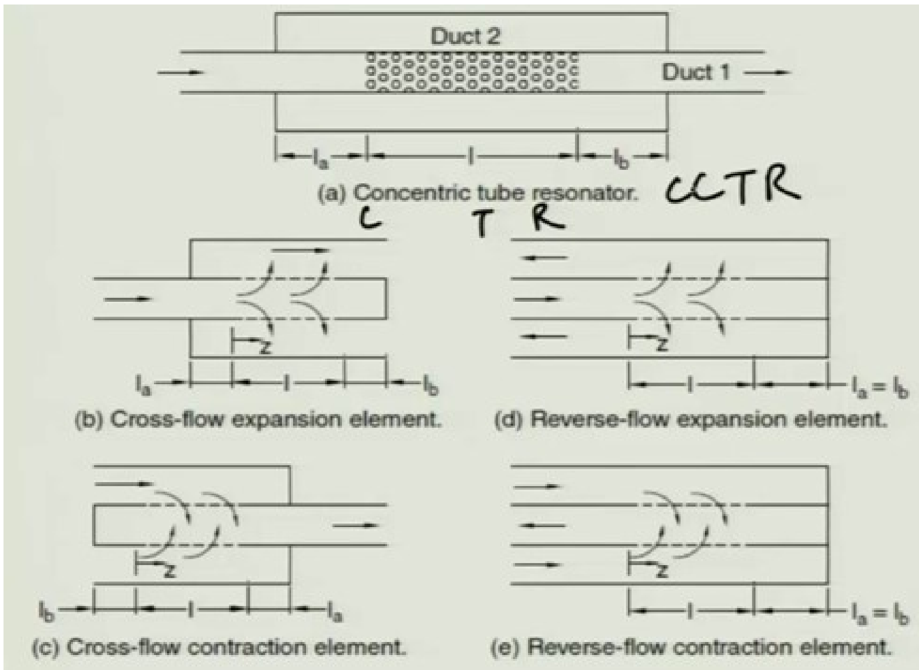
Actually, that is like we on the like we are discussing towards the end of the last lecture that most of the mufflers almost all of them have some sort of a perforated element, basically to guide the flow and not allow the flow to completely expand and which will cause you know lot of flow induced noise problems and that will be counterproductive. So, that is why we put a lot of focus on analysis.

So, this class what we are going to do week 7 and lectures 4 and 5 of week 7, they are combined. So, now in this week what we are going to do really is analyze two duct perforated systems. You know we will talk about the development of such equations right from the control volume approach and point out terms which really lead to the coupling of waves.

In all our analysis you know as we have been doing so far, we will be considering only the planar wave propagation along the axis of the duct. So, that is what we are going to do and let us see how we go about it. Well, to begin with what we will do let us consider. So, let me write down this class the objective of this class two-interacting duct system. So, different configurations that are possible.

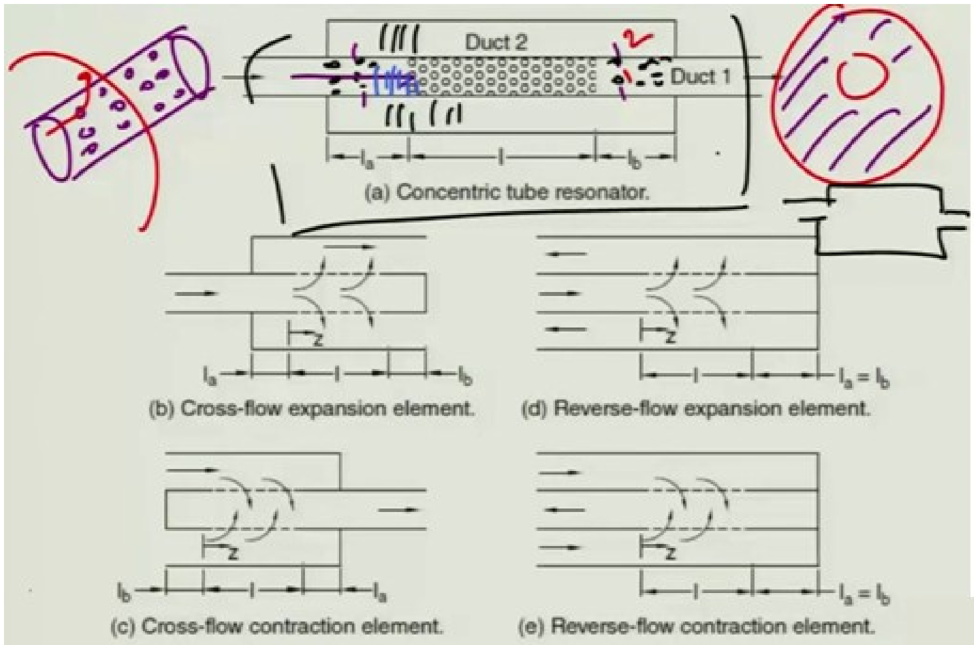
- **Two-Interacting Duct System**
- **Review Perforated Perforate Expressions**
- **Fully Perforated Fully Perforated CTR (**Concentric Tube Resonator Concentric Tube Resonator**)**

We will also do review very briefly though review perforated perforate expressions that is developed by people or different times perforate expressions. And, then we will possibly try to solve one example, I guess in MATLAB of a fully perforated fully perforated CTR that is, what is CTR? It is concentric tube resonator concentric tube resonator, alright.



So, let us analyze different systems. So, here we are. So, all these figures the set of figures what they really show you is a typical Concentric Tube Resonator CTR that is what is called C here, T here and R. Later on, probably sometime in the beginning of week 9, I reckon we adding an extra C that is a conical concentrative resonator what we you know I just kind of briefly alluded to that in the last week's lectures.

So, for now let us keep the discussion very simple we have a basically this is nothing, but a it could be in principle you know this is a circular duct. This could be circular outer section or a circular chamber or it could also be elliptical in principle. Lot of commercially used mufflers are also elliptical in shape but, does not matter as long as we are considering planar waves.



The cut on frequencies depending on the cross-section will change of course, but the main idea is to analyze the coupling that happens between the inner pipe let us say 1 and the outer annular cavity that is you know the outer cavity this guy so, when planar waves propagate. So, there are different elements. So, before we analyze different elements let us you know let us first talk about what happens. So, this is a schematic diagram for a concentric tube resonator.

So, this is partially perforated. So, and clearly you can see neck extension. So, suppose you know this guy were not there, suppose you know this part is not there. So, you would get back your simple extended inlet and outlet element. So, this perforated bridge what I was mentioning in the last class, this acts like the real purpose is to guide the flow.

But, in the process what it does is basically the flow as of as the flow passes through it, the waves that are generated the incoming disturbances that that occur from the engine that propagate downstream of the engine, they are they get a chance, they get an opportunity.

Note that this is a solid section you know these way. So, there is the waves cannot interact directly with the anode cavity in this region. It is only when the waves when they up when they approach this perforated section through the small holes you know something like this they get a chance to go out of these waves and interact with the annular cavity surrounding it, is not it? The waves are interacting now.

So, basically by now you can sort of very well guess what is happening in the duct 1. So, it is a planar wave propagation and in the angular cavity you can also consider a planar wave only. So, when you talk about when you mean planar waves you really mean this kind of a wave front and here you mean this kind of a thing, is not it?

So, where does the magic happen? How does the coupling happen? So, one thing that mentioned at the very outset you know at the very beginning is that coupling between the planar wave propagation here in the duct 1 and the annular cavity duct 2, really happens through the mass exchange that is basically from the continuity equations in each of the duct.

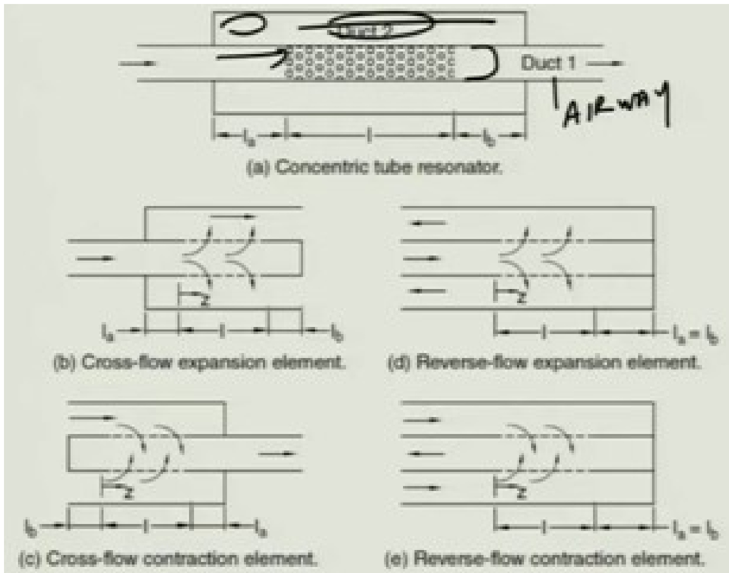
You know because of the annular cavity, some part of the fluid is going inside the control volume here we will do a thorough analysis and some part you know can also come inside. So, it is like the exchange between the acoustic masses exchange between the ducts 1 and 2 of the acoustic masses leads to an additional term in the continuity equations in each duct, duct 1 and duct 2.

And, we will see how this kind of affects the development of equation will have four first order linear differential equations and you can sort of couple them or combine them into eliminate some variables velocity variables acoustic particle velocity to get two coupled second order linear differential equations ordinary differential equations, but they are second order and coupled in $\tilde{p}_1\ and\ \tilde{p}_2$ So, that is what happens.

And, you know note that this can be. So, this partially perforated bridge you know certain length l, this is exactly like a counterpart. This entire configuration what you seeing here this one this is like a this is like a counter part of your this configuration, is not it? So, now if you were to make this thing fully perforated, what would have happened?

So, that so, and especially when it has a high porosity as we are discussing that would basically you know act like your expansion chamber a concentric expansion chamber and, but it would still be a thousand times better than you know just leaving it aside leaving it just like this because of flow separation a full.

Because you know even if it is fully highly porous, but it is fully perforated it would do it is job at least it will provide you some attenuation it will not be counterproductive. So, that is one thing we should be able to analyze such things. So, apart from all these the other elements also, which is kind of shown in these figures. So, this is now one thing before I just sort of forgot to mention that you know in this configuration that we are seeing; that we are seeing here.

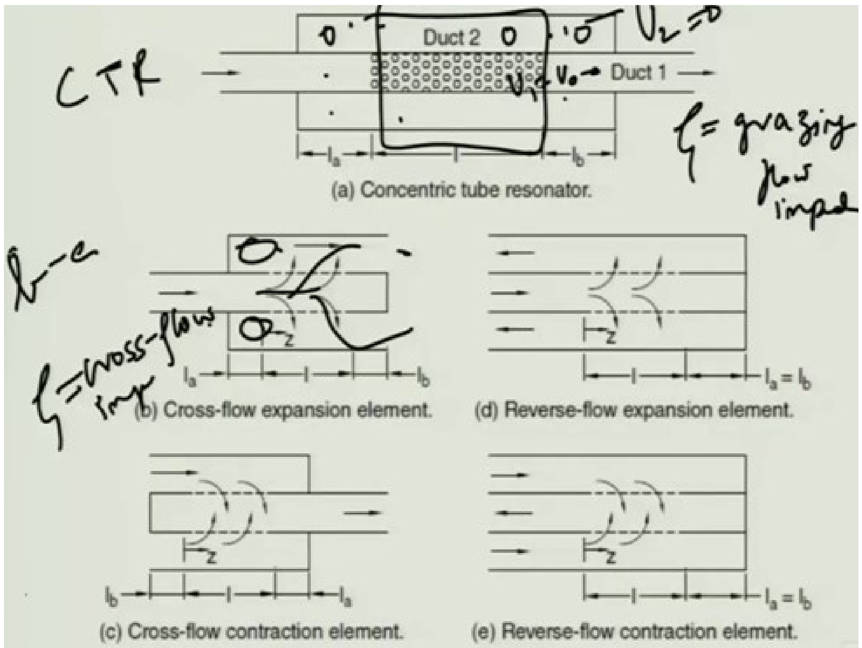


(a) Concentric tube resonator.
(b) Cross-flow expansion element.
(d) Reverse-flow expansion element.
(c) Cross-flow contraction element.
(e) Reverse-flow contraction element.

You know hang on. Here we are seeing that the flow is going like this. So, and so, there is a flow in the central duct or it is also called airway. This duct is also called airway ok. So, airway definitely has a flow mean flow along the z direction or yeah and it is typically although it has a profile something like this with boundary layers one will consider it like a plug flow only.

Now, but however, in the annular cavity we for you know for at least for this configuration CTR whether it is a partially perforated or fully perforated does not really matter. We are assuming there will be no flow, zero mean flow. That may not be 100 percent correct there might be some mean flow at least in these regions annular cavity does definitely does not have mean flow, but in the middle of the section depending on the length you could have some nonzero mean flow.

One really needs to do a CFD analysis to resolve the steady state flow fields, may be ran simulation or something like that would help to resolve the flow fields. So, now, that is what is happening, but for our cases for to keep our analysis plane of analysis simple yet reasonably accurate. We will assume at least for this configuration zero mean flow.



That is basically let me sort of write it if you have a mark or velocity U_0 here uniform velocity all this region in all these regions we have 0 mean flow that is U_0 or U_2 is 0, capital U_2 is 0 here. But, capital you know U_1 is some value here typically less than 0.15 or so.

The other configurations also like is called a cross flow. The difference between this one and this guy is that flow just grazes in this surface and so, it is called a grazing. So, whatever perforate impedance expression ς what you will be seeing that will be for a grazing flow grazing; flow just grazes the surface grazing flow impedance.

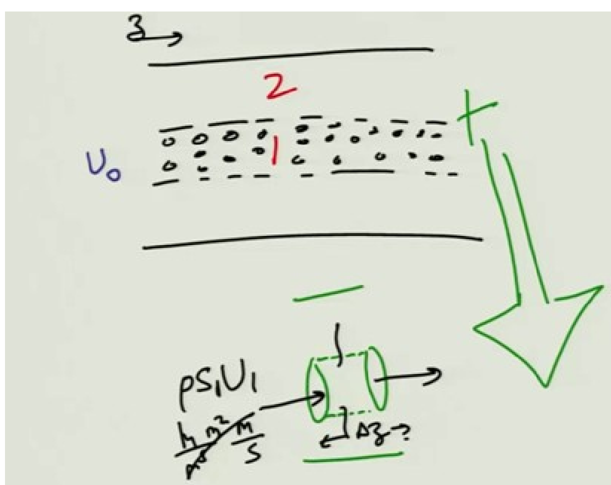
For this one there will be a cross flow because the flow you know this is like a dead end. This really is a dead end the flow it has to have the flow is left with no choice, but to go through the perforated tube and then it sort of interacts without outer cavity. So, here in these you know you can do some lot of simplified analysis, but what I am saying is that for the cross flow configuration you know the annular cavity assume the flow to be 0.

But, in this part there could be some nonzero flow we probably can assume some reasonable values and in this part they will be flow and the flow will be distributed like this. But, the essence is that the flow has to cross over and that is why it is called cross flow expansion chamber.

Similarly, here also the flow has to you know come out of the cavity mean flow and is called a reverse flow expansion chamber. This is yet another configuration cross flow contraction element. This was an expansion, this was a contraction in the sense that the flow has to come it meets a dead end. Before it does so, it has to go inside.

So, we assume certain suitable values of that and here we have a reverse flow contraction element with that you know these are really building blocks of the much more complicated stuff that I had shown you in the last class. So, because combining all this one can really very complicated muffler system with certain desired transmission loss and all that.

But, in all the configurations you know b to e that shown here configurations b to e we must use expression for perforate impedance which has a cross flow impedance cross flow impedance, ok. We must do that. We will probably look at it at in the next weeks lecture. For now, let us get back to our main thing which is your CTR Concentric Tube Resonators alright, let us get to that. So, how do we how do we begin our analysis? Let us draw a simple figure.



So, basically this is let us say this is your angular cavity and inside you have a perforated section which is something like this you know you have holes here. So, if you go back to your last slide, you know analysis of this particular part, this part this all this is simple. You know we have by now it should be a little proficient in analyzing that you know this is like a $-jY_0$ into cot k_0l a. Similarly, this is $-jY_0$ into cotk_0 l b the impedance is seen here and here and then you have tubular elements with transfer matrices.

Main thing what you have you have a fourth order differential equation if you combine if you were to combine things, but then there are similar techniques of using coupled analysis of second order ODE's. So, we will soon get to know all that. So, this is an element a fundamental element what we will consider now? I am considering this guy as 1, this guy is 2 annular cavity.

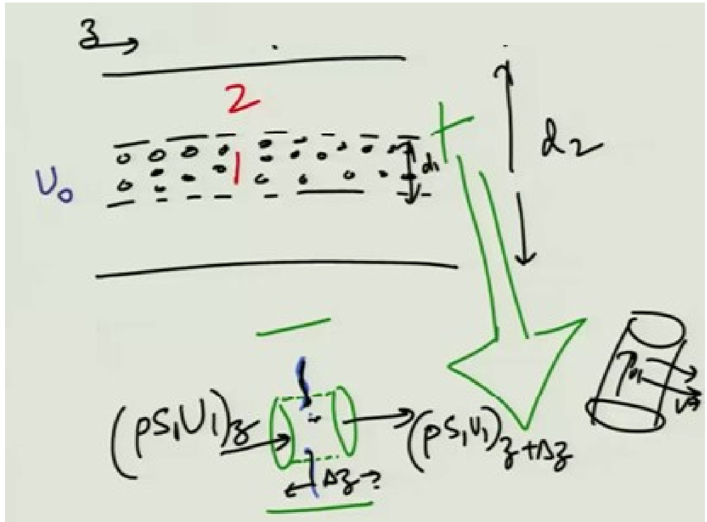
So, how do we go about analyzing this thing and here of course, I am assuming like I said the mean flow U_0. So, let us begin with you know considering the continuity and the momentum equation in an element in a small control volume let me sort of do an exaggerated version which is your this thing.

So, we will consider a small control volume and analyze the flux that comes in, the flux that goes out and what is the flux that is going out through these? You know side surfaces or the cylindrical surfaces that also has to be accounted for. All this was not there you might have to go back to your week 1 lectures to you know kind of contrast this with the derivation there.

So, in that case for a 1D case at least there was no flow through the cylindrical surfaces. There was only 1D flow, ok. So that, that is how we led to the derivation of the 1D equation ok. So, now, in this case of course, we have flow through the cylindrical surfaces also. So, we need to account for that. So, let us quickly write down the fluxes that one would sort of get when such a thing is considered.

So, now, here I am assuming let us say this is rho you know this is your let the density be rho and this is your cross section area let me call this S_1 is all uniform by the way S1 and S_2 are uniform with distance z. So, this is z. S_1 and S_2 are uniform with z ok. So, ρ into S_1 and this element is let us say Δz.

So, what is the flux that goes inside this thing? So, this has a unit of kg by meter cube, this is meter square ok and velocity you need to multiply, is it not? Let us call this velocity U_1. So, this has unit of meter by second. So, these guys cancel you are left with the unit of kg per second. So, that is your mass flow rate you know rho SU; remember your elementary fluid mechanics.



So, $(\rho\, S_1 U_1)_z$ and the one that goes out from the opposite surface is also pretty much the same form, but this is evaluated at $z + \Delta z$ ok. So, the flux you know basically density into the cross-section area into the velocity that will give you the mass flow rate. So, the difference of that, that will give you the net mass flow rate. So, basically I guess I have to use another slide. So, the one that goes inside in flux is

$$(\rho S_1 U_1)_z - (\rho S_1 U_1)_{z+\Delta z}$$

So, but you know. So, that has to be balance with the net change in the mass flux within this control volume with time, but before we you know directly jump on to writing that expression and equating things on the whole side of the equation, we must also remind ourselves that there are things that happen now from this surface.

It is like a porous or permeable surface this perforates that is you know basically there is one thing I want to make a very small digression, they are basically they are different approaches of dealing with perforates. The one that we will be following and what others have also followed you know throughout the literature, throughout this development of perforates is that this surface is considered as a kind of a porous surface with certain porosity, is like a permeable surface allowing waves to go through them.

You know with a certain some sort of a resistance or impedance what we call. So, it is like wherever we have a perforate surface we take the difference we often define a quantity named as a difference in the perturbation pressure across this element divide by the normalized velocity that is $\rho_0 C_0 U$, which is assumed to be constant across this at the interface.

So, this ratio is called the perforated impedance. It is like you know the waves are attributed they are allowed to go through the surface as though you know the holes kind of serve the purpose of making the surface very permeable or you know allowing the waves to go through that. But, then there is also yet another approach in which you know these perforates these small holes that we are seeing.

They are like small cylindrical legs and outside the annular region is a big cavity. So, in a way you can consider will there is more thought to it I mean more analysis needs to be done, but you can consider you know wave propagation through individual such holes through very complicated analysis by considering wave propagation through each of these holes and you know analyzing that and maybe as array of resonators.

But, this the approach that we are following in right now, is very easy and it is used a lot. The other approach well I am not quite sure of that people might have done it, but this is the most sort of popular approach of modeling the perforates by means of impedance expression. It is like taking it like a permeable surface.

So, anyhow coming back to the discussion the waves that go through this surface I mean what is the mass flux that goes out? So, what is the cylindrical the curved surface area of the cylinder. So, this is your let us say the length obviously, we know Δz and let us say the diameter is d_1 and the outside diameter is d_2 small d_2 ok.

So, what is the curved surface area of this guy? That is your πd into $d\Delta z$ you know πd is the circumference multiplied by Δz which is your elemental length. So, this is the area and rho is the density into U into U well, we need to use some different variable. So, let us call this as U^*.

Now, again this one another thing that I want to talk about here is that we will eventually linearize this equation we have not done that so far. So, U^* will not be having a mean flow component. It is just a perturbation velocity because it is like the acoustic mass that is being going out or coming in inside of this volume.

So, U^* is basically very small it is orders of magnitude smaller than the mean flow velocity whatever is present. So, in a way $U^*$is really your acoustic particle velocity, but in the radial direction right. So, this is your you know if you draw it isometric view of this and you know. So, it is something like this along the radius is going.

Otherwise U 1 is going like this, U^* is going like this. So, anyhow so, the point here is that this mass this particular guy this needs to be sort of added to this equation. So, basically the mass that sort of leaves this thing is also

$$-(\rho\pi\, d_1 \Delta z)U^*$$

So, this then must be balanced whether net changes of net temporal change. So, what is the total mass that is present at any time in the control volume that is your volume

$$= \ (\rho S_1 \Delta z)_t$$

So, time change with respect to this ok. This is what you get. Now, let us do one thing let us divide throughout by this guy. So, we should also divide the entire thing by delta z first and or probably we can divide throughout by S_1. Let us do that first and so, what we do dividing by

$$=> \qquad (\rho U_1)_z - (\rho U_1)_{z+\Delta z} - \frac{(\rho\pi d_1 \Delta z)}{S_1} U^*$$

$$= (\rho \Delta z)_t$$

Now, we again divide throughout by Δz. Here we get, Here we divide throughout by delta z you know. So, this guy goes away, this guy goes away ok. So, we do that. So, now, as a result and we take the limit delta z tends to 0 ok. So, then we get this guy inside this bracket plus you have

$$=> \ -\frac{\partial(\rho U_1)}{\partial z} - \frac{\rho\pi d_1}{\pi d_1^2} S\, U^* = \frac{\partial p}{\partial t}$$

$$=> \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_1)}{\partial z} + \frac{4\rho}{d_1} U^* = 0$$

So, let us strike out things. So, once we do that let us now also transpose out all the terms. Now, we have this U_1 we transpose this on the other side ok and then we transpose this guy also on the side. So, we finally, arrive, but we are not quite there I mean there are few more things to be done and no prizes for guessing.

We need to now linearize,

$$\rho = \rho_0 + \tilde{\rho}_1$$

let us say rho is this thing for the duct 1, remember we are still in the duct 1 region. So, that is basically we are in the region somewhere here ok. And U_1 is nothing, but obviously,

$$U_1 = U_0 + \widetilde{U}_1$$

but like I said will have a mean flow ok. So, we can just say let us say this is U_0 because we promise there will be no flow in the annular cavity at least for the configurations at least for CTR configuration. Other configurations we are like you know cross flow elements, different cross flow expansion contraction element of the reversed expansion elements they may have some nonzero mean flow in the annual cavity.

But, right now let us not sort of bother too much about that let us just sort of analyze these configurations that will be good. So, let us do that we get this. So, basically what will happen when you do that? $\partial\partial\ \rho_0$ of t this will be 0 because this does not change. So, we will just be having rho 1 here and again if we use our old techniques you know

$$(\rho_0 + \tilde{\rho}_1)\left(U_0 + \widetilde{U}_1\right)$$

and you know let us go back to our week 1 lectures where we talked about the principles of linearization I mean different arguments that we put forth to linearize things. So, we would basically get this is definitely a second order quantity this one, but this will cancel and this does not vary with a space. So, this will also sort of a vanish, but what we are sort of really left with is a

$$\frac{\partial}{\partial}\rho_0 \not{U_0} + \rho_0\widetilde{U}_1 + \tilde{\rho}_1 U_0$$

$$+\tilde{\rho}_1\widetilde{U}_1$$

So, in this particular thing and here of course, will have an extra term $\rho_0 + \tilde{\rho}$ and all that sort of a thing. So, we will worry about that in a bit. What we have to do really is that just simplify this equation.

Here we will get this thing in because where the arguments that I put forth and this entire thing you know will go inside this bracket. So, Δz of this guy is 0, Δz of rho naught. So, rho naught you can take it out. So, I will just directly write ρ_0 and what do we get here? $\widetilde{U}_1$ is not it?

We get this guy and here we get U_0 is again taken common and we get this is rho ok and this particular term is a too small quantity. So, we choose to sort of ignore this term. And, then we have additional term,

$$\frac{\partial\tilde{p}_1}{\partial t} + \rho_0\frac{\partial\widetilde{U}_1}{\partial z} + \frac{U_0}{C_0^2}\frac{\partial\tilde{p}_1}{\partial z} +$$

$$\frac{4\rho_0}{d_1}U^* = 0$$

But the radial velocity. So, here we finally, arrive at the equation. So, you can also have a look at the book by professor Munjal in which these equations are sort of directly given, but what we did now was a detailed proof of the equations. Similarly, now one thing I will not go to the derivation again, but directly because of lack of time I will just directly write down the momentum equation in the duct 1.

So, momentum equation will not change it is again like I am repeatedly saying the perforate just serves to act like a medium of transferring the acoustic fluxes. So, basically whatever changes or additional terms arise they will arise in the continuity equation. That is why only the continuity equation is affected. So, if $U^* = 0$ in case of a solid wall, you get back your old equation.

So, let me just expand this guy out. So, this is your this thing. So, this is something like this and what we can also do concurrently is assume isentropicity. So, when we do such a thing what we basically get is we sort of get

$$\rho_0 \frac{\Delta \widetilde{U}_1}{\Delta t} + \frac{\partial \tilde{p}_1}{\partial z} = 0 \qquad (1)$$

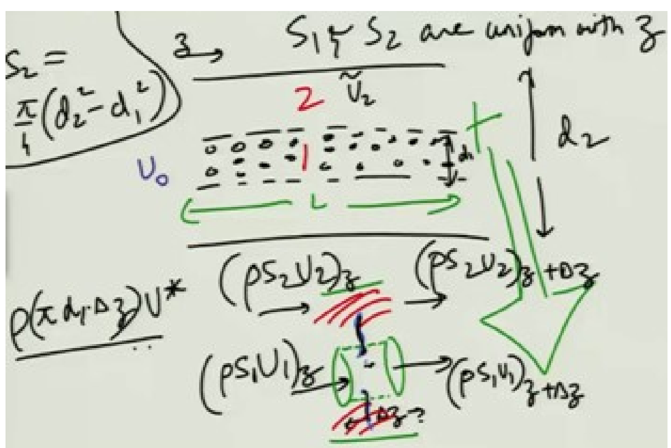
$$\rho_0 \left\{ \frac{\partial \widetilde{U}_1}{\partial t} + U_0 \frac{\partial \widetilde{U}_1}{\partial z} \right\} + \frac{\partial \tilde{p}_1}{\partial z} = 0 \qquad (2)$$

And here also we get C naught square and instead of this thing we get p_1. So, some directly sort of writing this (1) and (2). Now, basically what we do before we actually move ahead you know it is sort of a pretty instructive if we could further simplify equations (1) and (2) at this stage.

You know by assuming **Time Harmonicity**; time harmonic solutions and another thing is directly talk about what

$$U^* = \tilde{p}_1(z) - \tilde{p}_2(z)$$

So, if you go back to our you know control volume, the median to fortunately in terms of algebraic simplifications.



This will have only the $\widetilde{U}_2$ term present ok and this will not have any mean flow at least for CTR. So, you know additional terms will sort of drop off. $\widetilde{U}_2$ will only be there, but what about U^*? You know this should be a relation between U_1, U_2 or actually $\tilde{p}_1 - \tilde{p}_2$ and U^*.

So, basically as it turns out you know this U^* is nothing, but the difference here it is the first time that we are introducing the expression of perforate impedance. So, you know we will talk about all this thing go for an impedance later on, but

$$U^* = \frac{\tilde{p}_1 - \tilde{p}_2}{\rho_0 C_0 \varsigma} \qquad (3)$$

I am not this is supposed to be z something like this, but I am not writing z, it is understood. So, in other words, the perforate impedance that is the opposition that the perforate surface offers to flow

$$=> \qquad \zeta = \frac{\tilde{p}_1 - \tilde{p}_2}{\rho_0 C_0 U^*}$$

So, this is what we what we get. Now, we will use this equation let us call this as equation (3), is equation (1) and (2) already I have talked about by putting or insisting time harmonicity and using equation (3) in equation (1).

$$\frac{j\omega}{C_0^2}\,\tilde{p}_1\,\rho_0\frac{d\tilde{U}_1}{dz} + \frac{M_0}{C_0}\,\frac{d\tilde{p}_1}{dz}$$

$$+ \frac{4\rho_0}{d_1}\;\frac{(\tilde{p}_1 - \tilde{p}_2)}{\rho_0 C_0 \zeta} = 0$$

$$\rho_0\left\{j\omega\tilde{U}_1 + U_0\,\frac{d\tilde{U}_1}{dz}\right\} + \frac{d\tilde{p}_1}{dz} = 0$$

So, basically we need to simplify these equations I have already written down these equations. So, how did I get this? So, term by term we need to sort of simplify if we go back if we put $\tilde{p}_1 e^{j\omega t}$ is your nothing, but insist time harmonicity. So, you once you assume this and obviously, you need to solve in space.

So, when you do the derivative it is pretty simple say it straight forward that you will get this term with p 1 tilde to be understood as a function of z and e jω t all the variable always have this ejωt. So, that will factor out and same thing will apply here. So, we wherever temporal derivatives are there we will replace that by jωt and with understanding that I have talked about now.

And, so, as a result there is only one variable that is left that is z. So, the PDEs will then be converted to ODEs. So, del del z term will become d dz ok and another thing that we have done directly in the equations that I have shown here is that you can clearly see this guy $\tilde{p}_1 - \tilde{p}_2$ to theta divided by rho naught $C_0ς$. So, this is substituted equation (3) in the equation 1 and we have just left it in this form for now, but we will obviously, need to process the equations further.

And, another thing is that U_0 by c is, M_0 divided by C_0 that is what sort of we have. Let me do some live simplifications here. So, we multiply by sort of C_0 everywhere, C_0 and this guy will go away and here will be just C_0. So, what is ω / C_0? That

$$jk_0\tilde{p}_1 + C_0\rho_0\,\frac{d\widetilde{U}_1}{dz} + \ M_0\frac{d\tilde{p}_1}{dz}$$

$$+\frac{4}{d_1}\,\frac{\tilde{p}_1 - \tilde{p}_2}{\zeta} \ = \ 0 \qquad (4)$$

$$C_0\rho_0\left\{jk_0\widetilde{U}_1 + M_0\frac{d\widetilde{U}_1}{dz}\right\} + \frac{d\tilde{p}_1}{dz} = 0 \qquad (5)$$

And, we multiply it throughout by C_0, is not it? So, basically your this term will also go away and this will cancel out, is not it? Because if you look back at your this expression rho naught then here we had this expression. So, rho naught, rho naught directly sort of goes away and we are left with this. So, we can just leave it like this for now, this probably one of the most simplified forms and we probably would have to rearrange a few things here and there.

So, that is what we will do. In this particular momentum equation what we can do is multiply the term outside by C divide by C_0. So, this will become k_0 this will become M_0 ok, in we can you know further divide the entire thing by rho naught C_0 that is like a normalization values. So, we can sort of do that.

But, you know there is one good thing about say equation 4 and 5 is that it is a in the form of a coupled linear ODE. So, what are the variables in 4 and 5? p_1 U_1 ok and p_2 is of course, there and then we would also need you know as it turns out we have 3 variables and you know p_1 p_2 and U_1. But, then in the annular cavity there will be continuity and momentum equation also and U there will be a term containing U* in the continuity equation.

We need to eliminate that to get two more you know such equations. So, then we will be having four equations and four variables. So, then we will you know sort of will be in a decent position to put that in the coupled you know x dot is equal to ax kind of a form, x is a state vector then *x* dot means with derivative spatial derivative. So, let us go back to our control volume what I had drawn here. So, you know what? So, basically we are referring now to this region you know just outside. So, what is the, you know the other thing remains the same.

So, we need to speed up our derivation. There was zero mean flow like I have been insisting for CTR and the flow that goes inside now this is rho S2. Now, what is S 2 by the way? S_2 is let me make it clear here. This is ok this is here. So, this is what it was now, but S_2 remains uniform as I have been mentioning. So, we will keep this guy aside and go about our business of simplifying things.

So, rho S_2 let us say you know U_2 z. Now, you know I will just write down few things assuming that you guys are now sort of very well acquainted with the derivation the way it goes. The one that enters the stuff here, the one that goes out here and then there is a contribution that there is a term that goes that comes inside is not it?

And, then there is we need to balance it with your the temporal change is that is happening inside ok. So, as a result of all these things let me just finally, write down you know I am I will not do the control volume thing again in the sense I will not do this business again for lack of time. So, what I will probably do is directly write down the linearized form for the annular cavity of the continuity equation assuming zero mean flow in the annular cavity. So, here we

$$\rho_0 \frac{\partial \widetilde{U}_2}{\partial z} - \frac{4 d_1 \rho_0}{d_2^2 - d_1^2} \, U^* + \frac{\partial \tilde{\rho}_2}{\partial t} = 0$$

Now, we as usual we put our this expression p_1 - p_2 divide by this thing in here and also sort of simplify things by assuming the following form.

So, this is rho naught, what was it? It was p_1 - p_2 by and this guy becomes $j\omega\rho_2$, but keep in mind that ρ_2 is nothing, but your p_2 by C^2 $\tilde{p}_2$ by C_0^2. So, let us do some simplification and multiply all the side and the entire thing by C_0.

$$C_0\rho_0 \frac{d\tilde{U}_2}{dz} - \frac{\cancel{C_0} 4 d_1 \cancel{\rho_0}}{d_2^2 - d_1^2}\left(\frac{\tilde{p}_1-\tilde{p}_2}{\cancel{\rho_0}\cancel{C_0}\zeta}\right) + \frac{j\omega\tilde{p}_2}{C_0} = 0$$

$$=> \quad \rho_0 \frac{dC_0\tilde{U}_2}{dz} - \frac{4d_1}{d_2^2 - d_1^2}\left(\frac{\tilde{p}_1-\tilde{p}_2}{\zeta}\right) + jk_0\tilde{p}_2 = 0 \qquad (6)$$

So, multiply by C_0. So, this goes away and this thing is like this and this will become ok, plus j times k_0 p_2. This is what we will get ok. Initially you had ${C_0}^2$ we divide whether we multiply throughout by C_0 to get this kind of a thing ρ_0 here cancels out and you multiply by C_0.

So, C_0 is cancelled out once you multiply C_0 here, this gets cancelled with this one. And, there is only a ${C_0}^2$ only C_0 survives and omega by C_0 is k_0 in this term. So, well, we get this equation and let me number this as equation (6). So, it is clearly an ODE and we have you know 3 variables $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{U}_2$.

So, you know we have basically four variables $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{U}_1$, $\tilde{U}_2$. We have managed to successfully eliminate U^* the coupling the radial velocity that goes radial acoustic velocity that goes out of the perforate thing ok. So, because of that we now have your equations here. Now, what happens to the momentum equation? You know momentum equation will be the same as in duct 1 the inner duct.

But, just that your mean flow is not there. So, we just have this guy. So, again you know let us sort of simplify things.

$$\rho_0 \left\{\frac{\partial\tilde{U}_2}{\partial t}\right\} = -\frac{\partial\tilde{p}_2}{\partial z}$$

$$\rho_0\, j\omega\tilde{U}_2 = -\frac{d\tilde{p}_2}{dz}$$

$$=> \quad \rho_0\, C_0\, jk_0 U_2 = -\frac{d\tilde{p}_2}{dz} \qquad (7)$$

We probably can name this is equation (7). The probably the most simplified equation of all the equations that we have seen so far. So, here of course, this was also simple, but only thing was that mean flow velocity was there. Now, we clearly have four equations (4, 5,6) and (7).

So, for this thing, how do we go about doing the stuff? So, what we probably can do is that put that in the form of

$$\frac{d}{dz}\{\dot{X}\} = \underset{4\times 4}{[A]}\{X\} \quad \{X\} = \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{U}_1 \\ \tilde{p}_t \\ \rho_0 C_0 \tilde{U}_2 \end{Bmatrix}$$

So, dot means with respect to or you can actually instead of dot a better way dot usually means with respect to time, but I guess this is a better way. So, d /dz of this thing that is the, this particular form has to be solved. So, it is d /dz of x is equal to a where clearly a is your 4 rows 4 matrix you know let us populate the matrix. So, basically from equations (4, 5, 6, 7) that is (4 - 7) you know from (4 – 7) we need to get to this form basically to get the form of A.

$$A = \begin{bmatrix} -jk_0 - \frac{4}{d_1\zeta} & 0 & \frac{4}{d_1\zeta} & 0 \\ 0 & -jk_0 & 0 & 0 \\ \frac{4d_1}{(d_2^2 - d_1^2)\zeta} & 0 & \frac{-jk_0 - 4d_1}{(d_2^2 - d_1^2)\zeta} & 0 \\ 0 & 0 & 0 & -jk_0 \end{bmatrix}_{4\times 4} \begin{matrix} C_1 \\ M \\ C_2 \\ M_2 \end{matrix} \begin{Bmatrix} p_1 \\ \rho C_0 U_1 \\ p_2 \\ \rho_0 C_0 U_2 \end{Bmatrix}$$

Let us write down the equation A like this. First is the continuity equation second is your momentum equation for duct 1. So, let us do a systematic job it is going to be a bit tedious. So, I would request you guys to bear with me I may make some occasional sort of mistake. So, in that case I will have to do it some part again let me sort of make equal spacing's three and four ok.

So, this is you know it is understood that the first row is the one continuity in duct 1, momentum in duct 1, continuity in duct 2, momentum in duct 2. So, well this is nothing, but I should have drawn dotted lines ok. So, now, let us fill in the different entries ok. So, basically what we need to do is that find out A 11 element.

So, for that we probably have to go back to slide 8, where we you know one thing another thing that I want to tell is that you know do not worry about the terms this one and these two terms here. These are the mean flow terms which you know kind of poses a little bit of difficulty in solving the thing in the sense that, that d/dz of the state variable that will that also will be a matrix there will be slight modification to that.

I will get back to that particular part in a while, but here you have jk_0, alright. So, now, let us worry about populating the a matrix and then whatever suitable adjustment we need to make to the d/ dz term on the left hand side will do that at a later stage. So, jk naught you know will be coming in here plus there is a clear contribution by the term you know the encircled term in here, ok.

What is going to happen you know, you really have to take this on the right hand side. So, this will become minus jk naught and minus jk naught let me write using this blue color minus jk_0 and minus 4 d_1 by ς, ok. We will sort of get that ok. So, remember one part that this is the entire thing is going to be multiplied.

We will get that sort of a thing. So, this is what will happen in here. Now, going back to the slide 8 remember we have we are done with the p1 terms here, do not worry about the derivative here. You know another thing that I must mention in here because we are choosing the variable something like you know rho naught C_0 U here. So, we will take this entire thing inside.

So, because ρ_0 C_0 of the characteristic impedance will not depend on the derivative. So, you know that is like a constant. So, so the variable will be d dz of ρ_0 C_0U_1. So, rho naught C_0 d_1 is essentially taken as one variable. So, well you know this term also will not have any contribution in the a matrix. This term will not have this will come in the left hand side I will get back to that.

Other term that probably have a contribution is you know plus mind you plus 4 divide by d_1 zeta because plus because you have to take this on the other right hand side. So, here it will be 0 because you know this will be multiplied by this guy and like I said this will be 4 divided by d_1 ς I guess I have made a mistake here.

So, sorry, for that it is 4 divided by d_1ς, ok and here will be 0 because there will be no U_2 term. We will worry about this term and this term sort of at a later stage, but now let us also worry about the equation (5), that is the momentum equation which constitutes the second row.

So, d dz and these two terms will be on the left hand side only term that contributes is minus jk_0. So, basically your minus jk_0 has to be transposed on the right hand side. That is why I will become minus rho naught C_0U_1. So, basically you know here you will have minus jk naught everything else will be 0, is not it? Just sort of let us verify and because we are calling this $\rho_0 C_0 U_1$ as one variable, is not it. So, we are done with the second equations (2) C_1 M_1.

Now, let us worry about the other part that is your the continuity equation in the duct in the annular part. So, basically that is your I guess this equation, is not it? So, what are the terms that are contributing in for the third row? Well, this we will worry about at a later stage on the left hand side and this. So, only terms sort of here that are contributing is minus jk_0 you know minus jk_0 for p_2 we have to write that.

Let us get back to your this thing minus jk_0. So, what else is contributing to p_2? So, here you have your minus minus plus; the plus will become sort of minus when you sort of transpose it on the other side. So, - 4d 1 / d2 square minus d_1 square ς ok. So, you will sort of have this thing here. What about the other terms that are going on in your equation (9).

So, p_1, p_2 is there; p_2 we have sort of taken care of. So, the term that is associated with p_1 tilde. So, here is minus. So, once we take it on the other side it will be just plus. So, plus on this matrix so, here we need to write 4 times d_1 divide by d_2 square minus d_1 square and zeta ok. Everything else will be 0. Everything else will be 0 why? Because there will be no contribution from U_1 term and U_2 term and only thing now is remaining is your this guy, this equation.

Now, this is fairly simple. What you do is that you sort of put a minus sign in here and so, they will become minus jk_0 and where should minus jk_0 come? It should come on somewhere here because this is multiplied to this 0 0 0 is multiplied here 0 here 0 with p_2 minus jk_0 with this. So, we have your A matrix, but remember there is more to be done. Now, this you have a particular form let us make some suitable amendments.

$$\frac{d}{dz}\{[B]_{4\times4}\,\{X\}_{4\times1}\} = [A]_{4\times4}\,\{X\}$$

We will eventually be getting with 4 cross 1 matrix. So, this entire thing is d dz operators is operating upon this. So, what is the B matrix really? So, I just need to rub this guy and let me define B matrix here itself.

$$B = \begin{bmatrix} M_0 & 1 & 0 & 0 \\ 1 & M_0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} p_1 \\ \rho_0 C_0 U_1 \\ p_2 \\ \rho_0 C_0 U_2 \end{matrix}$$

So, something like this ok p_1 $\rho_0 C_0 U_1$, this is not this I am just writing to remind myself that things are you know that is how the things will happen. So, we need to get back what are the derivatives? You know all the derivatives the d dz terms were really in here. So, the continuity equation, is not it? So, here we have d dz of p.

So, only term I mean only thing that would matter here this is your M_0 and here will be 1. So, what I am saying is that M_0 has to be there, M_0 will be there and in place of this there will be just 1 will be just be 1. So, this is M_0 and this, this is 1, 0, 0, M_0 and 1. So, all the other things are not there.

Let us come to the second row. So, d dz of p; so, here of course, there will be; there will be 1 and in place of this will be M_0, but hang on we also need to do little bit of simplification. Remember, our variables are M_0 C_0. So, there is one thing we also need to figure out that rho naught C_0 is also multiplied by throughout here.

So, rho naught C_0 will go here as well. So, this will be clubbed in here. So, this will become 1 and this will become M_0. So, going back here in slide 8, these two terms are there and rho naught C_0 U_1 is sort of one variable and this guy anyhow will be transposed on the other side. So, that is how we get your minus jk_0 here.

But, in this thing you have your M_0. Now, we have to worry about this guy in I think there is no there are no prizes in guessing. For the equation (6) you know this was sort of fairly straightforward, the only contribution that was coming was from this thing. So, d dz of U_0 U_2. So, basically you know here we have your 0, 0, 0 and 1, this was 1 because you remember this is only thing is the term associated this once you take this thing inside here. The coefficient is 1 ok and for this guy this is minus transposed in here. So, again the coefficient is 1. So, one this is 1, 0, 0, 0.

Clearly when you have M_0 to be 0, then you get back your familiar sort of equations what we really get back is you know when M_0 is non-zero of course, it needs to be there, but in case it is 0. So, what we get back is 0 times p_1. So, and so that is 0 and all there will be no contributions only thing that will be there is one times or just $\rho_0 C_0$ U_1 here.

And, then your p 1 and this is also 0 0. So, this will be just be p 1 d dz of p 1 because of your momentum equation and these things are also fairly straight forward. So, basically in case of M_0 to be 0, which is your case of a stationary medium, we would sort of get back the state vector that is being given here.

In case M_0 is nonzero, we need to sort of invert this matrix. Since M_0 is not dependent upon the spatial variable z, so, what we can do is basically multiply throughout by B inverse because M_0 will not be singular matrix. So, multiplying throughout the B inverse we will get you know B inverse A. So, once you multiply B inverse B will be identity matrix I and B inverse A will get us thing.

$$\frac{d}{dz}\{X\} = [B^{-1}]_{4\times4}\,[A]_{4\times4}\,\{X\}_{4\times1}$$

$$\{X\} = \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{U}_1 \\ p_2 \\ \rho_0 C_0 \tilde{U}_2 \end{Bmatrix}$$

You know this was something like this, you know once there is also one more thing what I would like to you know just mention in here. So, in case we chose rho naught $C_0$1 as the first variable p_1 here.

So, you know this - jk_0 would have been here and everything else would be there and you know these things would have sort of shifted, but what other thing it would have done is that you know this would have been 1 and M_0 in here. You see M_0 is multiplied by p_1.

So, in case just if we switch things, so, 1 will be here M_0 will be here and same thing 1 will be here and M_0 will be here and this guy one would be here and this thing. So, it would have become a bit more easier to understand that well, M_0 is 0 then we would naturally get back your identity matrix, but this is just the manner we choose p_1 taking p_1 rho naught C_0 U_1 first.

The idea is that you know after doing all this algebra, we have A matrix and we have your B matrix. So, let me call this equation well, star and equation this is double star ok, equation star and equation double star and finally, we have your this matrix. Let me call this equation triple star. So, this is really a set of coupled ODEs in the four variables what I have mentioned in here p_1 $\rho_0 C_0 U_1$ p_2 $\rho_0 C_0 U_2$.

So, there are different ways of really solving it this is one way and you know if you were to do by the hand calculation it will become very tedious if four coupled ODEs like I said depending on the perforated impedance it might be a strong coupling. So, generally people go to computer programming MATLAB you know for instance or FORTRAN or whatever one is comfortable with and get inverse you know define the matrices you know compute basically your inverse.

You know in MATLAB what we would do is basically do inverse B multiplied by A ok A matrix. And, once we do that d dz of this then there are things like matrix exponential; matrix exponential one we can simply evaluate. So, the solution will be X the X vector will you know some exponential times matrix into some constants, which we need to eliminate by appropriate boundary conditions.

So, we can solve a simple MATLAB program at least for the case of a zero mean flow in MATLAB we can demonstrate that by kind of integrating this matrix using something like exponential e exp A or whatever length we are kind of integrating. So, length of this guy the perforated section that I am talking about here let us say the length is the total length is L ok.

So, since it is all uniform is a constant coefficient differential equation. So, we would get your e to the power exponential e to the power M exponential M exp M AL. We will demonstrate that shortly in MATLAB at least for the case of zero mean flow and possibly with some value of non-zero mean flow.

And, then we can for a fully perforated thing I will just demonstrate the MATLAB code talk about the boundary conditions, which are eliminated and explain them later and get the transmission loss curves for a fully perforated section using certain perforate impedance expression, but before I do that I would also mention couple of points.

First is, you know this is one way of solving another way is analytical kind of a thing in which basically you know what one does is really you know solves you get two coupled second order ODEs in where you eliminate your U_1 and U_2. So, the variables are p1 and p_2 and based on the mean flow present or the perforated impedance the coupling can be strong or they can be sort of weak.

And, you can get your that equation you know can be further combined into one equation that is your fourth order ODE and then you can go about solving that thing using your characteristic equation and all that, getting four different linearly independent solutions. So, that is going to be quite tedious specially when flow is there.

So, I would not even bother going into that, but at the end of the day when you are using your you know this function exponential, the MATLAB what it is doing? It is relying on the eigenvalues and the eigenvectors. So, the other way, of course, is to do not rely on this thing and find out the eigenvalues of this coupled equation by yourself and then do the simple linear algebra thing xi inverse lambda xi something like that. But we will probably to make it simple in this course we are using MATLAB.

So, we will do exponential M times AL, A is a matrix or B inverse A L L is the length of the perforated section. And, then this will be multiplied by some arbitrary constants to be determined. So, what we are going to do now that at least for the concentric tube chamber. Let us worry about the boundary conditions that will happen at the left and right boundaries whether it is a fully perforated thing or a partially perforated thing.

And then I will for a certain couple of by taking certain perforate impedance expressions I will do some MATLAB calculations or probably I will worry about the different perforated impedance condition and then present you some MATLAB results. So, all this we can have a look at in the lecture for the next week that is week 8 and that is how we will go about solving before doing some parametric studies even for the external and outlet case.

So, thanks a lot. I will see you in week 8.