Muffler Acoustics - Application to Automotive Exhaust Noise Control

Prof. Akhilesh Mimani
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture: 1
Introduction to Acoustic Wave Propagation

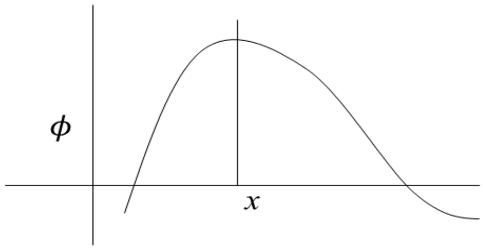
Welcome to the course on Muffler Acoustics. I am Dr. Akhilesh Mimani at Department of Mechanical Engineering, IIT Kanpur. We are going to talk about different aspects of muffler analysis, design, starting from the basic concepts of wave propagation in this lecture. Gradually building up the equations, governing equations, which includes the wave equation and different boundary conditions, the two-dimensional, one-dimensional, three-dimensional wave propagation inducts.

And then talking about different aspects of muffler elements, starting from simple elements like area discontinuities, expansion chambers, extended inlet and outlets, and how these are used to attenuate noise. And then we will go on different complicated elements like perforates, and dissipative elements, and cooperating mean flow effects, three-dimensional analysis using different methods like, analytical techniques and numerical analysis.

And little bit about experimental techniques and introduction in muffler acoustics and finally, talking about the design aspects. So, this course will basically cover the entire aspects of muffler design, and you most welcome for the course.

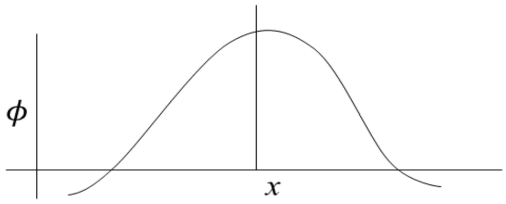
So, before we begin the muffler analysis, let us go to the basic concepts what is a wave? Well, a wave is a disturbance is a form which travels in space as well as in time. So, how do we define a wave? Let us consider a waveform like this, where this is

$t = t_0$



$\phi = p\ (x, t)$

$t = t_0 + \Delta t$



So, you see notice the peak, the peak shifts towards the right that is to say the pulse propagates towards the right-hand side. Similarly, we can also have a pulse that propagates towards the left-hand side. So, this is a kind of a wave which travels in a certain direction.

So, it is really a disturbance that basically disturbance or a form that travels through space as well as time. A formal definition is often hard, but we can basically illustrate different examples. For example, if you drop a pebble in a pond or basically clap your hands in a in a room which has fully cemented walls, you could see different things echoing; or if you drop a pebble in a pond, you could see waves going out in a in the water.

So, similarly another famous example is that of a tuning folk so that folks they vibrate in direction. So, wave propagation requires some medium to travel which can be air, water and then there are different types of waves. One is the wave that propagates in the direction in which the particles oscillate, and that is called the longitudinal wave, also called the p-waves in geophysics. And the other waves are called transverse waves.

For example, in that thing, the particle motion is in a direction perpendicular to that of the propagation medium. All mechanical waves require a medium to propagate. For example, sound waves they require air to propagate. Similarly, vibrations on a violin string requires the presence of string; the string itself is a medium through which the wave propagates.

Similar like I said if you drop a pebble in water on a calm surface of a pond or a lake, it creates disturbance that is waves which travel in the outward direction. So, these are water waves; the water itself is a medium. However, other waves like electromagnetic waves do not require any medium. For example, radio waves light and other waves infrared radiation, they require no medium. For them, vacuum is sufficient that is why we receive so much electromagnetic wave from space.

Another thing about wave motion is that the medium itself may or may not be moving. For example, going back to the problem of dropping a pebble in a pond, it does not require the water stream to be flowing in a certain direction. It can be calm quiescent medium, and the waves are still generated. And another case for example, flow in ducts which is really the central matter of this course.

For example, the duct work used in exhaust system of automobiles, there is always mean flow and plus their disturbances that are convicted along the direction of the flow, and some disturbances of course, go against the direction. So, in that case medium is essentially moving in a certain direction. So, depending on the application medium may be moving or not it depends.

Let us now consider what is the basic mathematical form that governs wave propagation.

So, for that; let us consider the form

$$\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial j(x-cot)}{\partial(x-\text{cot})}\right\}$$

$$\frac{\partial \tilde{p}}{\partial x^2}+\frac{\partial \tilde{p}}{\partial y^2}+\frac{\partial \tilde{p}}{\partial z^2}-\frac{1}{C_0^2}\frac{\partial \tilde{p}}{\partial t^2}=0$$

So, this entire thing can be written as

$$\Delta\tilde{p}-\frac{1}{C_0^2}\frac{\partial \tilde{p}}{\partial t^2}=0$$

So, this is your second order hyperbolic wave equation which governs the propagation of acoustic waves. In a general form, where del square is a Laplacian in the Cartesian coordinates *x y*.

Now, let us consider some special cases of this equation. So, as we saw Laplacian is given by

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So, suppose, if we drop the dependence on y and on z, what do we get?

$$\frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{1}{C_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2} = 0$$

We get the one-dimensional equation. The solution of that is the plane wave solution. We will talk about it what do you mean by plane wave and all those concepts. But before that, let us see if we can factorize this equation and get some more meaningful physically more meaningful forms. For that, let us go to another slide.

So, the form that I had shown in the last slide can be factorize in the following

$$\left(\frac{\partial}{\partial x} + \frac{1}{C_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{C_0}\frac{\partial}{\partial t}\right)\tilde{p} = 0$$

Now, let us consider let us put P tilde in this form to get things like this.

$$\left(\frac{\partial}{\partial x} + \frac{1}{C_0}\frac{\partial}{\partial t}\right)\phi = 0 \qquad (1)$$

where, $\phi = \frac{\partial \tilde{p}}{\partial x} - \frac{1}{C_0}\frac{\partial \tilde{p}}{\partial t}$

So Eq. (1), the solution of that is what we seek. And how do we solve for ϕ when ϕ satisfies the equation given by Eq. (1).

$$\frac{\partial \phi}{\partial x} + \frac{1}{C_0}\frac{\partial p}{\partial t} = 0 \qquad (1)$$

$$\phi = f(x - C_0 t)$$

$$\frac{\partial f(x - C_0 t)}{\partial x} = \frac{\partial f(x - C_0 t)}{\partial (x - C_0 t)}\frac{\partial (x - C_0 t)}{\partial x}$$

$$= f'(x - C_0 t)\ (1)$$

$$= f'(x - C_0 t)$$

So, we essentially have the first order equation in space and time is also called the advection equation, which basically governs the propagation of a disturbance quantity in one-dimensional space and time. In this case, one-dimensional because we have only x coordinate, only one coordinate involved.

$$\frac{\partial \phi}{\partial x} + \frac{1}{C_0}\frac{\partial \phi}{\partial t} = 0 \qquad (1)$$

So, now, how do we how do we solve for this one? Is there any way is there any means that we can guess the solution, at least some generic form of ϕ which would satisfy this? Any form, it could be ϕ can be some sinusoidal function, or some complex exponential functions, or some even algebraic forms like polynomials or something like that.

$$\phi = f\ (x - c_0 t)$$

So, it turns out that when you consider phi in the following form, this ϕ in the form this one will satisfy the equation (1). Let us see how. When you put this thing in the above equation and start differentiating, let us do it term by term. So, here I am bringing in the, this term substituting this in the first in, substituting this part in this term to get this term. We will work out term by term how things happen.

$$\frac{\partial y\ (x - c_0 t)}{\partial x} = \frac{\partial f(x - c_0\ t)}{\partial (x - c_0 t} \frac{\partial\ (x - c_0 t)}{\partial x}$$

So, applying the chain rule, where f' is derivative partial derivative with respect to $x - c_0$ t. And just like we have substituted the form f of x_0 - c_0 t in the above equation in the first term, to get finally this one. So, I will write this term again in the new slide. Similarly, we have to put this term in this equation as well.

$$\frac{1}{c_0}\frac{\partial \phi}{\partial t} = \frac{1}{c_0}\frac{\partial f(x - c_0)}{\partial t}$$

$$= \frac{1}{c_0}\frac{\partial f(x - c_0 t)}{\partial (x - c_0 t)} \frac{\partial (x - c_0 t)}{\partial t}$$

$$= \frac{1}{c_0} f'(x - c_0 t)\,(-c_0)$$

$$\frac{1}{c_0}\frac{\partial p}{\partial t} \qquad = -f'(x - c_0 t)$$

So, let us see what happens when you put this phi form in the second term to get what we get is $-c_0$. When you take the partial derivative with respect to time this term goes away and we have only minus c naught surviving, so this, these cancels. And what you are left with this is essentially this one.

So, one thing is clear now that this is what it is.

$$\frac{\partial \phi}{\partial x} = f'(x - c_0 t) \qquad (2)$$

$$\frac{1}{c_0}\frac{\partial \phi}{\partial x} = -f'(x - c_0 t) \qquad (3)$$

So, we need to now add these two things to get this equation (2), and (3).

So, these two terms cancel, and we get this.

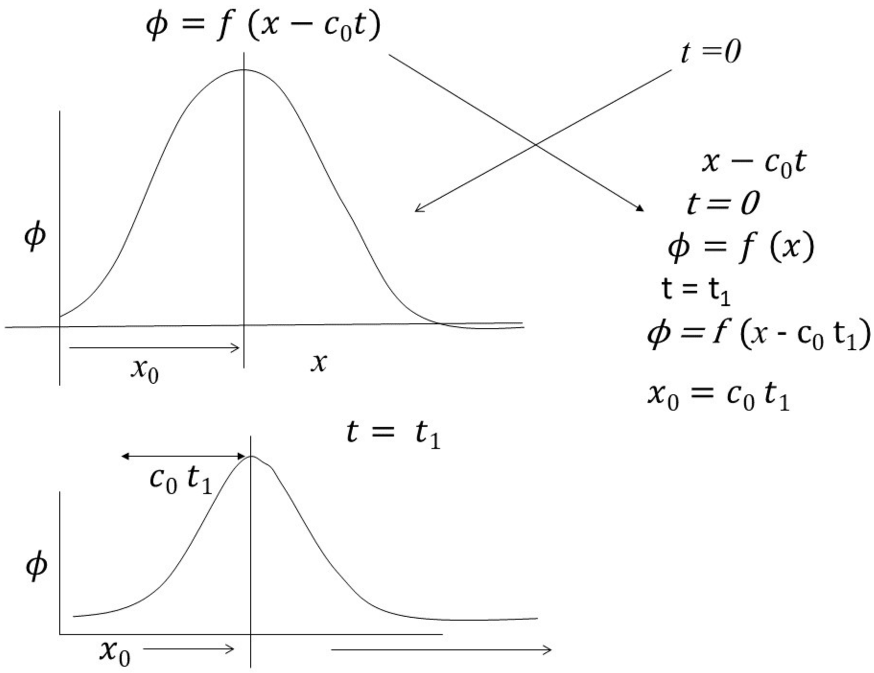
$$\frac{\partial \phi}{\partial x} + \frac{1}{c_0}\frac{\partial \phi}{\partial t} = f'(x - c_0 t) - f'(x - c_0 t)$$

$$= 0$$

Well, what does it mean then? This has some significance isn't it?

That means, f of any form, if ϕ is dependent upon x and t in such a manner that it can be clubbed together grouped together in the form $x_0 - c_0 t$ that is any function which can be written as in the form of $x_0 - c_0 t$ that will always be a solution of equation (1) that we are seeing at the top.

So, it can be sinusoidal function; it can be a complex exponential; it can be algebraic functions, it is very easy to see that. And one thing that we must note here that c0 [vocalized- noise] denotes the speed of sound that is the speed at which the disturbance propagates through the medium.



So, in the context of the first example that I mentioned sometime back, what does this solution mean *f* of x - $c_0\, t$. What does it mean? So, I will let us revisit that thing again. So, this is your space x, this is your variable ϕ.

What it means is that at a any generic point *x* if the value of phi is the same, now suppose here we are getting the peak at t = 0 at certain x_0 which is denoted here.

Now, for the same pulse to retain its shape that is to have the same maximum value but at a different time, the argument inside the function must be the same. That is to say for t_1 which is obviously, greater than 0 as we are defining *x* is given by $c_0\, t_1$ or in other words x_0 the value where the peak of the pulse happens occurs is shifted in space by a distance equivalent to the sound speed times the time elapsed that is *x* is $x_0 = c_0\, t_1$.

So, at t_1, the pulse becomes something like this, x naught shifts, and this is at t is equal to t_1. So, this difference is $c_0\, t\, t_1$. So, what it means is that the pulse essentially has shifted towards the right; obviously, for a greater time, the pulse continues to move in the forward in the positive *x*-direction that is in this direction. So, the significance of *f* of $x - c_0\, t$ means it is a forward propagating wave or the wave or the pulse that propagates in the positive *x*-direction. So, now, we know phi. So, ϕ is a pulse that propagates in the forward x direction like we found out now.

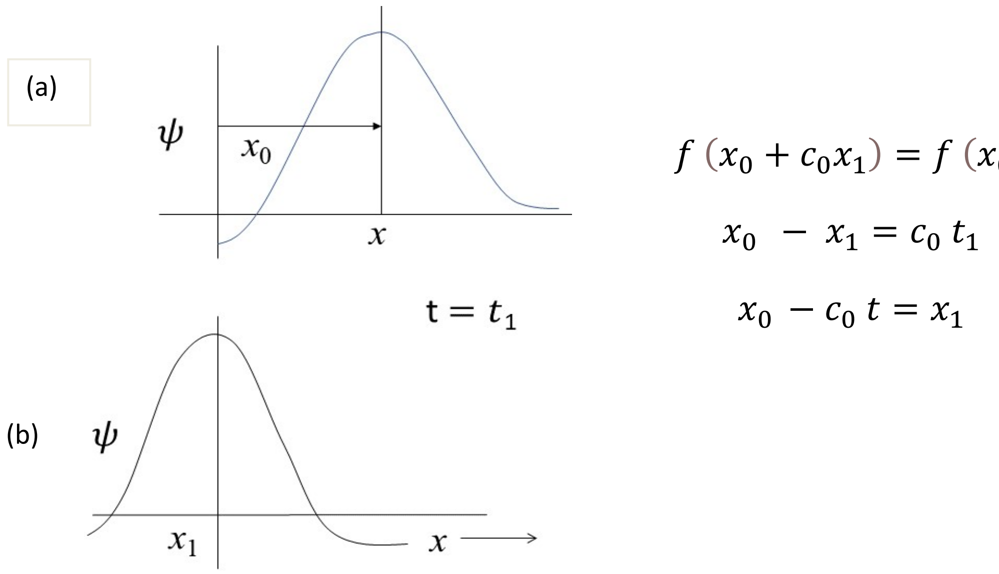
Similarly, if we were to consider the factorization just reverse the factorization that is to say this term would have been here and this term would have been here, essentially, we would have found out the solution of the following equation.

$$\left(\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)\psi = 0$$

$$\psi = \frac{\partial \tilde{p}}{\partial x} + \frac{1}{c_0}\frac{\partial \tilde{p}}{\partial t}$$

$$\psi = g\,(x + c_0 t)$$

Now, what does this mean? Basically it is just the opposite propagating wave. So, if t is increasing in time, time is increasing, then for the argument to remain constant, x must decrease.



So, if you have a pulse, again the same pulse that occurs at this point and time is increasing. So, for the argument to remain constant, x must decrease, that would mean the pulse $x + c_0 t$, t_1 is the same is the same value as the other one. So, this thing is the same as f of x_0, and let say this is x_1. So, the maxima will occur for this thing where $x_0 - x_1$ is equal to c_0 times t_1.

So, this quantity is obviously positive and x_0 is has some certain value. So, that would mean the position at which the peak occurs has shifted towards the left in this case. So, the waves would have gone somewhere like in here. So, this is your x, this is the function. And at time t

is equal to t_1, this is the figure (a), let us say this is figure (b); at this time instant, the peak is now somewhere here which is x_1 somewhere here maximum value. So, with the passage of time as the time increases this pulse begins propagating in the backward direction. So, the idea is that for the other factor for this equation.

$$\tilde{p} = f\,(x - c_0 t) + g\,(x - c_0 t)$$

$$\frac{\partial \tilde{p}}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial \tilde{p}}{\partial t^2} = 0$$

So, now the complete solution is given by for at least in the one-dimensional sense ok. So, the first term represents the outgoing wave or the wave that goes propagates in the positive x-direction as we saw. Similarly, the other solution represents a wave that goes or propagates in the backward direction or the negative x-direction.

So, both of them can be superposed to get complete solution of the one-dimensional wave propagation equation as we were discussing, representing the most general solution. And here there are no boundary conditions that are implemented in this solution, and hence this is also called the free wave solution.

So, another way to look at this thing, because the equation is linear, so we could basically get individual solutions and then superpose them as we have done in this thing to get the complete solution of this thing. As a check, it would be good if you could basically substitute P tilde the solution given by equation 4 in the one-dimensional equation here and differentiate term by term. So, what I will do? So, this is left as a exercise for the students here.

$$\frac{\partial^2 f(x - c_0 t)}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x} f(x - c_0 t)\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial f(x - c_0 t)}{\partial (x - c_0 t)}\,\frac{\partial (x - c_0 t)}{\partial x}\right\}$$

But I will just work out one term and rest students can do for themselves. That if you put f of x c_0 t in the equation and take a double derivative, what will you get? You can write it like this.

$$\frac{\partial}{\partial x}\left\{\frac{\partial f(x-c_0 t)}{\partial(x-c_0 t)}\right\} = \frac{\partial}{\partial x} f'(x-c_0 t)$$

$$\frac{\partial f'(x-c_0 t)}{\partial(x-c_0 t)} \frac{\partial(x-c_0 t)}{\partial x} = f''(x-c_0 t)$$

$$\frac{\partial \tilde{p}}{\partial x^2} = f''(x-c_0 t)$$

$$\frac{\partial^2}{\partial(x-c_0 t)^2}$$

Now, again applying the chain rule, what we get what we get? So, finally, what we get is this one, where the double derivative this thing means with respect to partial derivative with respect to $x - c_0 t$. So, like this in a similar manner what I suggest is that you could probably work out the derivative with respect to t, you will get c naught square in the numerator. And you will see these terms pertaining to f of $x - c_0 t$ getting cancelled out; and similarly g will also work out, they will also cancel out.

So, this basically what it means is that the solution gained by 4 is the most general solution or the one-dimensional equation. It represents plane waves that propagate in opposite direction. The first term propagates in the positive x-direction, and the other one in the negative x direction. No, boundary conditions have yet been enforced on the solution given by 4, and hence it is a free wave.

We will see in the next lecture that when we have certain initial conditions how does a solution look like initial condition in terms of acoustic pressure or velocity fields, and draw some parallel with vibrations in a string, and before we move on to the forced wave problems.

Thank you.