Advanced Engineering Mathematics
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Lecture – 57
Testing of Hypotheses - I

Hello friends, welcome to my lecture on testing of hypotheses, let us consider all possible samples of size n which can be drawn from a given population, the samples of size n can be drawn from a given population either with replacement or without replacement.

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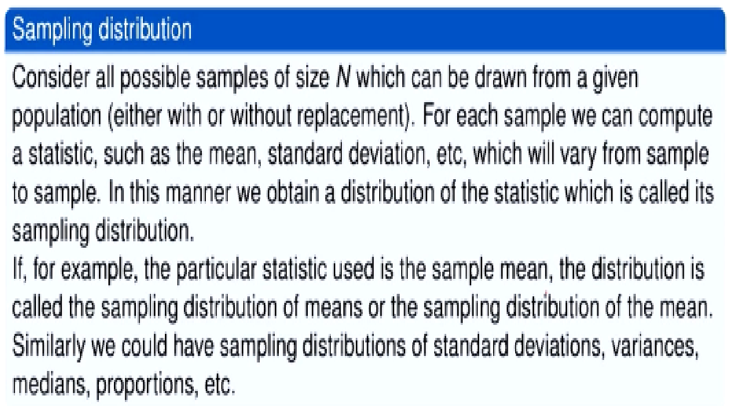

Now, for each sample, let us compute a statistics such as the mean or a standard deviation etc., which will vary from sample to sample, for different samples, mean and standard deviation will be different, so in this manner, we obtain a distribution of the statistic which is called its sampling distribution. If for example, the particular statistic used is the sample mean, the distribution is called the sampling distribution of means or the sampling distribution of the mean.

Similarly, we could have sampling distribution of a standard deviation, sampling distribution of variances, sampling distribution of medians, sampling distribution of proportions etc.

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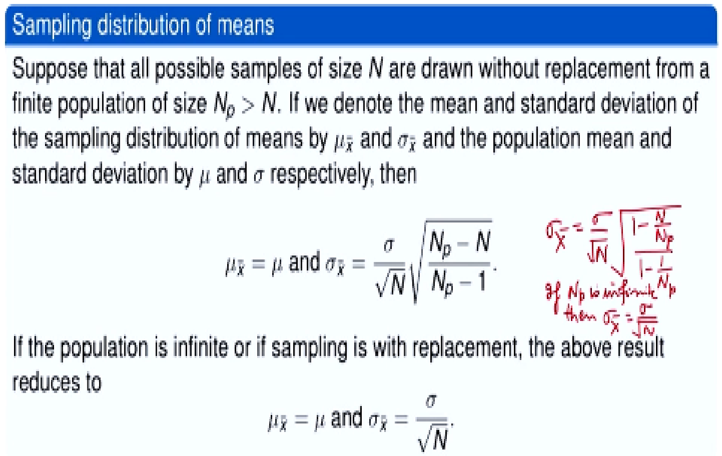

Now, suppose that all possible samples of size N are drawn without replacement from a finite population of size Np, okay, where Np is > N, if we denote mean and standard deviation of the sampling distribution of means by Mu x bar and sigma x bar and the population mean and standard deviation by Mu and sigma respectively then it follows that Mu x bar + Mu an sigma x bar = sigma over root N and then square root of Np – N divided by Np – 1.

If the population is infinite or if sampling with replacement, when the sampling is with replacement, okay, it is considered theoretically as a population as an infinite population because any samples; any number of samples can be drawn from the population without adjusting it, so when there is a sampling with replacement, it is theoretically considered as the infinite population.

So, here if the population is infinite or the sampling is with replacement then this sigma x bar = sigma / root N, we can write sigma x bar = sigma /root N and then square root 1 - N over Np divided by 1 – 1 over Np, so when Np is infinite, okay, sigma x bar =; so if Np is infinite, then sigma x bar = sigma y root N, so either Np is infinite or we are considering the sampling with replacement then we consider the population to be infinite.

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Sampling distribution of means cont...

For large values of $N (N \geq 30)$ the sampling distribution of means is approximately a normal distribution with mean $\mu_{\bar{x}}$ and standard deviation $\sigma_{\bar{x}}$ irrespective of population (so long as the population mean and variance are finite and the population size is at least twice the sample size). This result for an infinite population is a special case of the central limit theorem of advanced probability theory which shows that the accuracy of the approximation improves as N gets larger. This is sometime indicated by saying that the sampling distribution is asymptotically normal.

In case the population is normally distributed, the sampling distribution of means is also normally distributed even for small values of N (i.e. $N<30$).

Now, so in that case, Mu x bar = Mu and sigma x bar = sigma y root N. now, let us consider sampling distribution of means, for large values of N, when n is taking values > = 30, the sampling distribution of means is approximately a normal distribution "with mean Mu x bar and a standard deviation sigma x bar irrespective of the population, so long as the population mean and variance are finite and the population size is at least twice the sample size.

This result for an infinite population is a special case of the central limit theorem of advanced probability theory which shows that the accuracy of the approximation improved as N gets larger, this is sometimes indicated by saying that the sampling distribution is asymptotically normal. In case the population is normally distributed, the sampling distribution of means is also normally distributed even for a small values of N that is N < 30.

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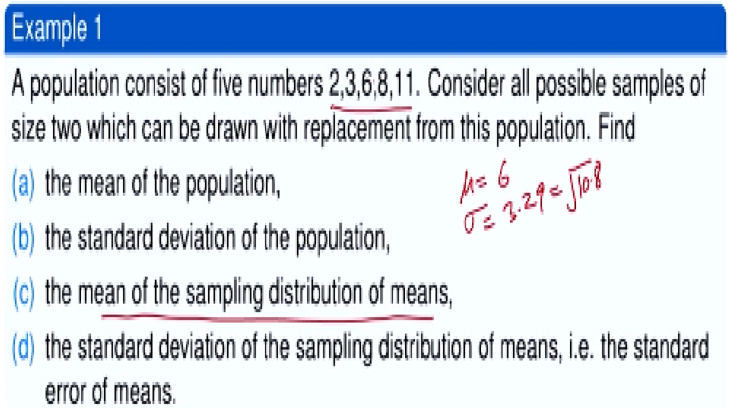

Let us consider an example, a population consist of 5 numbers; 2, 3, 6, 8, 11, let us consider all possible samples of size 2 which can be drawn with replacement from this population. So, finite; so when we are considering sampling with replacement then this population; we have to find the mean of this population, the standard deviation of the population, the mean of the sampling distribution of means, the standard deviation of the sampling distribution of means.

So, we are considering all possible samples of size 2, which can be drawn with replacement from this population, let us first find the mean of the population.

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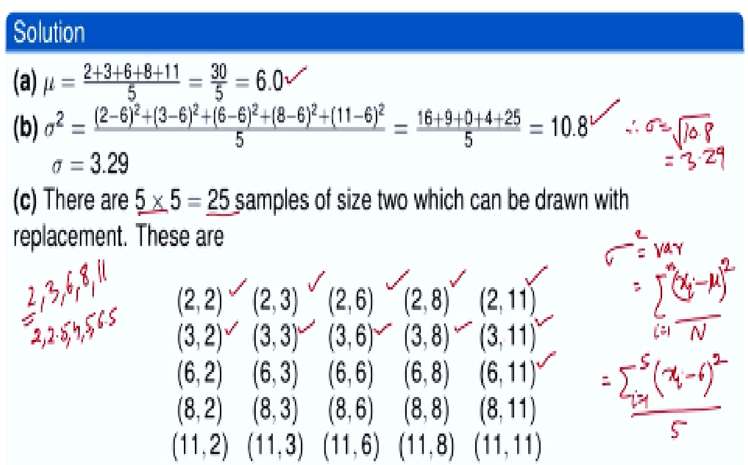

So, mean of the population we are denoting by Mu, so Mu = 2 + 3 + 6 + 8 + 11/ 5 which is 30/5 and this is = 6, okay, now sigma a square is the variance of the population and variance of the population we know is given by sigma a square which is variance of the population, variance is given by sigma i = 1 to n xi - Mu whole square divided by N, okay, so here we have xi's are the values are that the population is taking, 2,3,6,8,11.

So, we have x1 = 2, x2 = 3, and so on x3= 6, x4 = 8 and x5 = 11 and Mu we have already calculated 6, so we consider sigma i =1 to 5 xi – 6 whole square divided by N; N = 5 because there are population is a finite population, it consists of 5 numbers, so we have let us take x1 = 2, so 2 – 6 whole square, x2 is 3, 3 – 6 whole square, x3 is 6, so 6 – 6 whole square, x4 is 8, so 8 – 6 whole square, x5 is 11, so 11 -6 whole square divided by 5.

And when you compute this value, it comes out to be 10.8, so sigma = square root of 10.8 which = 3.29, now there are let us consider the c part. In c part, we have to find the mean of these sampling distribution of means, mean of the sampling distribution of means, so let us first find the sampling distribution of means, then we will find its mean. Sampling distribution of means, we are to take the samples of size 2 from this population with replacement, okay.

So, there are 5 * 5 that is 25 samples of size 2, okay, so 2, 3, now we have 2, 3, 6, 8, 11, we are to make samples of size 2, so let us pick 2 first, okay, then 2 we have picked up and then we have put it back, okay, so 2 can be paired with 2 and then 2 can be paired with 3, 2 can be paired with 6, 2 can be paired with 8, 2 can be paired with 11, so we get 5 pairs, these 5, okay, next time we pick 3, okay.

We have taken 3 and then put it back, okay, so 3 can be paired with 3, okay 3 can be paired with 2, 3 can be paired with 6, 3 can be paired with 8, 3 can be paired with 11, so we have again 5, okay, in a similar manner, we pick 6, okay we pick 6 and put back, okay, so 6 can be paired with 2, 6 can be paired with 3, 6 can be paired with 6, 6 can be paired with 8, 6 can be paired with 11, so in a similar manner, we can have 5 pairs with 8 and then 5 pairs with 11 in the first draw, okay.

So, in all there are 25 samples of size 2, now let us make the sampling distribution of their means, okay, so 2 + 2/2, so that will be 2, if you make the sampling distribution of means, so we will take the mean of each of the sample of size 2, okay, so mean of the sample size 2 that is 2,2 is 2 then this mean of the sample size 2, 2, 3 is 2 + 3/2 that is 2.5, then we have mean of the sample of size 2, 2, 6 is 2 + 6 divided by 2 that is 4.

And then 2 + 8 divided by 2 is 5, 2 + 11 divided by 2 is 6.5, so like that we can find the means of the other; all other samples of size 2 each, okay.

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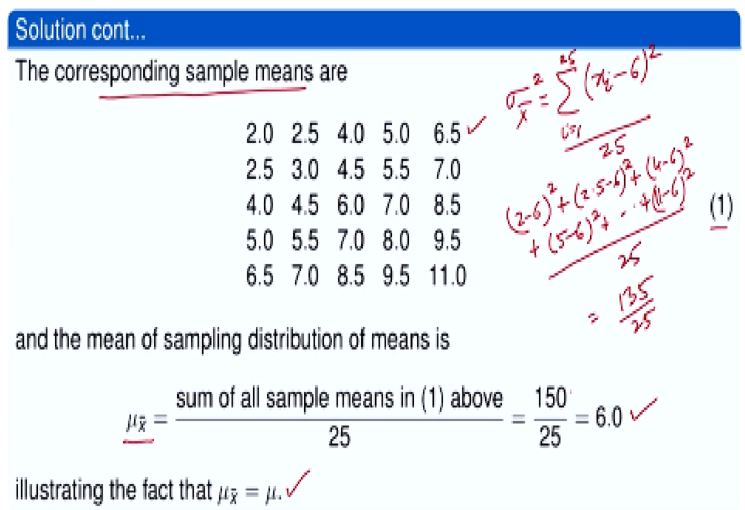

And then we have this distribution, okay this is for the means of the first row, okay, the first row and then we have the means of the second row, then we have means of the third row, we have means of the fourth row, the means of the samples of size 2 in the sixth row, okay, so this is sampling distribution of means, okay. Now, let us find the mean of this sampling distribution that is Mu x bar.

So, Mu x bar = sum of all these sample means in this equation 1, okay sum of all sample means divided by 25, they are all in 25, they are all 25 in number, so we let us add them all, add these 25 values and divide by 25, so sum of these 25 values is 150; 150 divided by 25 is 6, so you can see that Mu x bar = 6 and mean of the population is also 6, okay, so Mu x bar = Mu and then let us go to show that sigma x bar; let us find sigma x bar and then we will be showing that sigma x bar = sigma y root n.

Because it can be considered with this sampling with replacement, it can be considered as infinite in population. So, let us consider sigma x bar, the variance sigma x bar is square of the sampling distribution is obtained by; now, let us find sigma x bar, okay, so this is sampling distribution of means, okay, we have to find sigma x bar square here, so sigma x bar square will be = take sigma i = 1 to 25, okay and then we call this values as xi's, so xi – Mu x bar that is 6 whole square divided by we have 25, okay.

So, this can be written as 2 – 6 whole square, okay if you take this one, 2 – 6 whole square, then we have 2.5 -6 whole square, then we have 4 – 6 whole square, we have 5 – 6 whole square and so on, 6.5 - 6 whole square then 2.5 – 6 whole square and so on, 11 – 6 whole square divided by 25, so this is sigma x bar whole square, if you calculate this sum, okay, this sum is 135, so 135 divided by 25, okay.

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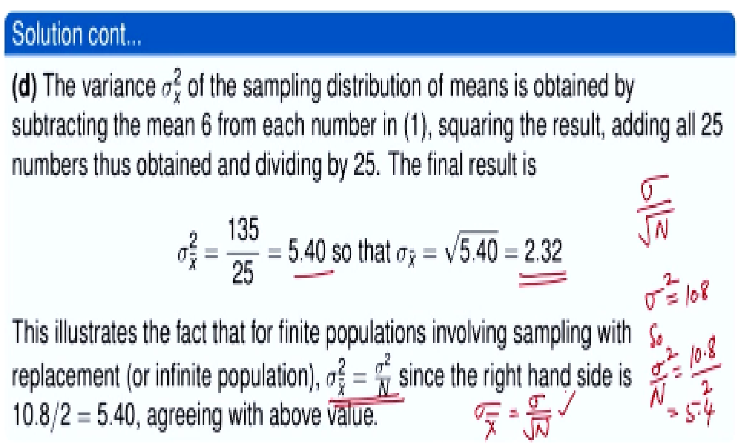

So, 135 divided by 25 is 5.40 and therefore sigma x bar is square root of 5.40 that is 2.32, now let us find sigma y root n, okay, let us see what is sigma, we had found okay, sigma square was = 10.8, okay, sigma square = 10.8, okay, so sigma square/n will be = 10.8 divided by 2 because we are taking sample size as 2, okay, so 10.8 divided by 2 that is 5.4, okay, 5.4 that is sigma square/ n , okay and here also you can see sigma x bar square is 5.4.

So, sigma x bar square = sigma square/ n, okay and thus we see that sigma x bar = sigma y root N, okay, this is because the; here we are dealing with sampling distribution with replacement, so the population can be considered as infinite, although it is finite population but when there is sampling with replacement, okay, any number of samples can be drawn from the population, so it can be considered as a sampling with replacement.

Now, it will; sampling with; sampling, infinite sampling, infinite population, so sigma x bar= sigma y root n we have to verify because in the case of infinite population, sigma x bar = sigma y root N.

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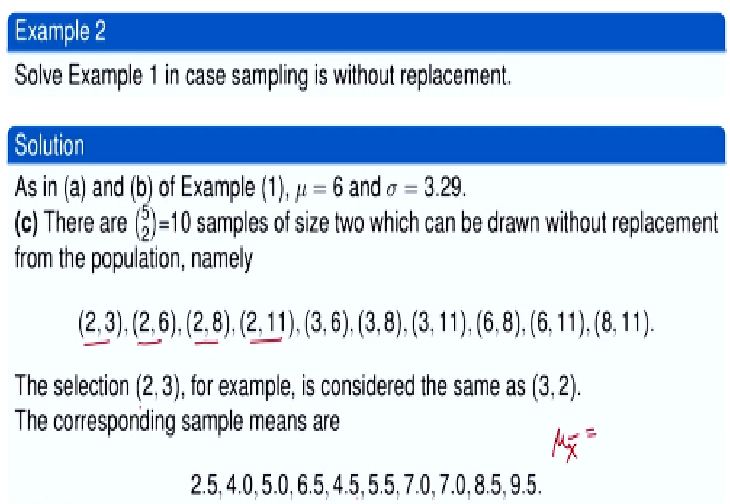

Now, let us solve example 1 in case sampling is without replacement, okay, so now, let us consider this problem, okay this problem where sampling is without replacement, okay, so mean of the population will remain the same, mean will be = 6, okay and standard deviation will also not change because it is not depending on sampling, there is no sampling here, so we have sigma = 3.29, okay that is square root of 10.8, okay.

Now, mean of the sampling distribution of means, now we have 5 numbers from; if there is sampling without replacement, then and we are making samples of size 2, then from 5 numbers, we can draw sample of size 2, okay that means there will be 10 such samples, okay and the 10 such samples are, these are 10 such samples; 2 3, 2 6, 2 8, 2 11, okay, thus we first we draw 2 in the first draw we take 2 and then 2 can be paired with 3, 2 can be paired with 6, 2 can be paired with 8, 2 can be paired with 11.

Now, we have drawn 2, okay, in the next draw, we now take 3, okay, so 3 then can be paired with 6, 3 can be paired with 8, 3 can be paired with 11, so 2 and 3 we have removed and then we take 6, 6 can be paired with 8, 6 can be paired with 11, so after that 6 is also drawn out, we have 8 and 11, so we get 10, 11 as the last sample of size 2, so there are 10 samples of size 2, okay. Now, here the selection 2 3 is considered same as 3 2.

The corresponding sample mean, now let us find the sample mean of the samples of size 2, okay, so mean of these 2 3 is 2.5, then mean of 2 6 is 4, mean of 2 8 is 5, and so on mean of 8 and 11 is 9.5, so we have sampling distribution of means were there are 10 numbers, means of these sample sizes of samples of size 2.

So, now let us find the mean of these sampling distribution, so mean will be = Mu x bar and Mu x bar =2.5 + 4.0 + 5.0 + 6.5 and so on + 9.5.

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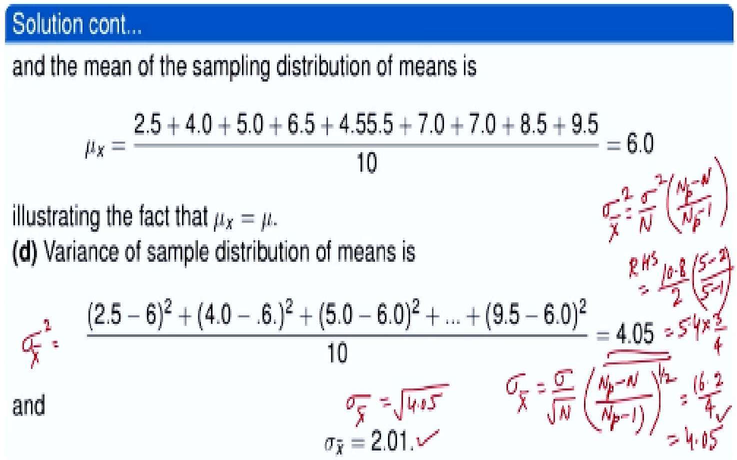

And it comes out to be of 60/ 10, this is 60/10, so it comes out to be 6, so in the case of sampling with replacement again Mu x bar = Mu, Mu = 6, now let us find sigma x bar, so sigma x bar = then sigma x bar square, sigma x bar square will be = 2.5 – 6 whole square, then 4 – 6 whole square, 5 – 6 whole square, 6.5 – 6 whole square and so on, 9.5 – 6 whole square divided by 10 because they are 10 in number, okay.

So, we have 2.56 – 6 whole square, 4.5 – 6 whole square and so on, 9.5 -6 whole square divided by 10 which is = 4.05, this is sigma x bar square, okay, now so if you find sigma x bar, sigma x bar will be = square root of 4.05 which is sigma = 2.0, now let us now here, we are considering sampling without replacement, so we have to verify the formula, sigma x bar = sigma y root N multiplied by Np – N divided by Np -1 square root raised to the power 1/2 okay.

Or we can say sigma x bar square = sigma square/ N * Np – N divided by Np - 1 okay, now let us say right hand side, okay, right hand side we have to calculate an then show that right hand side = the left hand side, okay, so right hand side sigma is we have found, sigma x square we found to be = 10.8, okay, sigma x square we found to be 10.8, so we get 10.8 divided by N; N is the sample size, sample size is 2, okay.

And then we have Np, Np is the size of the population, size of the population is 5, okay – 2 divided by 5 -1, okay, so what do we get here; this is 5.4 and what we get here, 5- 2 is 3, 3 divide by 4, okay, so how much is that? This is 3, 4, and 3, 4, 5, just 16.2, okay divided by 4 and this is nothing but 4.05, okay so sigma x square/ N * Np – N divide by Np - 1 comes out to be 4.05 which is same as sigma x bar square here, this 4.05.

So, sigma x bar = sigma y root N * Np – N divided by Np - 1 raised to the power 1/2 formula is verified, okay.

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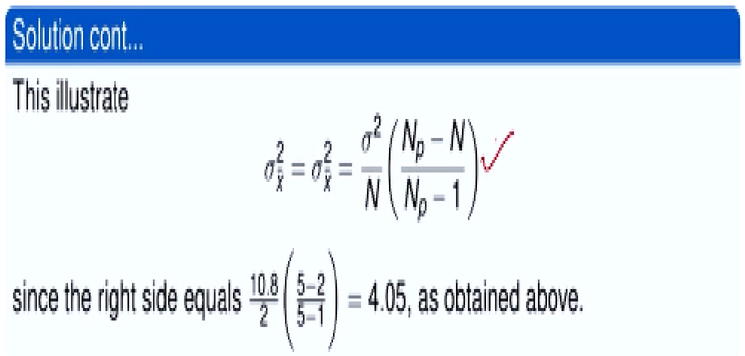

This formula is verified.

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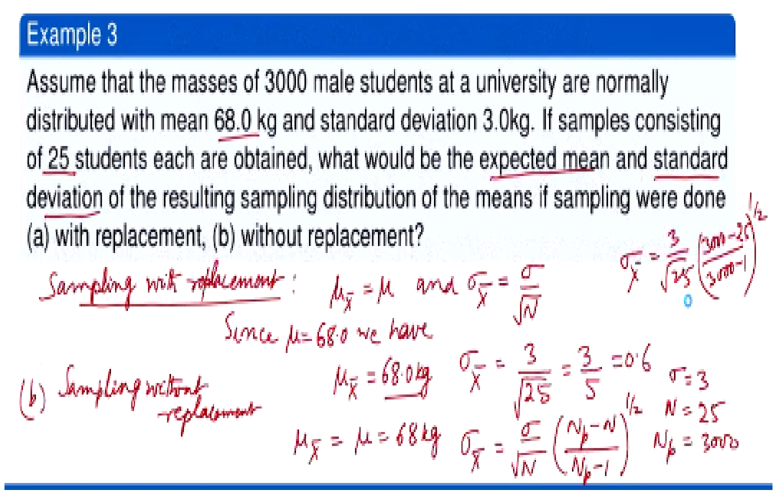

Now, let us assume that masses of 3000 male students at a university are normally distributed with mean 68 kg and standard deviation, 3 kg, if samples consisting of 25 each, if 80 samples consisting of 25 students each are obtained, what would be the expected mean and a standard deviation of the resulting sampling distribution of the mean, if sampling were done with replacement.

So, first let us consider sampling with replacement, okay in the case sampling with replacement, we know that Mu x bar = Mu, okay and sigma x bar = sigma y root N, let us assume that the masses of 3000 male students at a university are normally distributed, mean is sigma, mean of the population is 68.0, so since Mu is 68.0 we have Mu x bar = 68.0 kg, okay, we are given a standard deviation to be 3kg, okay.

So, sigma x bar = sigma is 3, 3 divided by; now, this standard deviation, the sample size is 25, okay, sample size is 25, so 3 divide by 25 square root, so we get 3/5 which is = 0.6, okay, sample sizes is 25 now, so this is in the case of sampling with replacement, expected mean is 68 kg and standard deviation is 0.6, okay. Now, let us consider sampling without replacement, so in the case sampling without replacement again, Mu x bar = Mu and Mu is 68, so Mu x bar is 68 kg.

Sigma x bar = sigma y square root N * Np – N divide by Np – 1 raised to the power 1/2, sigma is given to be; sigma = 3, okay, sigma = 3 kg and then N = 25, we are taking sample size is 25 and p = population size is 3000, okay, so we have sigma x bar = sigma divided by root N, okay, so sigma x bar will be = 3 divided by root 25 multiplied by Np is 3000 - N is 25 divided by 33,000 -1 raised to the power 1/2, okay, so this is what we have.

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Solution

(a)

$$\mu_{\bar{x}} = \mu = 68.0\ kg \ \ and \ \ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} = \frac{3}{\sqrt{N}} = 0.6\ kg$$

(b)

$$\mu_{\bar{x}} = \mu = 68.0\ kg \ \ and \ \ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}\sqrt{\frac{N_p - N}{N_p - 1}} = \frac{3}{\sqrt{25}}\sqrt{\frac{3000-25}{3000-1}} = 0.59\ldots$$

which is only very slightly less than 0.6 kg and can therefore for all practical purposes be considered the same as in sampling with replacement.
Thus we would expect the experimental sampling distribution of means to be approximately normally distributed with mean 68.0 kg and standard deviation 0.6 kg.

And if you calculate this 3/ root 25 is square root 3000 – 25/ 3000 – 1, it is very near to 0.6 okay it is something like 0.59 something okay, it is very near to 0.6 okay and can therefore, for all practical purposes we considered the same as in the case of sampling with replacement, thus we would expect the experimental sampling distribution of means to be approximately normally distributed with mean 68.0 and a standard deviation 0.6.

In the case of sampling with replacement, it came out to be 0.6, in the case of sampling without replacement, it came out to be 0.4559 which is slightly < 0.6 and therefore it can also be considered as; this case can also be considered as the sampling with replacement.

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Testing of hypotheses

Very often we are required to make decisions about the population parameters on the basis of a random sample from the population. Such decisions are called statistical decisions. For example, we may wish to decide on the basis of a sample data whether a new serum is really effective in curing a disease, whether one educational procedure is better than another.

Statistical hypotheses

In an attempt to make decision, we make assumptions or guesses about the parameter(s) of the population distribution(s). Such assumptions, which may or may not be true are called the statistical hypotheses. When more than one population is involved, statistical hypotheses consists of relationships between the parameters of the populations.

Okay, now let us consider testing of hypotheses, very often we are required to make decisions about the population parameters on the basis of a random sample from the population such decisions are called a statistical decisions. For example, we may wish to decide on the basis of a sample data whether a new serum is really effective in curing a disease, whether one educational procedure is better than the other; than another.

In an attempt to make a decision, we make assumptions are guesses about the parameters of the population distribution, now, such assumptions which may or may not be true are called the statistical hypotheses. When more than 1 population is involved, a statistical hypotheses consists of relationships between the parameters of the populations.

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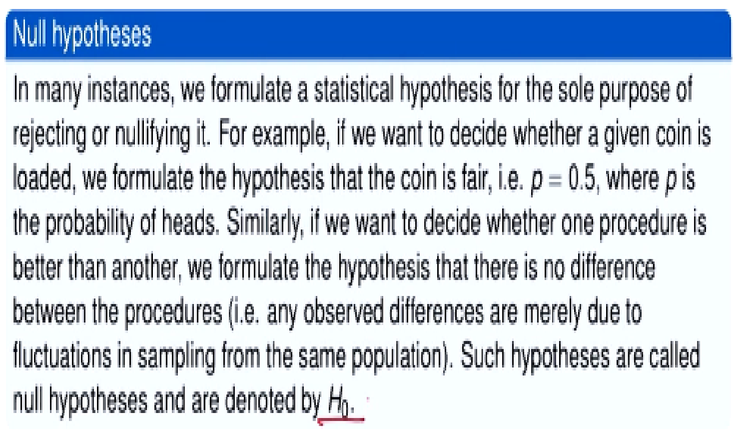

In many instances, we formulate a statistical hypotheses for the sole purpose of rejecting or nullifying it, for example if we want to decide whether a given coin is loaded, we formulate the hypotheses that the coin is fair that is p = 0 .5 where p is the probability of heads. Similarly, we want to decide whether one procedure is better than another, we formulate the hypotheses that there is no difference between the procedures.

That is any observed differences are nearly due to fluctuations in sampling from the same population, now such hypotheses are called null hypotheses and or denoted by H0.

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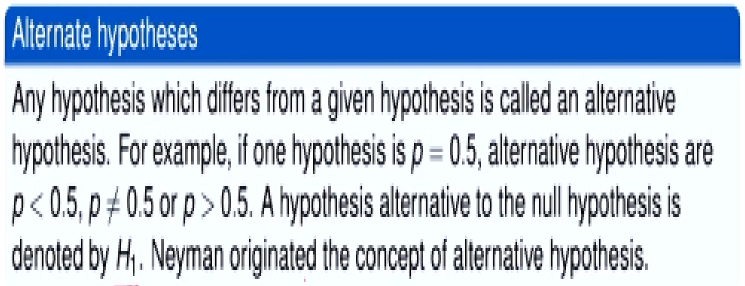

Now, any hypotheses which differs from a given hypotheses is called an alternative hypothesis. For example, if one hypothesis is p =0 .5 alternative hypothesis are p < 0.5, p0 = 0.5 or p > 0.5 a hypothesis alternative to the null hypothesis is denoted by H1, so H1 is the notation for alternative hypothesis. Now, Neyman originated the concept of alternative hypothesis.

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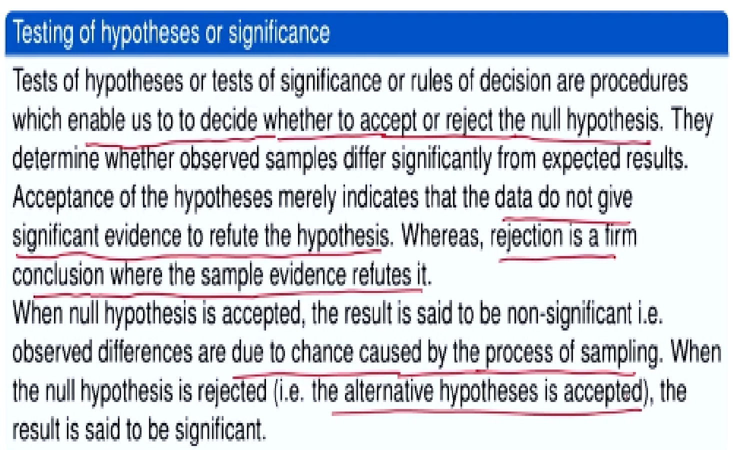

Testing of hypothesis or testing of significance; tests of hypotheses or tests of significance or rules of decision are procedures which enable us to decide whether to accept or reject the null hypothesis. They determine whether observed samples differ significantly from expected results, acceptance of the hypotheses merely indicates that the data do not give significant evidence to refute the hypotheses.

Whereas, rejection is a firm conclusion, where the sample evidence refutes it, so if the sample evidence refutes the hypotheses, we have to reject it. Now, when null hypotheses are accepted, the result is said to be non-significant that is observed differences are due to chance caused by process of sampling, when the null hypothesis is rejected, the alternative that is the alternative hypotheses is accepted, the result is said to be significant.

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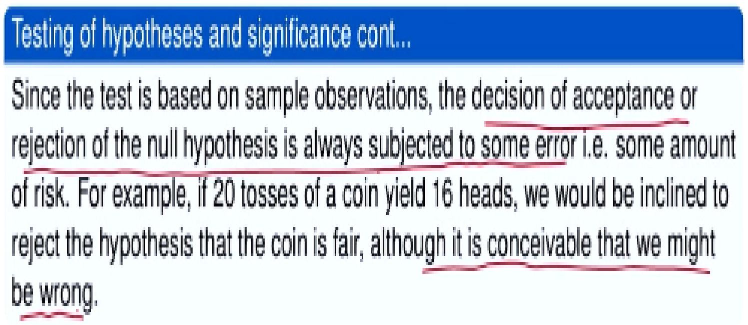

Now, let us consider further testing of hypotheses and significance, test of significance, since the test is based on sample observations, the decision of acceptance or rejection of the null hypotheses is always subjected to some error, okay, that is some amount of risk, for example, if 20 tosses of a coin yield 16 heads, we would be inclined to reject the hypotheses that the coin is fair, okay.

Suppose, our null hypotheses is that the coin is fair, okay and we carry out the experiment that is we take make 20,000 of the coin which yield 16 heads, we would be inclined to reject the hypotheses that the coin is fair, although it is conceivable that we might be wrong, okay.

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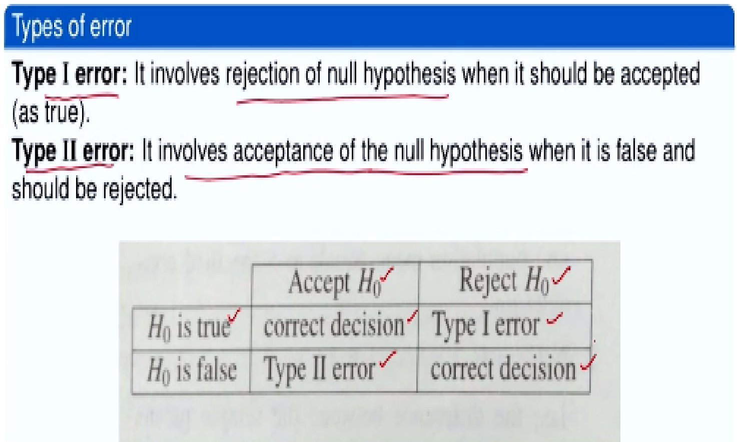

Now, let us consider types of error; type 1 error, it involves rejection of null hypotheses, okay, when it should be accepted as true, okay, so when it is; when H0 is expected to be true, but we rejected, we call; we have made type 1 error and when we accept the null hypothesis, while it is false and should be rejected, we make type 2 error, okay, so type 1 and type 2 error are given in the form of this table.

Null hypotheses H0 is true, okay, if we accept H0, we have made correct decision, if we reject H0, we have committed an error of type 1, H0 is false, okay, we accept H0 that means we have made type 2 error, if we reject H0, we have made a correct decision, so with this I would like to end my lecture, thank you very much for your attention.