Advanced Engineering Mathematics

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Lecture – 50
Exponential Distribution

Hello friends welcome to my lecture on exponential distribution let Nt be a personal process with rate lambda let x1 be the time of 1st arrival then p be the probability that x1> than t.

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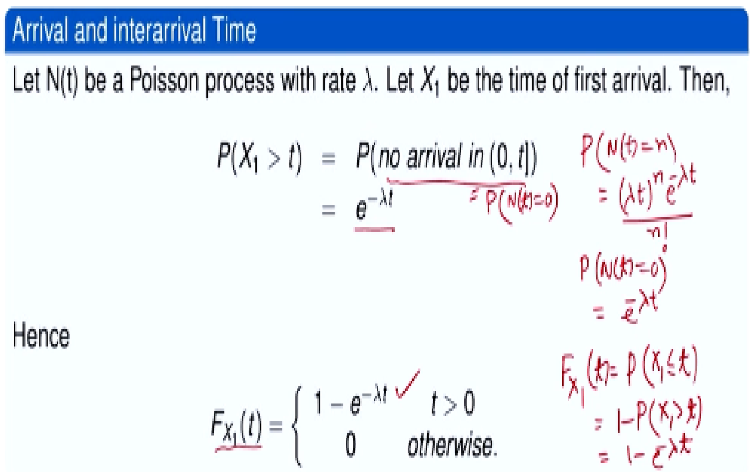

Will be = to probability that there is no arrival in the interval 0 to t because x 1 is the time of the first arrival and x1> than t so in the time interval 0to t there will be no arrival and probability that there is no arrival in 0 to t means probability that nt =0 means we have probability of in a Poisson process probability that Nt = n = lambda t rest to the power n e to the power – lambda t /n factorial so there is no arrival in 0t means probability that Nt=0.

Nt =0 means lambda t to the power 0 which is 1 then e to the power - lambda t /0 factorial so this will be e to the power – lambda t so this probability of no arrival in 0 t is actually probability of Nt = 0 there is no occurrence in the time interval 0 to t so this = e to the power – lambda t hence fx1t is probability that x 1 is < = t probability that x<=t and this = 1-probability that x1 is > t so probability at x1>t is –lambda t.

So, it should be 1 – e to the power – lambda t so we will get this 1-e to the power –lambda t when t is > than 0 and it will be 0 otherwise

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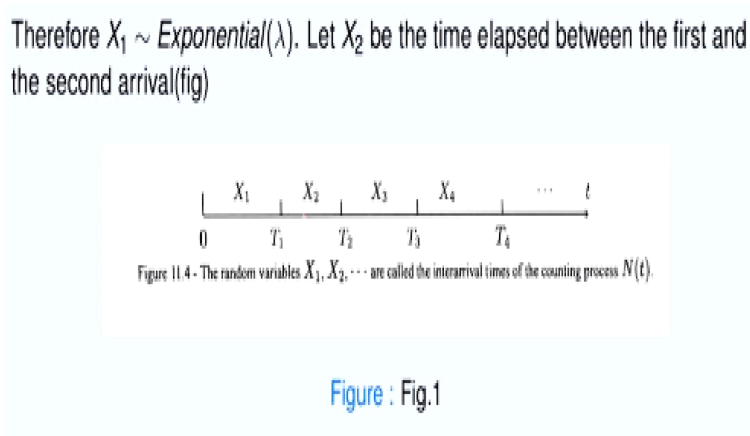

Now therefore x1 is an exponential lambda distribution and let x2 be the time elapsed between the 1st and the 2nd arrival so x1 is the time of the 1st arrival x2 is the time elapsed between the 1st and the 2nd arrival so then x3 is the time between the 2nd and 3rd arrival like that.

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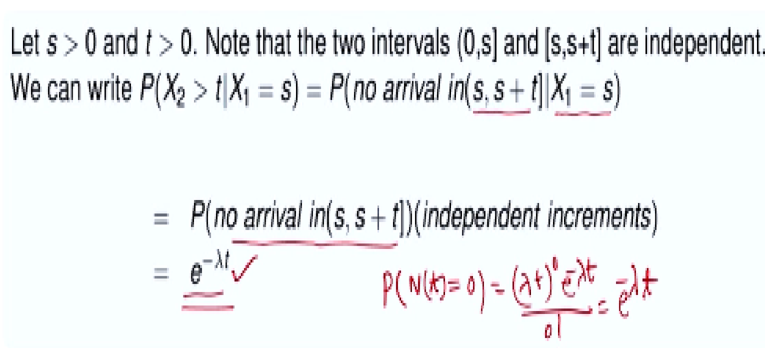

So, what we have let s be > 0 and t be > 0 let us note that the 2 intervals 0 as intervals 0 s and s s+t are independent we know that of the number of occurrences are independent of the non-overlapping intervals if there are 2 over an interval which are non-overlapping number of occurrences in 1 is independent of the number of occurrences in the other so let us note that 0 as s s+t are independent therefore probability that x2 is > t .

Given that X1 = s given that X 1 = x x1 is the time of the 1st arrival x1 = s that means s is the time of the 1st arrival x2 is t means there is no arrival in s + t X2 is the time elapsed between the 1st arrival and the 2nd arrival 1st arrival occurs at s at time s so that means and these 2nd the time elapsed between the 1st arrival and 2nd arrival is more than t it will mean that there is no arrival in the interval s s+ t.

Okay given that x1 = s now s s+t and 0 s are independent so probability that no arrival is there in s s +t interval given that x1 = s is system s probability that there is no arrival in the interval as s +t now there is no arrival in s s+t is same as the probability that there is no arrival in the interval 0 to t, we have already found the probability of no arrival in 0 t probability of arrival in 0t is same as probability of np arrival because it follows the same probability distribution.

So, no arrival s s+t means e to the power - lambda t it follows the same probability distribution means no arrival in s s +t will mean that probability that and t = 0 so this is the number of occurrences in the time interval 0 to t because of the independence of 0 to t s s+t number of no occurrences in the interval s to s +t will be no occurrence in the interval 0 to t so that is probability of nt = 0.

We can find so this will be lambda to the power 0 e to the power – lambda t/0 factorial so this is e to the power - lambda t so we get the probability of x2> than t given that x1= s s e to the power - lambda t .

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We conclude that $X_2 \sim Exponential(\lambda)$, and that X_1 and X_2 are independent. The random variable $X_1, X_2, \ldots$ are called interarrival times of the counting process N(t). Similarly, we can argue that all X_i are independent and $X_i \sim Exponential(\lambda)$ for i=1,2,3,....

Interarrival Times for Poisson Process

If N(t) is a poisson process with rate λ, then interarrival times $X_1, X_2, \ldots$ are independent and

$X_i \sim Exponential(\lambda)$ for i=1,2,3,....

If X is exponential with parameter $\lambda > 0$, then X is a memoryless random variable, that is

$P(X > x + a | X > a) = P(X > x)$, for $a, x \geq 0$.

And thus we can conclude that x2 is exponential lambda of following exponential lambda distribution thus X1 and X2 are we conclude that X to follow the exponential lambda distribution and that X1 and X2 are independent random variables X1 X2 are called inter arrival times of the counting process and we have seen that X1 is the time of the 1st arrival X2 is the time elapsed between the 1st first and 2nd arrival x3 the time elapsed.

Between 2nd second and 3rd arrival so they are called inter arrival times of the counting process Nt similarly we can argue that all X i are independent and Xi are follow exponential lambda distribution now inter arrival times for Poisson process if Nt is the Poisson process with the rate lambda with the rate lambda then inter arrival times X1 X2 are independent and follow exponential lambda distribution if X is exponential with parameter lambda > 0.

Then X is a memory less random variable so x is a memory less random variable means the probability that x is > X+ a given that X is > a is same as the probability that X > x so whatever has happened given that X > a only depends on the length of a to x + a that is x so probability that X is > x later we shall in the lecture we shall discuss in detail about this memory less property of the exponential distribution.

So, we will see that it depends on the length of the interval not on the values of this a and x + a we it only depends on the length of the interval.

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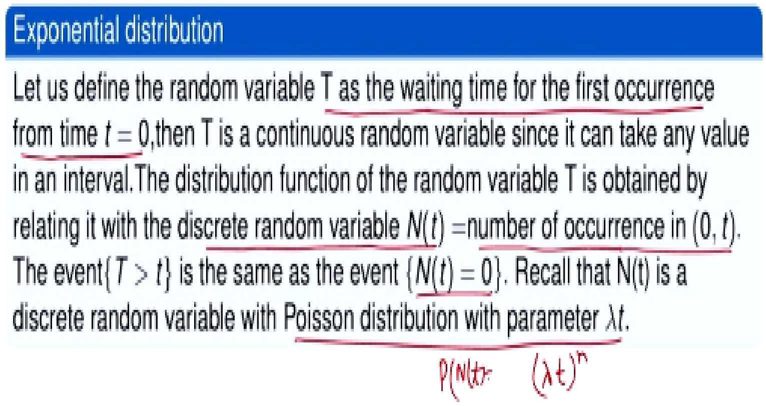

Now let us define the random variable T as the waiting time for the 1st occurrence from time t = 0 so t is the waiting time for the 1st from time t = 0 then T is a continuous random variable since it can take any value in an interval the distribution function of the random variable T is a obtained by relating it with the discrete random variable t and we know that N t is the discrete random variable of a continuous variable T and t denotes the number of occurrences.

In the interval 0 to t the event T > t is the same as event t =0 because t is the time of the first break even time for the 1st occurrence and if T > t it is same as and t = 0 there is no occurrence in the interval 0 to t now recall that Nt is the discrete variable with Poisson distribution with parameter lambda t this we have discussed in the previous lecture that Nt is a discrete n variable and follows Poisson distribution.

With parameter lambda t because the probability here in is given by lambda t rest to the power n and probability that Nt = n = lambda t to the power n e to the power – lambda t / n factorial so it follows the Poisson distribution with parameter lambda t.

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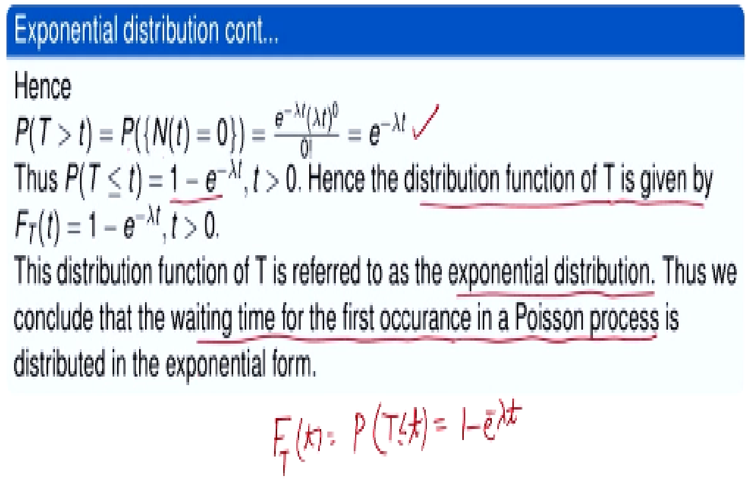

Probability that T is > t is same as we have just now discussed t is the time left between t = 0 tot for the 1st occurrence so if the time for the 1st occurrence is t and it is > t that means in the interval 0 to t there is no occurrence so probability that Nt=0 probability of Nt = 0 is e to the power – lambda t lambda t to the power 0/0 factorial so it is e to the power – lambda t thus probability of T< = t probability that T<=t is 1- probability that T>t.

That is it is 1 - r to the power – lambda t t>0 hence the distribution function of T is given by because FT t denotes the distribution function of t, so it is given by probability that T is <= t and we have seen t probability T<= t is 1 - e to the power – lambda t so the distribution function of T is given by Ft t and it = 1- e to the power – lambda t now this distribution function of T is referred to as the exponential distribution.

Thus we can conclude that the waiting time for the 1st occurrence in a Poisson process is distributed in the exponential form it is given by 1 - e to the power – lambda t.

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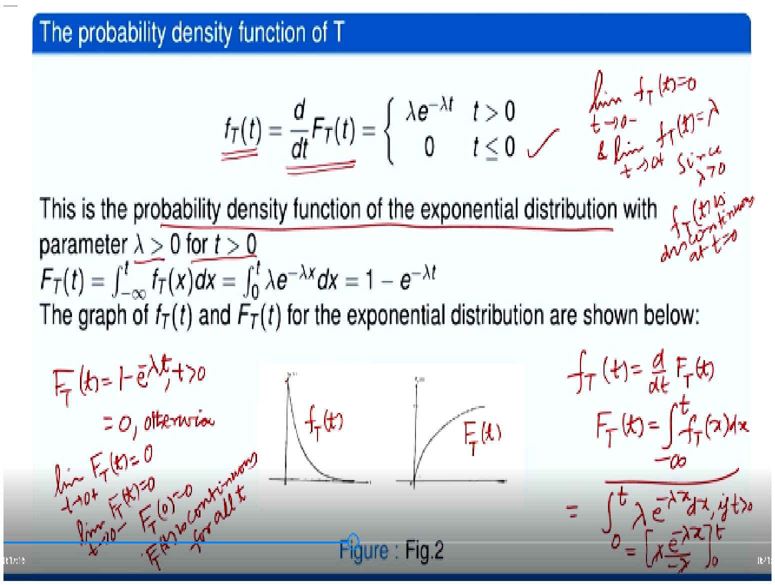

Figure : Fig.2

Now the probability density function of T we know that the derivative of the distribution function FTt the cumulative distribution function gives the probability density function so probability density function if we denote by FTt it is d/dt of FTt and d/dt of Ftt we can find from here. FTT is = this 1 1 – e to the power – lambda t when t is> 0 and 0 other wise so from here d/dt of Dt t we can determine this = lambda r to the power lambda t when t>0.

And o otherwise sp as we have just now seen probability density function probability density function is = derivative of the distribution function so a probability density function of the exponential distribution is given by FTt = lambda P to the power lambda t. When t is positive and 0 otherwise so we get this now this is the probability density function of the exponential distribution with parameter lambda > 0 T> 0 FT t if you integrate FT t = this.

We earlier also discuss this FT t when you integrate this equation FTt is given by the integral/ - infinity to t Ftx dx so the ct of the cumulative distribution function of t the random variable t is given by integral/- infinity to t FT tx dx. And here you can easily see that if you put the value of FT x is = 0 when x is <= 0 so from here we can see this is = integral/ 0 to t FT x = as lambda e to the power – lambda xdx if t is >0 if t <0 then FTx will be 0.

So, this cumulative distribution function FT t = 0 and when you integrate this what we get is e to the power - lambda x/ - lambda * this lambda and we take 0 to t so this gives you when you can cancel this lambda. With this lambda and when you substitute the limits you get one minus E to the power - lambda so FTt = 1- e to the power – lambda t when t>0 and 0 otherwise so we get back the cumulative distribution function FTt = integral/-infinity FTx dx.

Now here you can see the graph of this is the graph of density function FTt and this is the graph of we have taken FTt here we have taken cd f you can see FT t is given by lambda e to the power –lambda t. When t is >0 and 0 when t is <=0 this means that limit t tends to 0-fTt=0 and limited t tends to 0+fTt=when t goes to 0 e power-lambda t goes to 1 so this goes to lambda. SO, the value of fTt at t =0 so this 0 point is here.

So, this means that, and lambda is we have lambda to be positive. So, since lambda is >0 okay fTt is discontinuous at t=0 left and right hand limits are same. You can see the graph also okay when t tend to 0+ it goes to the value lambda this is lambda here okay and on the left side okay on the negative t axis fTt is everywhere 0 . So, it is continuous on the whole t axis except at t=0 while the function fTt okay.

FTt is continuous for all values of fTt=1- e to the power – lambda t when t is >0 we can find the limit here s t tends to 0+. So, when t goes to 0+ e to the power -lambda goes to 1 so 1-1 is this means 0 we get, and limit t tends to 0-FTt is also 0 okay and t tends to 0 also so FT0 is also 0 okay so we have FTt is continuous for all t. Okay this is clear from the graph also okay on the negative t axis FTt takes 0 value for all t.

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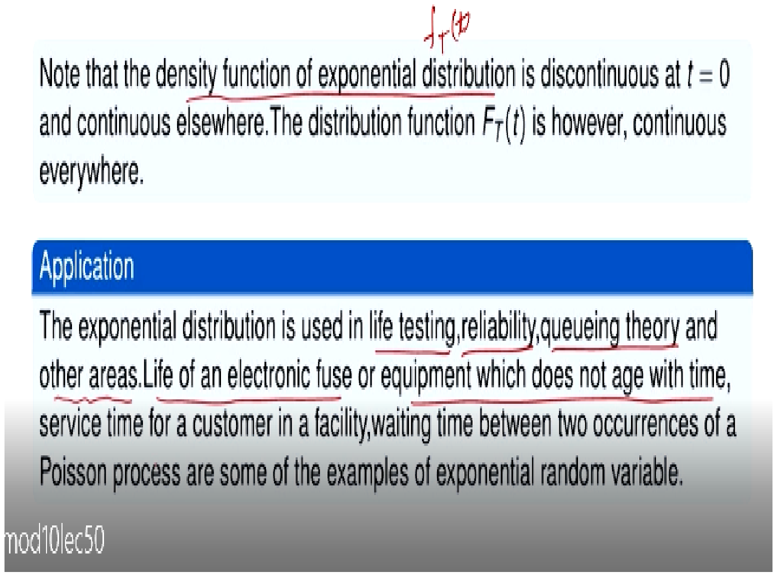

So, the density function of the exponential distribution that is FT okay FTt this is continuous everywhere except at t=0 okay. The distribution function FTt is however continuous everywhere. Now the exponential distribution is used in life testing, reliability, queueing theory and many other areas. Okay life of an electronic fuse or equipment which does not age with time, service time for a customer in a facility.

Waiting time between two occurrences of a Poisson process are some of the examples of exponential random variable.

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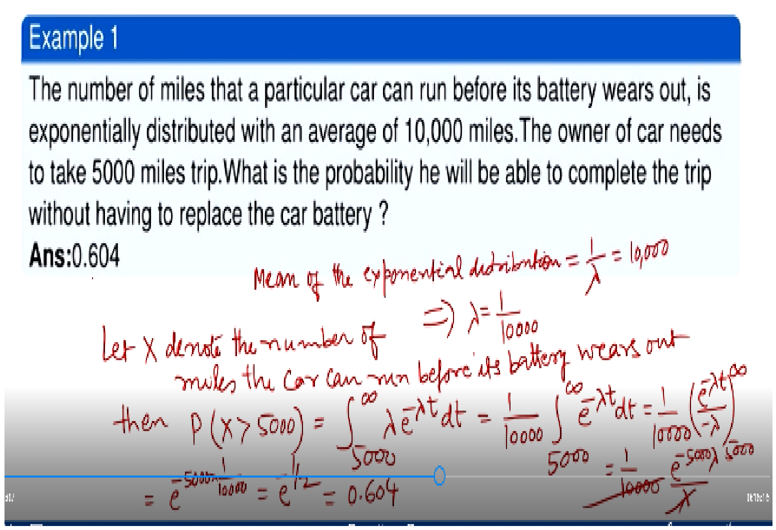

Now let us take some examples suppose we have to find we have to solve this problem, the number of miles that a particular car can run before its battery wears out is exponentially distributed with average of 10000 miles . So, the owner of car needs to take 5000 miles trip . Okay what is the probability he will be able to complete the trip without having to replace the car battery.

Mean =1/lambda mean of the exponential distribution is =1 lambda so 1/lambda=10000 miles okay which implies at lambda=1/10000 now the owner of car needs to take a 5000 miles trip what is the probability he will be able to complete the trip without having to replace the car battery. So, let us say that X denote the number of miles the car can run before the batters wears out okay.

Okay then we want the probability that X is >5000 if X is more than 5000, he will be able to complete the trip without the battery wearing out. So, this is = integral over 500 to infinity okay the FTt means lambda e to the power – lambda t dt okay that is the probability density function of the exponential distribution so lambda is = 1/10000 integral/5000 to infinity e to the power -lambda t .

So, this is =1/10000 e to the power -lambda t/-lambda and we have 5000 to infinity okay. Now when T goes to infinity lambda is >0 lambda is 1/10000 so e to the power -lambda t will go to 0 and therefore this is 1/10000 and when it is 5000 okay then, but we will get e to the power - 5000 lambda/lambda okay we will get this okay? Now lambda is 1/10000 so this lambda and this 10000 will cancel okay?

And what we get e to the power - 5000 lambda so this is = e to the power – 5000*1/10000. And this is =e to the power -1/2 okay? you get to the power - 1/2 comes out to be 0.604 so here we will be able to complete probability the trip with the probability of . 6 05 that means there are 6 more than about 60% chance that he will be able to complete the trip without the battery wearing out.

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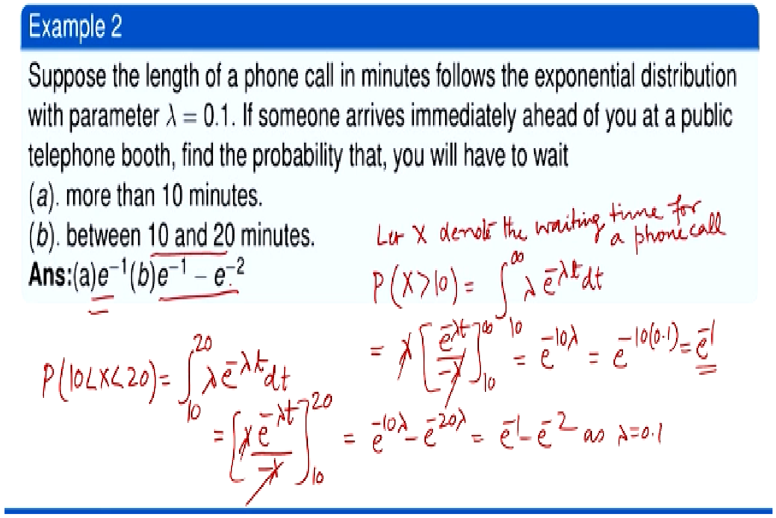

Now let us go to our next question support the length of phone call in minutes follows the exponential distribution width parameter lambda = 0.1 if someone arrives immediately ahead of you at a public telephone booth find the probability that you will have to wait more than 10 minutes so let X denote the waiting time okay for a phone call okay so then we want the probability that you have to wait more than 10 minutes that is actually X is >10.

So, this will be integral 10 to infinity okay? Lambda e to the power -lambda t dt and lambda=0.1 okay so we can write lambda here and then 10 to infinity e to the power -lambda t so integral will be e to the power -lambda t/-lambda and we have this situation this will be = this lambda will cancel with this lambda and we will have when t goes to infinity e to the power -lambda t will go to 0 and when it is 10 it will be e to the power - 10 lambda is 0.1 okay.

So, e to the power – 10*0.1 which means e to the power - 1 so we will have to wait more than 10 minutes the probability will we e to the power - 1 okay now we have to wait between 10 and 20 minutes. So, probability that 10<X<20 okay so this means we need the probability that means we have to calculate the integral 0 to 10 to 20 lambda e to the power – lambda t dt .Probability that a <x<b is given by integral/a to b FTt dt that is the density function.

Okay so we have here – e to the power – lambda t/-lambda*lambda and we have 10 to 20. Okay so this lambda will cancel with this we will get e to the power 10 lambda – e to the power -10 lambda -20 lambda okay when we shall put the limits okay. So, this is e to the power -1- e to the power -2 as lambda=0.1 okay so we get this probability. This is the probability that we have to wait between 10 and 20 minutes to make the call.

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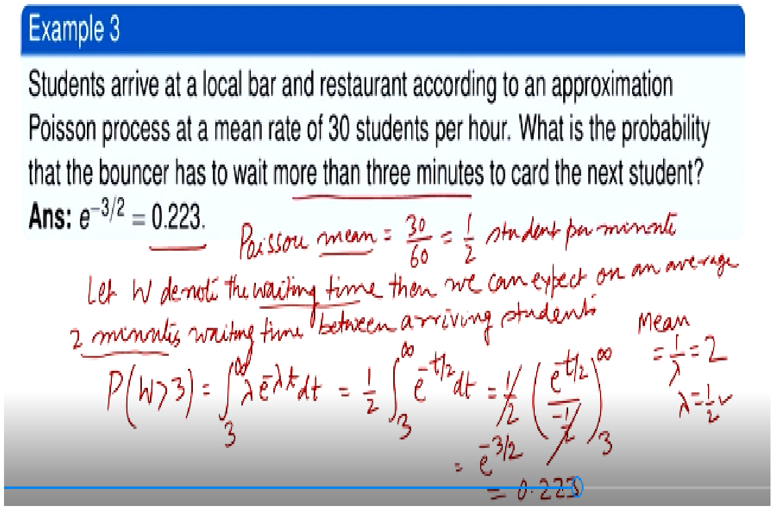

Now suppose we have this problem students arrive at a local bar and restaurant according to an approximation Poisson process at the mean rate of 30 students per hour. What is the probability that the bouncer has to wait more than 3 minutes to card the next student? So students arrive at a local bar and turned according to an approximation Poisson process at a mean rate of 30 students per hour meaning that the Poisson mean is 30/60.

Okay 30 students per hour means 1/2 student per minute okay now let us say let W denote the waiting time okay then we can expect on an average 2 minutes waiting time between arriving students . Because we have got 1/2 strength means in 2 minutes 1student so there will be a we can expect that that will be a time interval of 2 minutes between arriving students. So, we want the probability that the bouncer has to wait more than 3 minutes.

Okay on average the time is 2 minutes between arriving students. But we want the probability that the time I mean waiting time is more than 3 minutes. So, we want the probability that the waiting time is W is more than 3 okay that means integral 3 to infinity lambda e to the power -

lambda t dt. Okay now as i said mean of the exponential distribution is been by lambda okay now on an average what we are getting is the betting time is 2 minutes.

Okay so 1/lambda=2 minutes that means lambda=1/2 okay this lambda is same as the mean of the Poisson process the same lambda lambda =1/2 so ½ integral 3 to infinity okay e to the power -t/2 dt because lambda=1/2 okay so we have 1 /2 e to the power t/2 /-1/2 this will cancel with this and we get what when t goes to infinity it will go to 0 okay, so we get e to the power – 3/2 e o the power -3/2 is 0.223.

So, the bouncer will have to wait more than 3 minutes the probability for that is 0.223 okay now the expectation and variance of exponential distribution.

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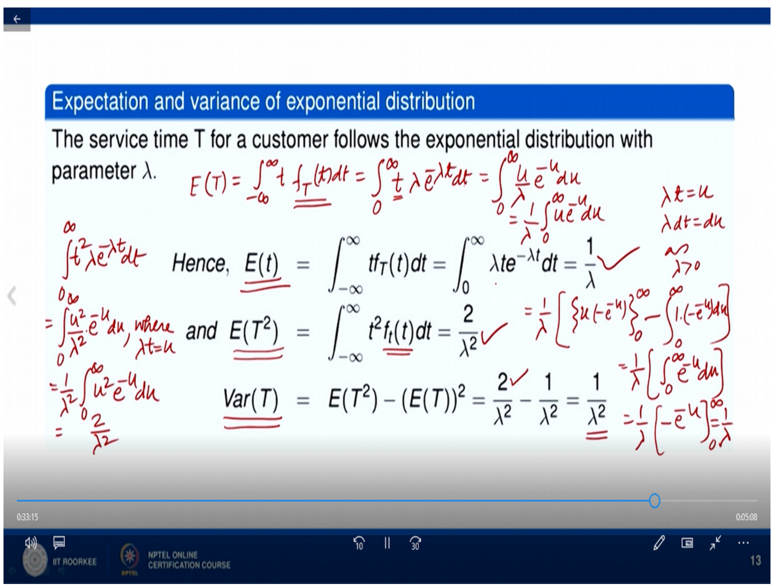

The service time T for a customer follows the exponential distribution with parameter lambda. So, here Et is the expected time of t expected time of t is=- infinity to infinity as we have written as t times Ft T dt okay this integral/0 to infinity because fTt the density function fTt it assumes value 0 when t is <0. So, this is 0 to infinity okay t times lambda e to the power – lambda t dt. Okay now take lambda t=u then lambda dt=du okay.

So, what do we get there and the limits because lambda is >0 as lambda is >0 okay when t goes to infinity u goes to infinity and when t is 0 u is 0 because the limits do not change okay? For t we put u/lambda okay and lambda dt we replace by du so we have e to the power - udu okay now this is 1/lambda, so we have 0 to infinity u e to the power - udu okay? Now we can integrate by parts, so we get 1/lambda u times - e to the power – u 0 to infinity.

And - derivative of u is 1 then - e to the power – udu okay this will be now u goes to infinity okay u times e to the power -u goes to 0 okay s we have this 0 and then when we put 0 here for u, we get this 0 okay so what do we get ultimately 1/lambda and - - becomes + so 0 to infinity e to the power - udu and this is 1/lambda we have e to the power -u- sign okay? 0 to infinity so when u goes to infinity t is 0 when u=0 it is .

So. with – and – we get because of power limits we will get it as 1/lambda and we get 1/lambda as the mean of the exponential distribution okay in order to determine variants of T we need to determine ET square ET square is integral/- infinity to infinity T square FTt dt and then this we can similarly determine we have determined ET okay so this will be FTt is again 0 when T >0 so we can out it as integral 0 to infinity t square FTt will be lambda e to the power -lambda t when T is positive.

Okay we can put again lambda t=u this will be = putting lambda t =u this will be =integral 0 to infinity t square will be u square/ lambda square kay *lambda dt will be du so e to the power -udu where lambda t=u okay this will be 1/lambda square integral 0 to infinity u square-u . so, because of u square here you have to integrate it twice with respect to by integrating by parts and you will see that its value comes out to be 2 okay 2/lambda square.

We get okay we get 2/lambda square variants of t then ET square – ET the whole square okay that is 2 /lambda square ET square means 1/lambda square, so we get the variants as 1/lambda square okay . So, variants is 1/lambda square and mean is 1/lambda okay so mean is 1/lambda we will use in the examples.

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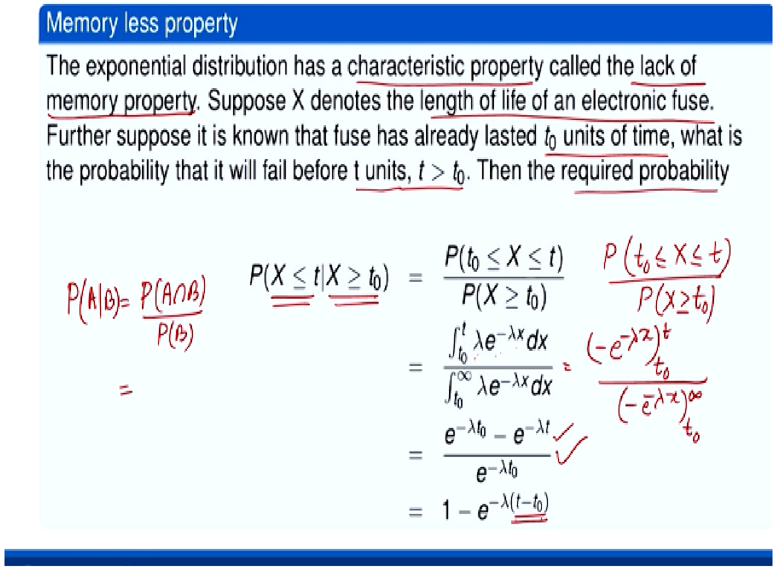

Now let us discuss memory less property the memory the exponential distribution has a characteristic property okay there is a characteristic property of the exponential distribution it is called lack of memory property okay? Suppose X denotes the length of life of an electronic fuse okay? Length of life often electronic fuse further suppose that it is known that fuse has already lasted t0 units of time.

What is the probability that it will fail before t units t>t0 then the required probability is PX<=t/X>t0 X denotes the length of life often electronic fuse we want the probability that a it fails before t units t units of time so X must be < or = t but given that it has already lasted t0 units of time. So, X is > or = t0 now this is conditional probability that X is < or = t given that X is >or = t.

We know that probability of A given B= probability of A intersection B/ probability of B okay so using that we will have this is event A this is event b okay intersection of A and B will be t0>=X <=t0<=X <=t so we will have the probability that t0<=X<=t/PX>=t0. Okay now so this in the numerator what we will have integral t0 to t and then the probability density function of the exponential distribution which is lambda e to the power -lambda x.

Okay lambda e to the power -lambda x dx and the denominator will be we have to find the probability that X is >=t0 so t0 is >infinity lambda e to the power lambda X dx when we integrate this what we get e to the power - e to the power - lambda x we get okay and the limits are t0 to t/ here we have again -e to the power -lambda x the limits are t0 to infinity . Okay, So numerator will be e to the power - lambda t0 -e to the power -lambda t and denominator.

Because X lambda is positive X when goes to infinity it will go to 0 e to the power- lambda x will go to 0. So, it will be e to the power -lambda t0 so we get this okay and this we can write as 1- e to the power -lambda times t-t0. Okay now you can see the probability that the length of life of the electronic fuse will be at most t units okay given that it has already lasted t0 units of time is depends on the length of time t – t0. Okay it depends on the length t – t0 okay.

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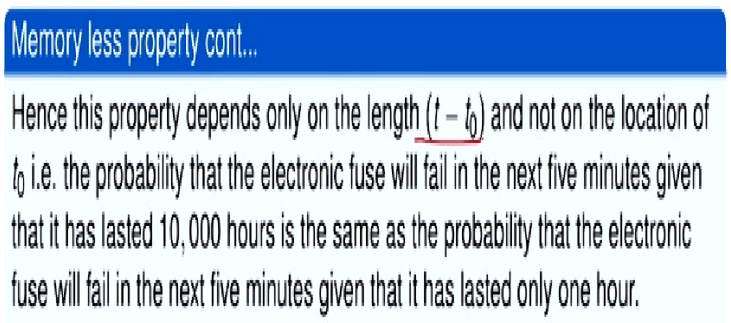

So, what we have this property depends only on this length t – t0 and not on the location of t0. Hence the probability that the electronic fuse will fail in the next five minutes given that it has already lasted 10000 is the same as the probability that the electronic fuse will fail in the next five minutes given that it has the lasted only 1 hour. Okay so suppose the electronic fuse has already lasted 1 hour.

And you have to find the probability that it will fail in the next five minutes that probability is same as the probability that it has already lasted 10000 hours, and it will fail in the next five minutes. Okay so this is called the memory less property the probability does not depend on what has happened before t0 units of time with only depends on the length t-t0 okay this is called memory less property of the exponential distribution. So, that is all in this lecture i thank you very much for your attention.