Advanced Engineering Mathematics
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Lecture – 25
Evaluation of Real Integrals Using Residues - V

Hello friends welcome to my last lecture on evaluation of real integrals using residues. So in this lecture again we are going to consider 2 real integrals which need different technique for evaluation and at different contour also in one case.

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Example 1

Show that

$$\int_0^\infty \frac{\cosh ax}{\cosh x}dx = \frac{\pi}{2\cos\frac{\pi a}{2}}, \text{ where } |a| < 1.$$

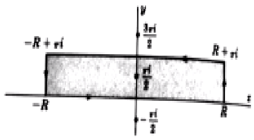

Figure : Fig.1

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Now let us consider for example the real integral 0 to infinity cosh hyperbolic ax/cosh hyperbolic x dx. We are going to show that it is value is pi/2 cosh pi a/2 where mod of a is <1. So in this example we will use this contour, where we take the rectangle with vertices at –r +r, r + i pi and –r + i pi. So we are moving along this rectangle in the counter clockwise direction and we will see that the corresponding contour integral which we consider here that has a simple pole at z = i pi/2 which lies in the upper half plane okay.

So that is inside the rectangle and therefore we can make use of the residue theorem to evaluate the value of the integral around this close contour. So let us consider this close contour. So consider the contour integral, so we are going to consider the contour integral, integral/C for cosh hyperbolic ax we write e to the power az and for the denominator cosh hyperbolic x we write cosh hyperbolic z.

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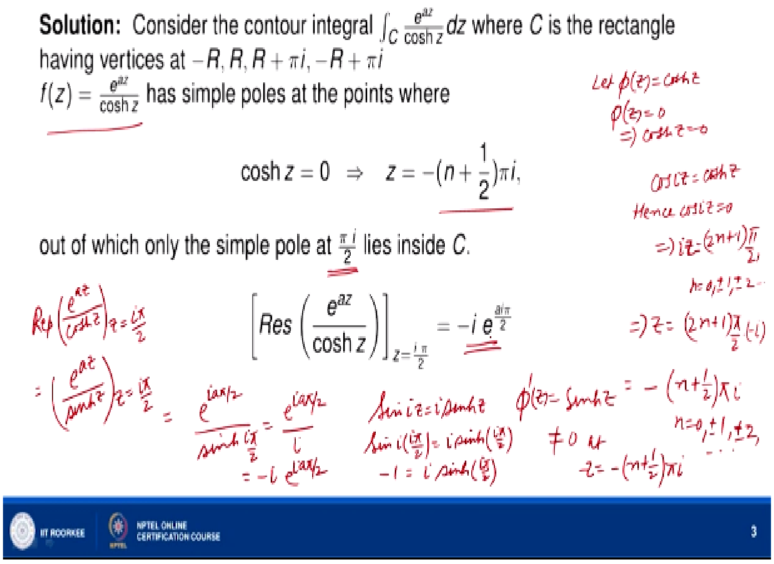

So integral over C e to the power az over cosh hyperbolic z, dz, where C is the rectangle having vertices at –r, r, r+ i pi and –r + i pi. Now fz = e to the power az over cosh hyperbolic z has poles where the denominator, cosh hyperbolic z = 0. So let us say let 5z = cosh hyperbolic z, then we notice that cosh 5z = 0 means cosh hyperbolic z = 0 and let us recall that cosh iz = cosh hyperbolic z.

So then cosh hyperbolic z = 0 implies cosh iz = 0, which means that iz = 2n + 1 * pi/2 where n = 0 +/- 1 +/-2 and so on and so z = 2n + 1 * pi/2 * 1/i, 1/i means –i okay. So we can write it as –n +1/2 * pi okay, this is = -n + 1/2 * pi i okay and n takes values 0 + -1 + -2 and so on, okay. Now out of these poles okay, now let us see one more thing that phi prime z = 5 z is cosh z, hyperbolic z.

So phi prime z is sin hyperbolic z and sin hyperbolic z is not 0, wherever cosh hyperbolic z is 0 okay. So sin hyperbolic z is not 0 at z = -n + 1/2 * pi i okay. So at z = - n + 1/2 * pi i, this phi z, this fz has simple poles okay. So fz = e to the power az/cosh hyperbolic z has simple poles at these points okay. Now out of the points z = -n + 1/2 pi i where n = 0 +/-1 +/-2 and so on okay.

Only the pole pi i/2 okay, only the pole pi i/2 lies inside the rectangle bounded by – r r r + i pi – r + pi i. So we will need to consider the residue only at z = i pi/2. So let us find the residue of e to the power az/cosh hyperbolic z at z = i pi/2. Now here it is easier if we differentiate the denominator of e to the power az/cosh hyperbolic z and put z = i pi/2. So this residue of e to the power az/cosh hyperbolic z at z = i pi/2 is same as is = e to the power az/ the derivate of cosh hyperbolic z that is sin hyperbolic z and we evaluate it at z = i pi/2.

So this is e to the power a i pi/2 okay and sin hyperbolic i pi/2, okay. Now let us recall that sin iz = i sin hyperbolic z. So when you put z as i pi/2 what we get? Sin i * i pi/2 = i sin hyperbolic i pi/2. This sin is i square pi/2, i square is -1, so sin – pi/2 which is -1 = i times sin hyperbolic i pi/2 okay and -1/i is i okay. So sin hyperbolic i pi/2 is i so this is e to the power i a pi/2 /i okay.

Sin hyperbolic i pi/2 = -1/i, -1/i is i, 1/i is –i so we get –i e to the power i a pi/2 okay. So we get this –i e to the power ai pi/2 as the residue of e to the power az/cosh hyperbolic z, at z = i pi/2.

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Now,

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_0^{\pi} \frac{e^{a(R+iy)}}{\cosh(R+iy)} idy + \int_R^{-R} \frac{e^{a(x+i\pi)}}{\cosh(x+i\pi)} dx$$

$$+ \int_{\pi}^{0} \frac{e^{(-R+iy)}}{\cosh(-R+iy)} idy = 2\pi\, e^{\frac{a\pi i}{2}} \text{ by residue theorem.}$$

Let us show that the second integral tends to zero as $R \to \infty$.

$$\left|\cosh(R+iy)\right| = \left|\frac{e^{R+iy}+e^{-R-iy}}{2}\right| \geq \frac{1}{2}\left\{|e^{R+iy}| - |e^{-R-iy}|\right\}$$

$$= \frac{1}{2}(e^{R} - e^{-R}) \geq \frac{1}{4}e^{R} \to (\text{it is true if } e^{2R} \geq 2)$$

Now we write the integral around the contour C. So integral around the contour C is integral from –r to r along the real axis that means we replace z/x. So integral along –r to r where z is replaced by x and we get this integral/-r to r fx dx, now going from r to r + pi i. So that means that when we go from r to r + i pi x is constant. So z will be = r+ iy, any point z you take on this segment r to r + i pi, z will be r + iy where y varies from 0 to pi okay.

So we will put that here, e to the power az upon cosh hyperbolic z that is e to the power a times r + iy/cosh hyperbolic r + iy and when z is r + iy we have dz = idy. So we put idy here and y varies from 0 to pi. Now when we move along this segment from r + i pi 2 – r + i pi then you can see that now we will have z = because now y is constant, x varies okay. So x + i pi okay, x + i pi y is = pi along this line.

So x + i pi and x varies from r2 -2 and dz will be = dx okay. So let us put this here in the third case, so x varies from r to –r, e to the power a times x + i pi/cosh hyperbolic x + i pi dx and then we move along the segment – r + pi i 2 – r. So when we move along – r + i pi 2 – r x is now – r. So z = - r + iy okay for this, z = - x + r + iy dz = -dy and y varies from pi to 0. So here you can see integral/pi to 0 e to the power –r + iy cosh hyperbolic –r +iy dz becomes idy and then righthand side is by residue theorem 2 pi i * okay.

So we multiply 2 pi i to the residue and 2 pi i * -i is – 2 pi i square that is 2 pi. So we get 2 pi * e to the power ai pi/2 okay. Now let us show that the second integral on the right side this one okay. This integral tends to 0 as r goes to infinity. So mod of first let us evaluate, let us take lower bound rather. Lower bound for cosh hyperbolic r + iy. So mod of cosh hyperbolic r + iy is mod of e to the power r + iy + e to the power –r –iy /2 and mod of z1 to z2 is >= mod z1 – mod z2.

So we have mod of this + this >= 1/2 e to the power mod of r + iy – mod of e to the power –r –iy. Now mod of e to the power iy is 1. So we have this as e to the power r okay and this is e to the power –r. Now 1/2 e to the power r – e to the power –r is >= 1/4 times e to the power r if e to the power 2r is >= this we can easily see. Let us say let e to the power r be =x okay.

Then what we want? 1/2 x – 1/x, this is e to the power –r is 1/x, we wanted to be >= 1/4 x okay. So what we want this is x square -1/x >= 1/2 x. This means we want 2 x square – 2 to be >= x square and this means x square should be >= 2, so e to the power 2 r must be >= 2, or e to the power r must be >= root 2 okay. It is not always true okay, but here we are taking r to go to infinity okay.

Here we want r to go to infinity so this inequality is valid okay. So 1/4 e to the power r is the lower bound for cosh hyperbolic r+iy and then we have.

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We have

$$\left|\int_0^\pi \frac{e^{a(R+iy)}}{\cosh(R+iy)} i\,dy\right| \le \int_0^\pi \frac{e^{aR}}{\frac{1}{4}e^R}\, d\theta$$

$$|e^{a(R+iy)}| = e^{aR}$$

$$= 4\pi\, e^{(a-1)R} \to 0, \text{ as } R \to \infty \text{ since } |a| < 1.$$

Similarly, let us show that the fourth integral

$$\int_\pi^0 \frac{e^{a(-R+iy)}}{\cosh(-R+iy)} i\,dy \to 0, \text{ as } R \to \infty.$$

$$|\cosh(-R+iy)| = \left|\frac{e^{-R+iy} + e^{-(-R+iy)}}{2}\right| = \left|\frac{e^{-R+iy} + e^{R-iy}}{2}\right| \ge \frac{|e^{R-iy}| - |e^{-R+iy}|}{2} = \frac{e^R - e^{-R}}{2} \ge \frac{1}{4}e^R \text{ if } e^{2R} \ge 2$$

$$\left|\int_\pi^0 \frac{e^{a(-R+iy)}}{\cosh(-R+iy)}\, dy\right| \le \int_0^\pi \frac{e^{-aR}}{\frac{1}{4}e^R}\, dy = 4\pi\, e^{(-a-1)R} \to 0, \text{ as } R \to \infty \text{ because } |a|<1$$

So we can now evaluate this integral, mod of integral 0 to pi e to the power times r + iy/cosh hyperbolic r + iy idy is now <= now mod of e to the power a r + iy as we have seen earlier also, this is e to the power ar because mod of e to the power aiy = 1. So e to the power ar for this and this is >= 1/4 e to the power r. So 1 up on cosh hyperbolic r + iy is <= 1 upon 1/4 e to the power r.

So this is now = 4 pi e to the power a-1 * r, but you see mod of a is < 1. So e to the power a-1 * r okay. This is always e to the power a-1, a-1*r is always negative and therefore this goes to 0 as r goes to infinity. Similarly, we can show that this fourth integral goes to 0 as r goes to infinity, but we do, let us consider first lower bound for cosh hyperbolic – r + iy. So mod of cosh hyperbolic – r + iy let us find.

So this is = mod of e to the power – r + iy + e to the power – of – r + iy/2 and this is = mod of e to the power – r + iy + e to the power r – iy, okay this is >= mod of this, let us first take this, mod of this – mod of that. So mod of e to the power r – iy – mod of e to the power – r + iy okay. Mod of e to the power, this is e to the power r * e to the power –iy mod of that is 1. So this is = e to the power r – e to the power – r/2.

Now as we have seen here e to the power r – e to the power – r/2 is >= 1/4 e to the power r. So here also it is >= 1/4 e to the power r provided if e to the power 2r is >= 2 okay. So this is true because r is going to infinity. Now mod of, okay so mod of pi to 0, e to the power a times –r + iy * dy/cosh hyperbolic – r + iy, this is <= integral pi to 0 become 0 to pi, because mod is there.

So we have e to the power – ar/1/4 e to the power r dy okay. So dy we have, so this is = 4 times pi * e to the power – a -1 * r okay. Again mod of a <1 okay. So this goes to 0 as r goes to infinity because mod of a is < 1 okay. So this is how we show that this integral okay and this integral, they go to 0 as r goes to infinity. Now as r goes to infinity what we have integral/-infinity to infinity fx dx + integral/infinity to – infinity. This dx = the residue, that is we found that to be this one.

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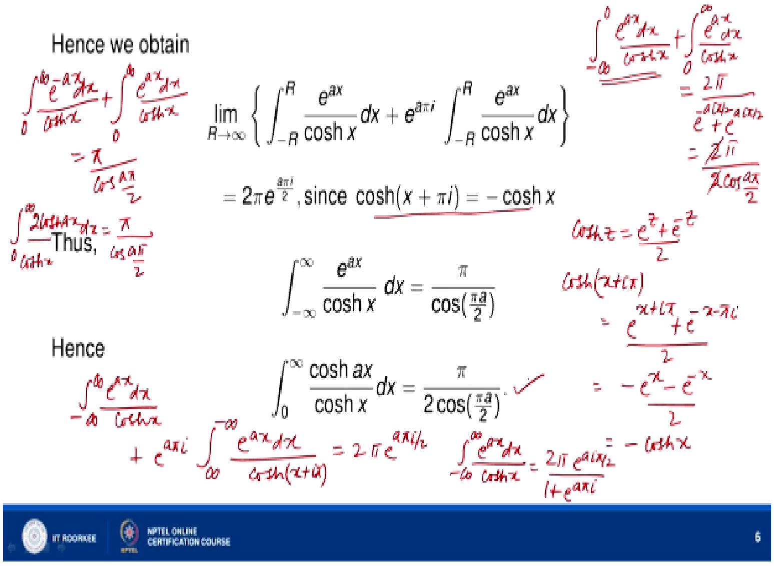

So integral limit r tends to infinity – r to r, e to the power ax cosh hyperbolic dx + integral over okay let us see how we get this. Okay here you can see when r goes to infinity this equation gives you integral over C, fz dz we found 2 pi i times this and 2 pi i – 2 pi i square means 2 pi times e to the power ai pi/2 okay. So 2 pi e to the power ai pi/2 = - infinity to infinity fx dx + integral over infinity to – infinity okay, e to the power ax * e to the power ai pi, dx/cosh hyperbolic x + i pi.

So this gives you this equation, limit R tends to infinity this equation we get. Here e to the power ai pi integral – R to R, e to the power ax cosh hyperbolic dx, now can be changed to integral over – infinity to + infinity because cosh hyperbolic x + i pi = - cosh hyperbolic x okay. So this is here limit R tends to infinity integral/ - R to R e to the power ax dx/cosh hyperbolic x + e to the power ai pi integral/- R to R, e to the power ax/cosh hyperbolic x + i pi and cosh hyperbolic x + i pi = - cosh hyperbolic x, this we can see.

Cosh hyperbolic z = e to the power z + e to the power – z/2 okay, so cosh hyperbolic ix + i pi = e to the power x + i pi + e to the power – x – i pi/2 and e to the power i pi is -1. So e to the power x, e to the power – i pi is also -1. So e to the power – x/2, this is – cosh hyperbolic x, okay. So this is – cosh hyperbolic x, so this will become what, integral/- infinity to infinity e to the power ax, dx/cosh hyperbolic x + integral/- infinity to infinity okay.

Integral/infinity to – infinity, it will be like this, this will be or this is integral/- infinity to infinity, e to the power ax dx upon cosh hyperbolic x + e to the power ai pi integral/infinity to –infinity. This will be integral/infinity to – infinity e to the power ax dx upon cosh hyperbolic x + i pi okay this is the = 2 pi e to the power a pi i/2 okay. Now because cosh hyperbolic x + i pi = - cosh hyperbolic x.

We can write it as integral/-infinity to infinity e to the power ax, dx/cosh hyperbolic x = 2 pi e to the power ai pi/2/1 + e raise to the power e pi i okay, thi sis what we get. So now we will go, this will break into 2 parts, integral/-infinity to 0, e to the power ax dx/cosh hyperbolic x + integral/0 to infinity e to the power ax/cosh hyperbolic x dx okay = 2 pi, e to the power ai pi/2 we can divide in the numerator and denominator we get e to the power –ai pi/2 + e to the power ai pi/2 okay.

So this is 2 pi/e to the power i theta + e to the power – i theta is 2 cosh theta, 2 cosh a pi/2, okay, so this we can cancel. We get pi/cosh a pi/2 okay. Now here what we do on the left side, replace x by –x in this integral okay, then what we will get, e to the power –ax here, this cosh hyperbolic –x is cosh hyperbolic x and dx will become –dx, so the limits of integration will change to 0 to infinity.

So we will get 0 to infinity e to the power – ax dx upon cosh hyperbolic x when we replace x by –x in the first integral + integral 0 to infinity e to the power ax dx/cosh hyperbolic x = pi/cos a pi/2, okay so this is left hand side is what, we get left hand side as 0 to infinity e to the power ax + e to the power – ax/cosh hyperbolic x, e to the power ax + e to the power –ax is twice cosh hyperbolic ax/cosh hyperbolic x dx = pi/cosh a pi/2 okay.

So we get the value integral 0 to infinity cosh hyperbolic ax/cosh hyperbolic x, dx = pi/2 times okay this 2 we can bring here. So pi/2 times cosh pi/2, so this is how we get the value of the integral using residue calculus.

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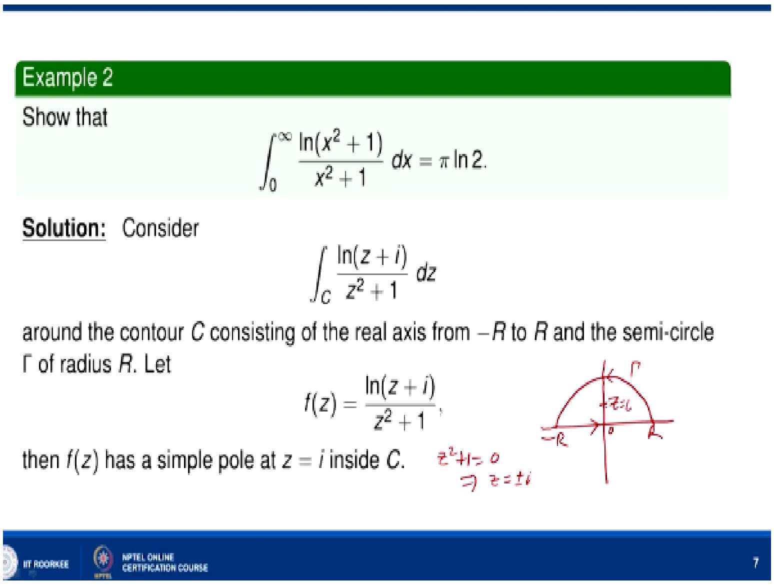

Okay so let us now consider another real integral which is integral/0 to infinity ln x square + 1/x square + 1 dx and we will show that it is value is pi ln 2 okay. Now let us consider the corresponding contour integral here we shall take as integral over C, ln z +i, for ln x square + 1 we shall be writing ln z+i/denominator when we have x square + 1 we write again x z square + 1, there is change in the numerator.

In the numerator we simply do not replace x,y,z, we rather consider ln z + i. So integral/C ln z+i/z square + 1 dz around the contour C consisting of the real axis from –R to R. So that contour are same as we have been considering earlier. So – R to R and this is 0 origin as semicircle gamma we are moving anti clockwise and then the function fz which is l and z + i/z square + 1 it has a simple pole okay.

It has actually poles, simple poles at z = +/- i, this gives you z = +/- i, but out of these 2 simple poles only 1 pole at z = i lies inside C okay. So we will consider the residue of fz at z = i only. So let us find the residue.

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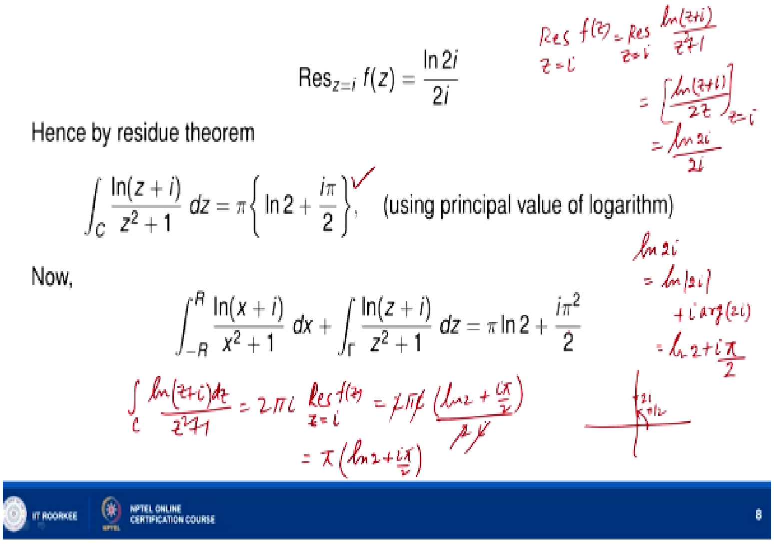

Residue of fz at z = i okay, so residue of fz at z = i let us find. So fz is ln z + i/z square + 1. So we can differentiate the denominator, the denominator derivative is 2 z ln z + i and then you put z = i. So what we get ln 2i/2i okay, because for z we are writing i, so ln 2i/2i, now ln 2i = ln mod of 2i + i times argument of 2i okay and you see 2i is here, okay, 2i is the point 0, 2 on y axis okay.

So argument of 2i is pi/2 okay, this is pi/2, so we have ln mod of 2i, this is mod of 2i is 2, so ln 2 + i times pi/2 okay. So we have here, now what we have integral/C, ln z + i/z square + 1 dz is 2 pi i times this is = integral/C ln z + i dz/z square + 1 = 2 pi i times the residue of fz and z = i, so this is 2 pi i times ln 2 + i pi/2, okay, ln 2 + i pi/2 / 2i okay, so divided by 2i. So we get this.

So we get pi times ln 2 + i pi/2, okay, so we get this, we are using here principle value of the algorithm. Now we can write the integral/C okay, integral/C fz dz = integral/-R to R fx dx because we are moving long the l axis + integral/gamma fz, dz. Let us take this R to be so large that the pole at z = i lies inside the contour okay. So by residue theorem we will have –R to R integral over –R to R ln x + i.

We are moving along x axis, so ln x + i/x square + 1 dx + integral/gamma ln z + i/z square + i dz = pi ln 2 + i pi square/2.

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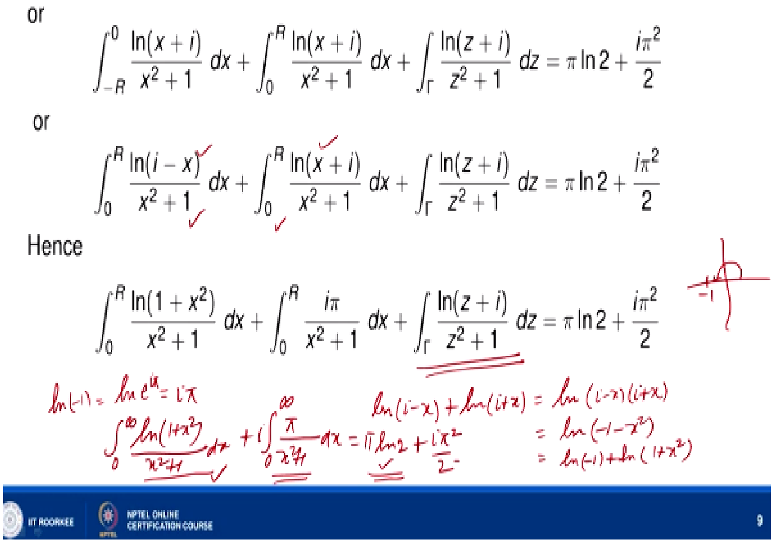

Okay now this integral when r goes to infinity. When r goes to infinity we shall show that integral along gamma goes to 0 and this integral becomes integral/-infinity to infinity ln x + i/x square + 1 dx, we will break into 2 parts. So integral/- infinity to 0 and then integral/0 to infinity. So integral/-r to 0 ln x + i/x square + 1 dx + integral/0 to r ln x + i/x square + 1 dx + integral/gamma ln z + i/z square + 1 dz = pi ln 2 + i pi square/2.

And this we can write as integral/, now here replacing x by –x, when you replace x by –x, what you get? ln i –x/x square + 1 dx becomes – dx, so the limits change to 0 to R. So integral 0 to R ln i – x/x square + 1 + integral/0 to R, ln x + i/x square + 1 dx + integral/gamma ln z + i/z square + 1 dz is pi ln 2 + i pi square/2 and then what we do, we can combine this integral and this integral okay. So noting that ln i – x + ln i + x = ln i-x * i + x.

So what we get ln – 1 – x square which is = ln – 1 + ln 1 + x square, ln – 1 -1 lies on the real axis here okay. So the argument is pi okay. So ln -1 = ln e to the power i pi it is magnitude is 1 okay. Magnitude of -1 is 1, so this is i pi, so what we get this is = i pi + ln 1 + x square. So this + this okay is ln 1 + x square + i pi, so the right integral 0 to r, ln 1 + x square/ x square + 1 + integral/0 to R, i pi/x square + 1 dx and integral/gamma ln z + i/z square + 1 dz = pi ln 2 + i pi square/2 okay.

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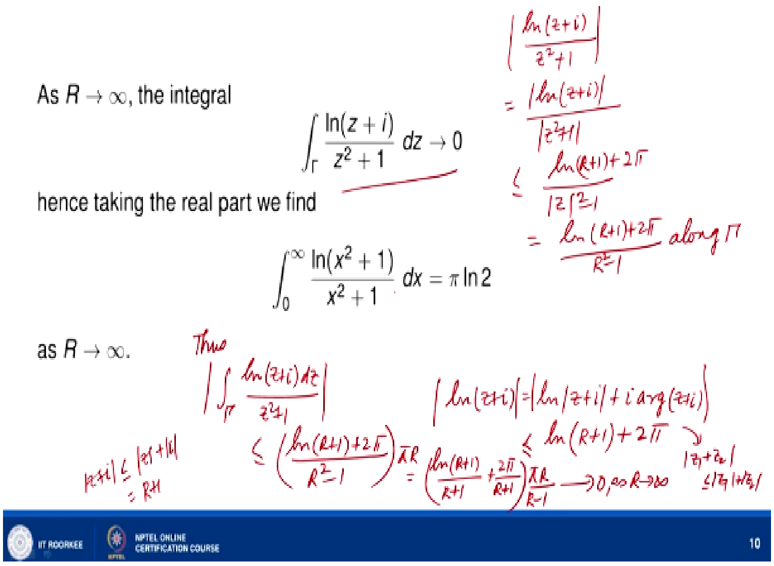

Now as R goes to infinity, as R goes to infinity these integral goes to 0, so what do we get integral over 0 to infinity ln x square + 1/x square + 1 what do we get here. This becomes, this integral become 0 to infinity ln 1 + x square/x square + 1 dx and this becomes integral/0 to infinity i we can write outside pi/x2 + 1 dx and this is 0 = pi ln 2 + i pi square/2, okay, so equating real parts both sides.

This is left side, we have this real part, so this real part = this real part okay and this imaginary part = this imaginary part. So you can see also the integral 0 to infinity pi/x square + 1 dx is i * pi square/2 because this is pi times integral 0 to infinity 1/x square + 1 dx which is pi/2 okay. So we get i pi square/2, so this is = this and this is = this and what we get here integral 0 to infinity ln x square + 1/ x square + 1 dx = pi ln 2 as R goes to infinity.

Now let us show that integral over gamma ln z + i over z square + 1 dz goes to 0 as r goes to infinity. So let us find the modulus of ln z + i/z square + 1, modulus of this. so this is mod of ln z + i/mod of z square + 1, this is <=, this is mod of z square – 1 and ln z + i = ln mod of z + i okay. Mod of z + i + i times argument of z + i. Okay mod of z + i is <= mod of z + mod of i okay.

So this is R + 1 okay, so this is <= ln R + 1 okay and here what we will get i times 2 pi because argument of z + i cannot be more than 2 pi if we are considering principle value of the logarithm. So we get here ln R + 1 + 2 pi. Magnitude of this, let us put magnitude of this and then here we can write, let us take mod of ln z + 1. So mod of ln z + i is mod of ln, mod of z + i + i times argument of z + i.

Ln mod of z + i is <= ln R + 1 and mod of this is <= by triangle inequality mod of ln z + i + argument of mod of argument of z + i, argument of z + i cannot exceed 2 pi okay. So this actually we are using here triangle inequality. You can say we are using here this triangle inequality. Here we are using mod of z1 + z2 <= mod of z1 + mod of z2, okay, so this is ln R + 1 + 2 pi/R square -1 okay along gamma.

Now what we have, so mod of thus mod of ln integral/gamma, ln z + i dz/z square + 1, mod of this is <= ln R + 1 + 2 pi/R square – 1 * length of gamma that is pi R okay. Now we can write it as ln R + 1/R + 1 + 2 pi/R + 1 * pi R/R-1 okay. When R goes to infinity pi R/R-1 goes to pi okay. So this goes to pi, now inside the bracket 2 pi/R + 1 goes to 0 and ln R+1/R+1 also goes to 0 because ln x/x goes to 0 when x goes to infinity.

So this goes to 0 as R goes to infinity. So this is how we show that this integral goes to 0 as R goes to infinity and thus the integral/0 to infinity ln x square + 1/x square + 1 dx = pi ln 2. So with this we come to the end of this lecture and we have finished the discussion on evaluation of real integrals using residue theorem. Thank you very much for your attention.