Advanced Engineering Mathematics
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Lecture - 13
Uniform Convergence of Series

Hello friends, welcome to my lecture on Uniform Convergence of series of complex functions. So let us consider a series of complex functions sigma n=1 to infinity fn(z) we shall say that the series converges uniformly to a function fz in a region R which may be open or closed of the z-plane.

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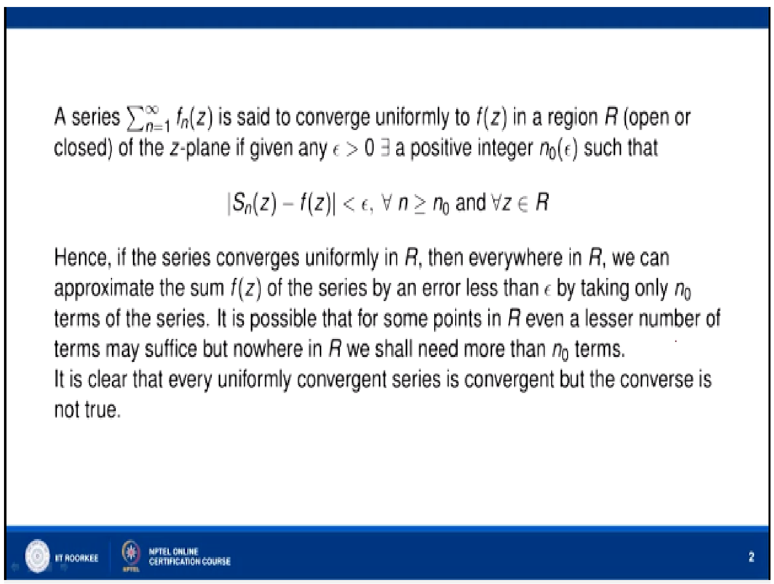

If for a given epsilon > 0 we can find an integer n0 depending only on epsilon such that mod of fn(z) mode of –fz is < epsilon for all n >= n0 and for every z belonging to R. Hence, if the series converges uniformly in R then everywhere in R we can approximate the sum f(z) of the series by an error < epsilon by taking only n0 terms of the series. It is possible that for some points in R even a lesser number of terms may suffice but nowhere in R we shall need more than n0 terms.

It is clear that every uniformly convergent series is convergent but the converse is not true as we can see later.

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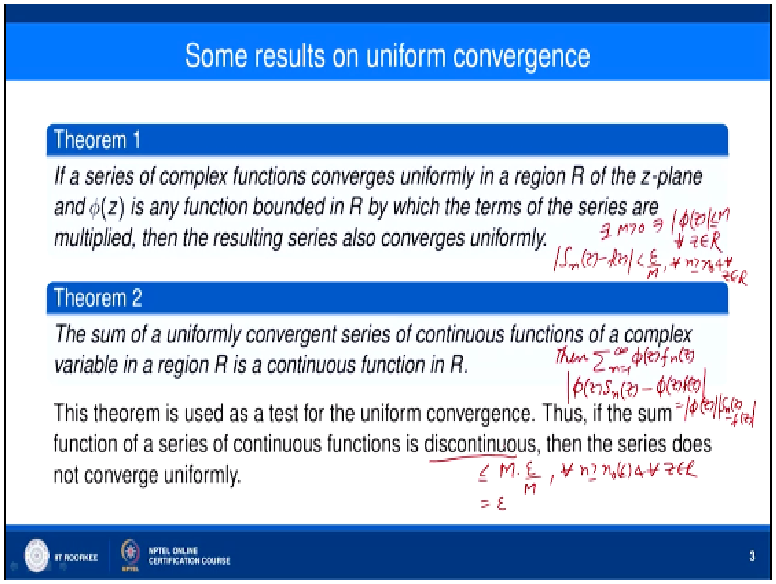

If a series of complex functions converges uniformly in a region R of the z-plane and phi z is any function which is bounded in R by which the terms of the series are multiplied, then the resulting series also converges uniformly. This is very simple. Suppose phi z is bounded function in R then their exist in constant time > 0 such that mode of phi z is <= m for all z belonging to R, okay. Now the series converges uniformly in R so by the definition of uniform convergence mode fn(z)-fz is < epsilon so mod fn(z)–fz is <, let us say epsilon by m, okay.

We can take epsilon to be epsilon by here, so mod of fn(z)-fz is < epsilon by m for all n >= n0 and for every z belonging to R. Then what we do then we consider, sigma n=1 to infinity phi z * fn(z) series, okay. So now fn(z) will be sum of first n terms of this series, so it will be phi z times fn(z) and fz will be phi z * fz, okay. So mod of this will be = mod of phi z * mod of Sn(z)-fz, okay, which is <= m times epsilon by m, okay for all n >= n0 and for all z belonging to R, which is equal to epsilon.

So if the terms of the series are multiplied by bounded function phi z okay then even then the series converges uniformly. So the sum of a uniformly convergence series of continuous functions of a complex variable in the region R is a continuous function in R; this theorem can be used as a test for uniform converges, if the sum functions of a series of continuous function is discontinuous then we can say that the series does not converge uniformly in R.

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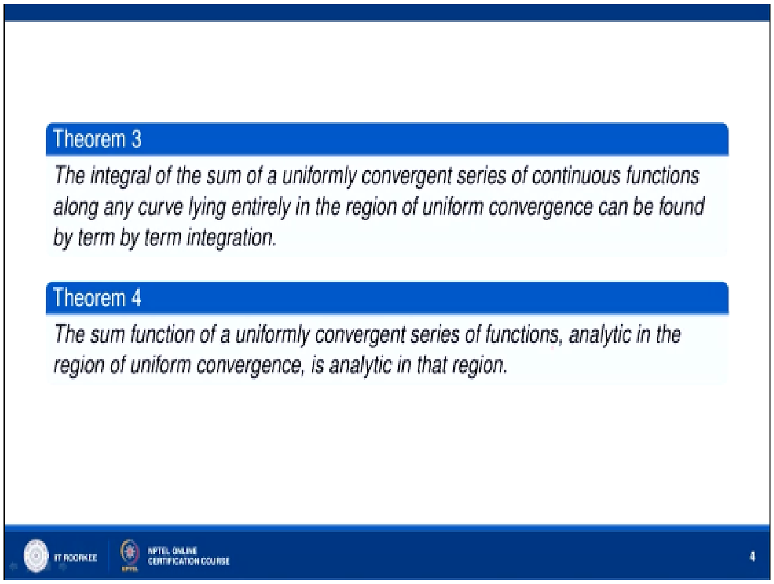

The integral of the sum of the uniformly convergent series of continuous functions along any curve lying entirely in the region of uniform convergence can be found by term by term integration; the series can be integrated term by term along any curve which lies in the region of uniform convergence. Now the sum function of a uniformly convergent series of functions, analytic in the region of uniform convergence, is also analytic. So we are going to prove this.

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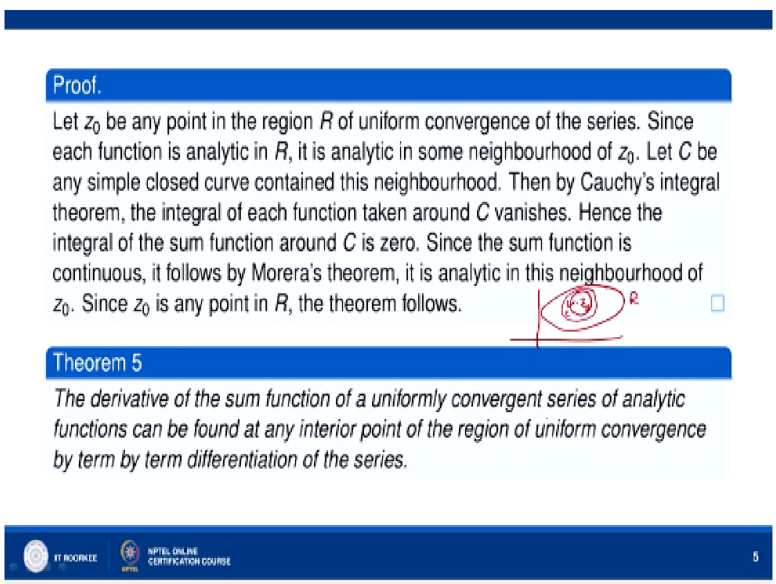

Let z0 be at the any point in the region R of uniform convergence of the series. Since each function is analytic in R it is analytic in some neighbourhood of z0. Let C be any simple closed curve contained in this neighbourhood. So let us say, take any region R, suppose this is origin R, z0 in any point here, okay. Take as neighbourhood of z0. Let us see we any simple closed curve contained in this neighbourhood, so we can take a neighbourhood like this okay and this is your curve C. Okay.

So let us C be any simple closed curve contained in this neighbourhood of z0 then by Cauchy's integral theorem the integral of each function taken around the C vanishes. Hence, the integral of the sum function around C is 0. Since the sum function is continuous it follows by Morera's theorem. In the Morera's theorem we have said that, if the function fz is continuous in a domain D and integral over C fz dz = 0 around any simple closed curve which lies completely inside D then the function fz is analytic in D.

So since the sum function is continuous it follows by Morera's theorem that it is analytic in this neighbourhood of z0 and now z0 is any point in R so the theorem holds. The next theorem is the derivative of the sum function of a uniformly convergent series of analytic functions can be found at any interior point of the region of uniform convergence by term by term differentiation of the series. So let us move this result.

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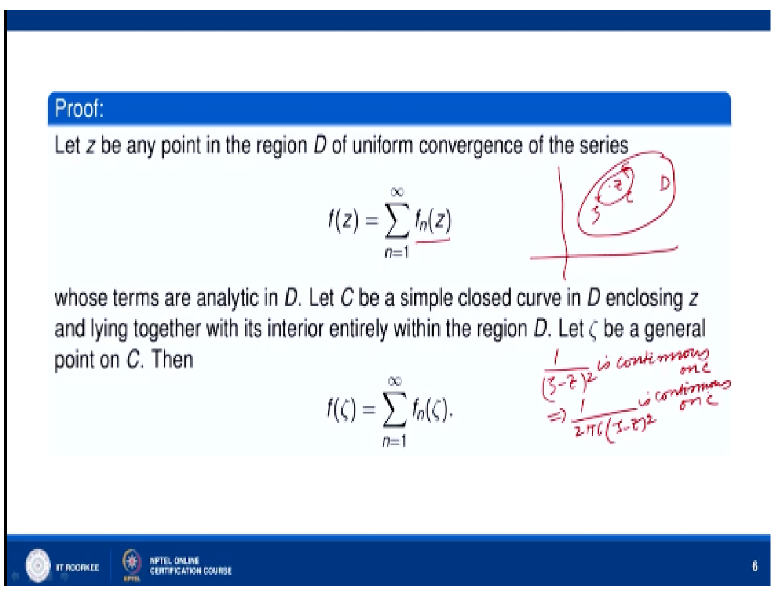

Let z be any point in the region D of uniform convergence of the series. Let us take a region domain D okay and let us take any point z which lies in D, so let z be any point in the region D uniform convergence of the series sigma fn(z), f(z) is the sum of the series. The terms of the series are analytic functions in D, so fn(z) is analytic for every n in D and let C be any simple closed curve in D which encloses z, okay, so take any simple closed curve in D which encloses z and line together with this interior entirely within the region D.

And let zeta be a general point on C, okay. Okay so let us take any point zeta on C. Then f zeta will be equal to sigma n=1 to infinity fn zeta since the function 1/2pi i * zeta – z whole square is bounded on C. Now how it is bounded? You can see zeta lies on the curve C; z is a point in the interior of C, okay. So 1/zeta – z whole square is then analytic function on C okay so therefore it is continuous, okay.

So 1/zeta-z is continuous on C and which imply that 1/2pi i zeta – z whole square is continuous on C. And so it is bounded because C is a simple closed curve. So it is bounded on C and therefore now let us apply the theorem 1, okay this theorem. If a series converges uniformly in a region R of the z-plane and phi z is any bounded function in R by which the terms of the series are multiplied, then the resulting series also converges uniformly.

So this series which convergences uniformly to f(z), if we multiply this series by a bounded function then the resulting series will also converge uniformly.

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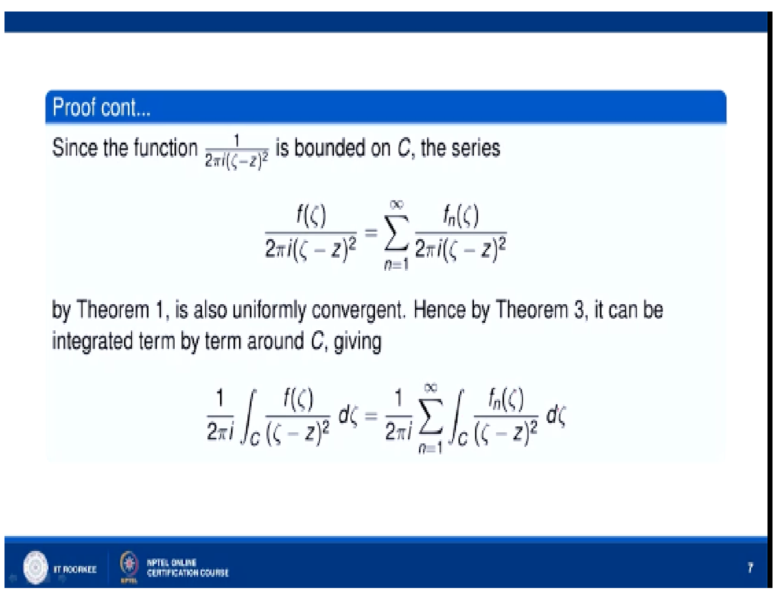

So we are multiplying this series by the bounded function 1/2pi i zeta – z whole square and then the resulting series will also converge uniformly to f zeta over z – z whole square into 2pi i. So sigma n=1 to infinity, fn zeta over this series, let us we are multiplying this series, okay. This series they are multiplying by one/2pi i * zeta – z whole square. So we get this okay by theorem 1 and this series converges uniformly to this function.

Now by theorem 3 this series; now this series convergence uniformly to this and therefore we can integrate this series term by term around the simple closed curve C which lies entirely inside D, okay around this simple closed curve C and after we integrate around the curve C what we get is 1/2pi i integral/C f zeta d zeta/zeta – z whole square = 1/2pi i sigma n=1 to infinity integral over C fn zeta/zeta – z whole square D zeta.

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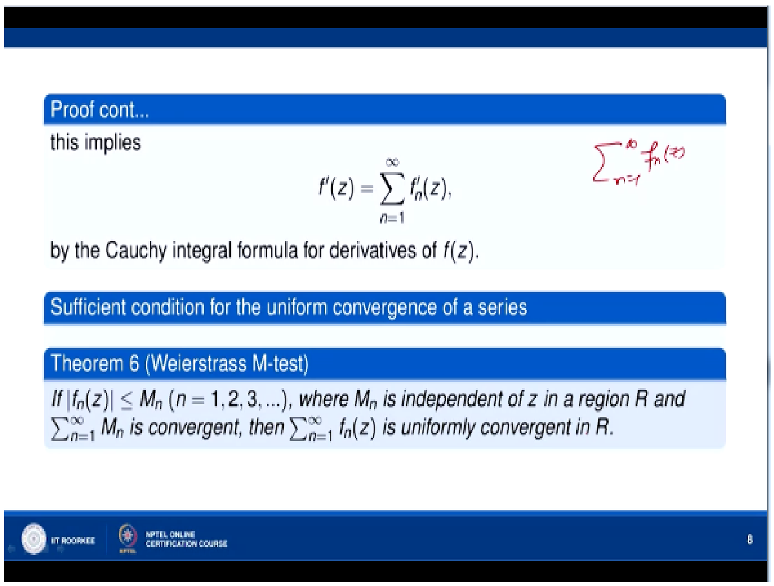

Now let us apply the Cauchy integral formula for higher derivatives okay from that it follows that, the left hand side is f prime z, right hand side is sigma n=1 to infinity fn prime z, okay. So fn prime z = sigma n=1 to infinity fn prime z by the Cauchy integral formula for derivatives if z, and therefore we can see that the series can be differentiated term by term. Now sufficient condition for the uniform convergence of a series.

Like we have a uniform, like we have a Weierstrass M-test for series of real functions here also we have a we have the Weierstrass M-test. So if mod of fn(z), if you take the series sigma n=1 to infinity mod fn(z) series of complex function fz then if mod of fn(z) is <= mn for all n=1, 2, 3…

where Mn is independent of z in a region R and sigma n=1 to infinity Mn in convergent then sigma n=1 to infinity fn(z) converges uniformly in R, so we have this result.

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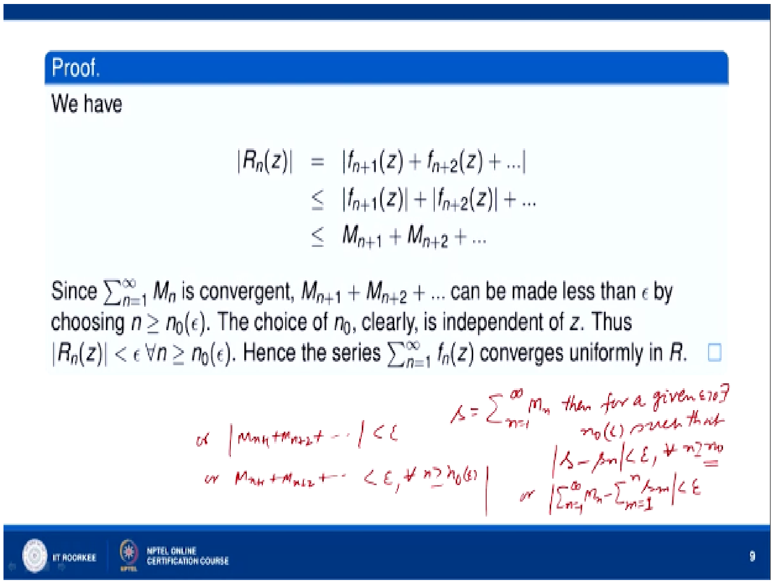

And we can easily prove this like we prove in the case of series of real functions. So here we have mod of Rn z = Rn z let us recall, it is the remainder after n terms of this series, so mod of Rn z is mod Fn+1 z, Fn+2z and so on and this is <= mod of Fn+1z, mod Fn+2z and so on which his <= Mn+1, Mn+2 and so on. Now, since the series sigma Mn=1 to infinity Mn is convergent okay, Mn+1 Mn+2 + and so on can be made < epsilon, okay.

Suppose, you can say that the sum of the series say S, S=sigma n=1 to infinity Mn then we know that for a given epsilon > 0 we can find n integer n0 depending on epsilon such that S-Sn, okay mod of S-Sn is < epsilon for all n > r=n0. So this is we consider mod of sigma n=1 to infinity Mn-sigma M=1 to n Sm okay < epsilon. So we are subtracting from sigma n=1 to infinity Mn the sum of first n terms of the series, okay. So this is the thing but mod of Mn+1 + Mn+2 and so on. Since Mn+1 and Mn+2 are positive we can write Mn+1+Mn+2 and so on.

So this is less than epsilon for all n > r=n0 epsilon, okay. So since the series sigma Mn is convergent Mn+1 Mn+2 can be made < epsilon for; by choosing n to be >= n0, the choice of n0. Now, this n0 does not depend on any z, okay, it depends only on epsilon okay, so, because it has come from the series of constant. So this series sigma n=1 to infinity fn(z) converges uniformly in R.

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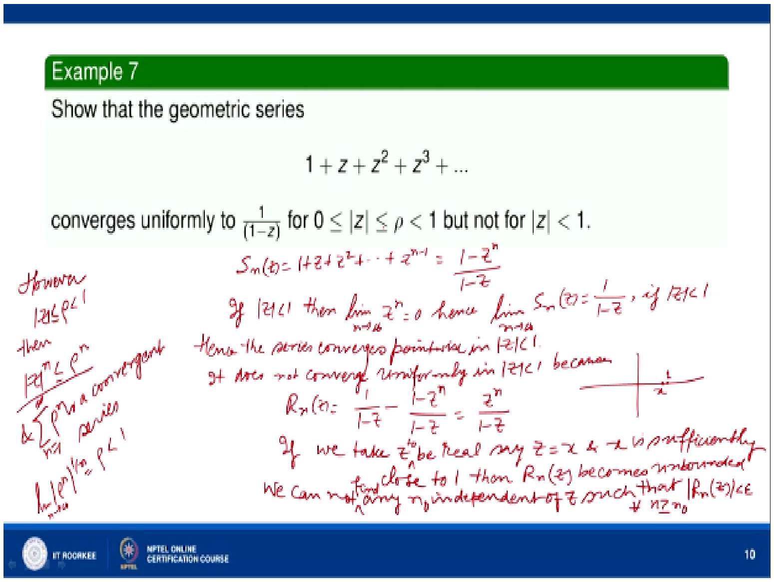

Now let us consider the geometry series. 1+z+z square+ z cube and so on. Let us show that it convergences is uniformly to 1/1-z for 0 <= mod <= rho < rho < 1 but not for mod z < 1. Let us consider the nth partial sum of this series, Sn(z)=1+z+z square and so on z to the power n-1; this we know it is a geometry series so we can write it sum 1-z to the power n/1-z, okay. Now if mod of z is < 1 okay then limit n tends to infinity z to the power n goes is = 0 hence, limit n tends to infinity Sn(z)=1/1-z if mod of z is < 1.

So this series converges point wise, okay. Hence the series converges point wise in the disc mod z <1. Okay. It does not convergent uniformly in mod z <1, how? It does not converge; this is because Rn(z). Okay, let us look at the Rn(z). Rn(z)=1/1-z f(z)- the sum of the first n terms should be 1-z to the power n/1-z. And this is equal to z to the power n/1-z, okay. So Rn(z) is the remainder after the n terms of this series. Okay. So this is z to the power n/1-z.

Now let us see, if you take real z to be real, okay let us if we take z to be real say z=x, okay and x is sufficiently close to 1 then what do we notice? Then, we notice that Rn(z) becomes unbounded. What I am saying is that, Rn (z=z to the power n/1-z. Let us take z to be equal to x, okay, z to be equal to x and say x is very near to 1, okay x is very near to 1, so then it is z=x we will have x to the power n/1-x and when x is very near to 1 what will happen, Rn(x) will tend to infinity, okay.

So Rn(x) will become unbounded and therefore we can say that we cannot find any n0, we cannot find any n0 independent of z such that mod of Rn(z) is < epsilon okay for all n >= n0 and so the series does not converge uniformly in mod z < 1. Now however, if we take mod z <= rho where rho is < 1 then what we notice is that, then mod of z to the power n is <= rho to the power n, okay.

So what will happen, here the n+1th term you can see in the series, n+1th term is z to the power n, so I am considering n+1th term so n+1th term the modulus of n+1th term is mod of z to the power n which is <= rho to the power n and we know that the series sigma rho to the power n, n=1 to infinity is a convergent series. This is the convergent series. This we can prove by applying the ratio test or the root test.

If you to apply the root test then mod of rho to the power n raise to the power 1/n limit n tends to infinity = rho, okay and rho is < 1. So the series sigma n=1 to infinity rho to the power n is a convergent series therefore, by Weierstrass M-test we can say that the series 1+z+z square and so on converges uniformly in the region 0 <= mod z <= rho, okay but not in mod z < 1. So this series converges uniformly this series converges point wise in mod z < 1 but not uniformly in mod z < 1. Let us take another example.

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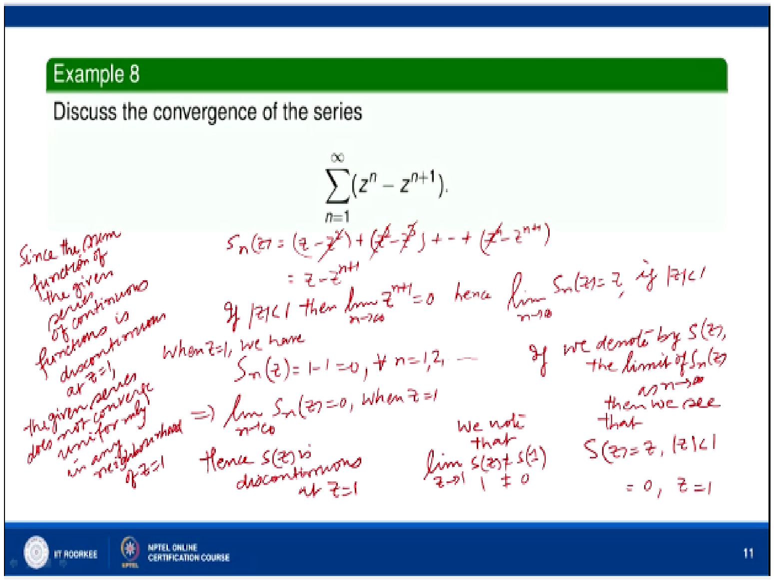

Suppose we take the series sigma n=1 to infinity z to the power n – z n+1. Here let us find the nth partial sum, so Sn(z)=z-z square z square-z cube and so on z to the power n - z to the power n+1 and we can see these terms cancelling okay and what we get is Sn(z)=z-z to the power n+1. Now if mod of z is < 1 then limit n tends to infinity z to the power n+1 goes to 0, okay. Hence, limit n tends to infinity Sn(z)=z if mod z is < 1, okay. Now let us notice the following.

If you find Sn(z) for z=1, Sn(z) when z=1 we have Sn(z)=1-1, okay. So we get 0, okay. So Sn(z)=0 for all values of n, n=1, 2, 3 and so on and therefore, limit n tends to infinity Sn(z)=0 when z=1, okay. So what do we notice, if we denote by f(z) the limit of Sn(z) as n goes to infinity then we see that f(z)=z when mod of z is <1, okay. And f(z)=; when we take z=1 f(z)=0 when z=1, okay. So you can see from here that f(z) is discontinuous at z=1 because limit z tends to 1 okay f(z)=1 okay we notice that this is not equal to S1 okay.

S1=0 and limit z tends to 1 Sz, Sz=z in every neighbourhood of z=1 okay, so this is equal to 1, so this 1 not equal to 0 and hence, f(z) is discontinuous at z=1. Now, from here we can say, the given series consistent of continuous functions at z=1. They are all continuous functions at z=1 but the some functions of the series is discontinuous at z=1 and therefore we can say that the 1 is the point of non-uniform convergence of series, that is the series does not converge uniformly in neighbourhood of f(z)=1, okay.

So since the sum function is discontinuous of the given series of continuous functions is discontinuous at z=1 we can conclude that the given series does not converge uniformly in any neighbourhood of z, z=1, okay. So this series does not converge uniformly in mod z < 1. Okay.

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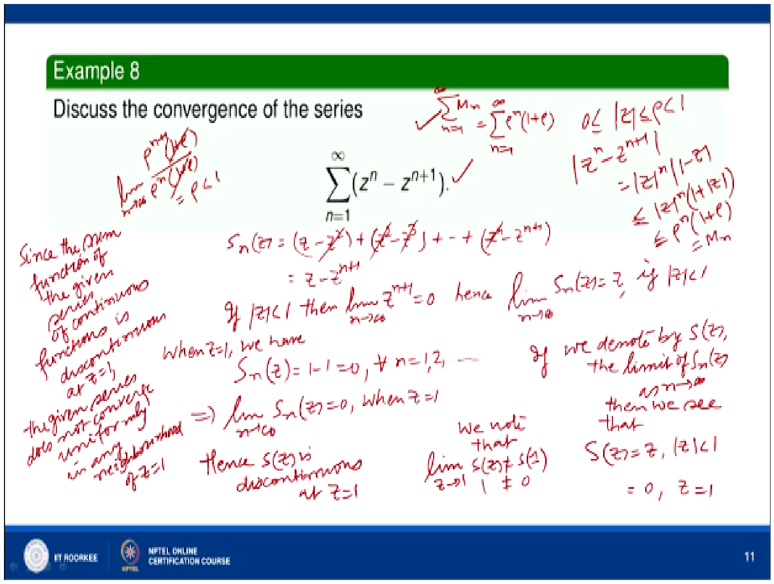

Let us consider the series sigma n=1; now here again we can conclude that if instead of mod z < 1 we consider this region, okay. If we consider 0<= mod z <= rho where rho is the 1, rho is <1 then this series converges uniformly because, why because mod of z to the power n let us take nth term, z to the power n+1 = mod of z to the power n * mod of 1-z okay. So this is = mod of z is <= rho so this is <= mod z to the power n*1+mod of z which is <= rho to the power n*1+rho, okay. So now this is let us take Mn, okay.

Now consider the series sigma Mn. So the series sigma Mn is sigma n=1 to infinity rho to the power n*1+rho, okay. If we apply the ratio test here then rho to the power n+1*1+rho/rho to the power n+rho, limit rho n goes to infinity. This will cancel with this and will get the limit as rho. Now this rho is <1. So the series sigma Mn converges uniformly, okay. The series sigma Mn converges; its convergent and therefore, the series sigma n=1 to infinity z to the power n-z to the power n+1 convergences uniformly in the region 0 <= mod z <= rho, okay and where rho is < 1.

So we can apply this Weierstrass M-test here to put the uniform convergence.

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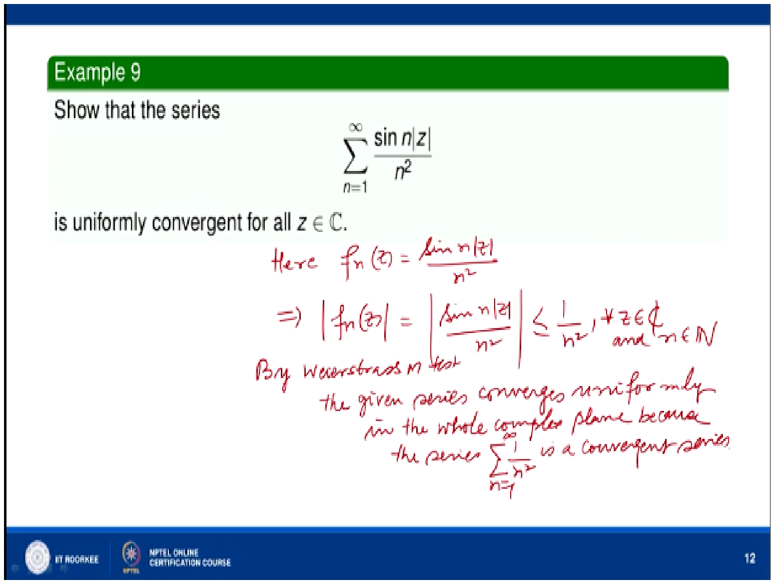

Now let us consider this series, sigma n=1 to infinity sin n mod of z over n square, okay. So we notice that here fn(z)=sin n mod of z/n square, okay. So, mod of fn(z)=mod of sin n mod of z/n square. Okay, now z is a complex number and mod of z is a real quantity so this is a real function, sin n times mod of z is a real function. And we know that mod of sin theta is <= 1 when theta is real so this is <= 1/n square for all z belonging to c and n belonging to set of natural numbers n, okay.

And so by Weierstrass M-test the given series convergent uniformly in the whole complex plane because the series sigma 1/n square is a convergent series. So we can apply the Weierstrass M-test to decide what the uniform convergence of this series. So we have proved the uniform convergence of this series sigma n=1 to infinity sin n mod of z over n square by applying the Weierstrass M-test.

In our next lecture, we shall discuss the power series. We have heard of power series for in real calculus. We have a power series here in complex also, so we shall discuss that in our next lecture. Thank you very much for your attention.