Dynamical Systems and Control
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Lecture – 60
Relation between Continuous and Discrete Systems - II

Hello viewers. Welcome to this lecture on the relation between continuous and discrete systems. So in this lecture, we will see some results on the controllability and the observability of continuous and discrete system and their relations.

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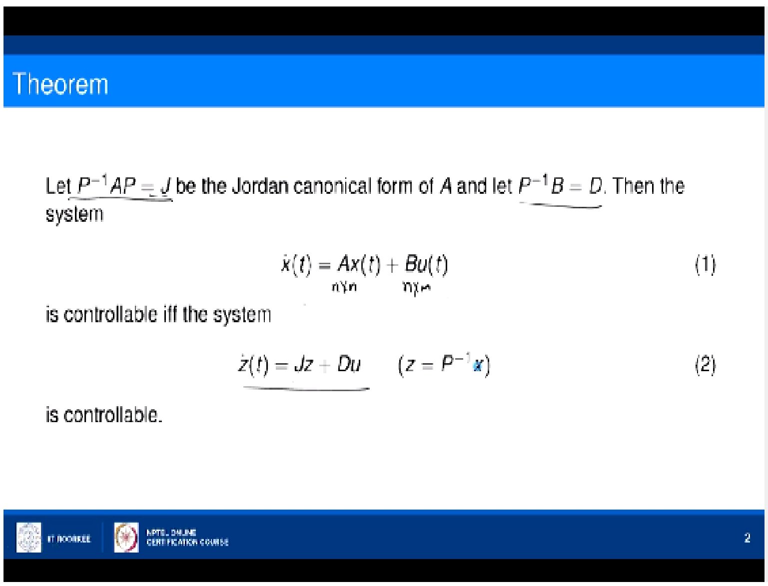

So let us consider the control system x dot=Ax+bu where A and B are constant matrices. A is an n*n matrix and B is n*m matrix. So let J denotes the Jordan canonical form of the matrix A. So there exists a non-singular matrix P such that P inverse AP is the Jordan form J. So let P inverse B is the matrix D. Then we can show that the system x dot=Ax+Bu is controllable if and only if the canonical system z dot=Jz+du is controllable, where z, the variable z is P inverse x. So it is a transformation which we are making on the state variable.

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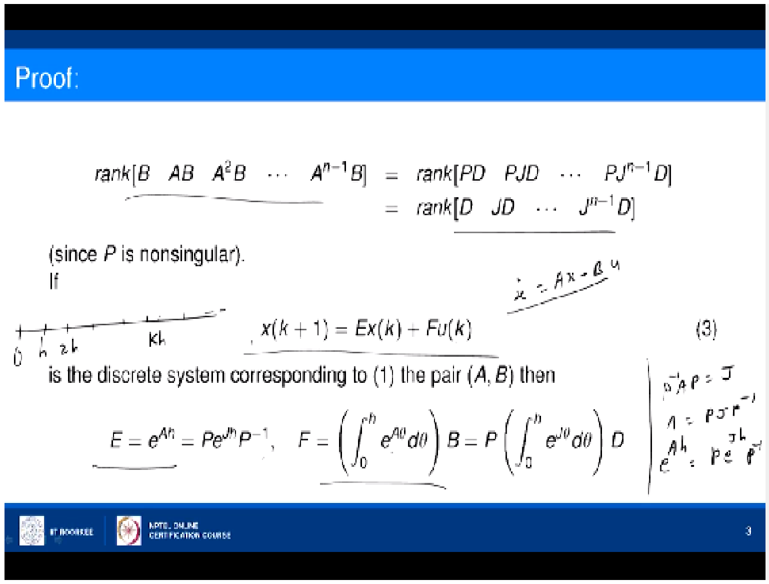

So we have already seen that the system, the controllability of both the systems are equivalent because of this rank condition. Rank of B AB A square B A power n-1B is n if and only if rank of DJDJ power n-1D is n. Now we consider the corresponding discrete systems. So first corresponding to x dot=Ax+Bu, the discrete system is given in the equation 3 as x of k+1=Ex of k+Fu of k where the matrix E and F are given by this. E=e power Ah and which can be written as Pe power Jh*P inverse.

Similarly, the matrix F is given by integral 0 to the power A theta*d theta*B. And A, because A can be written as P inverse AP is J. So we can write A as PJP inverse. So we get this expression. e power Ah can be written as pe power JhP inverse. These, whereever A is there, it can be replaced with the Jordan canonical form by using the P matrix like this.

So this we have seen already. How to convert a continuous control system into the discrete control system and this h value is the increment in the time, discrete time. So if the initial time is 0, then we consider 0 h 2h etc., *nhk, so *k*h etc. These are the discrete time intervals. At this discrete points, we get the value of x, the state variable and the control variable U of k.

They are related by this relation. So this has been seen in the previous lecture. So corresponding to the Jordan equation, this expression z dot=Jz+Du, we get the corresponding, the discrete form as z of k+1 is E bar zk+F bar u of k where E bar is given by this, F bar is given by this expression.

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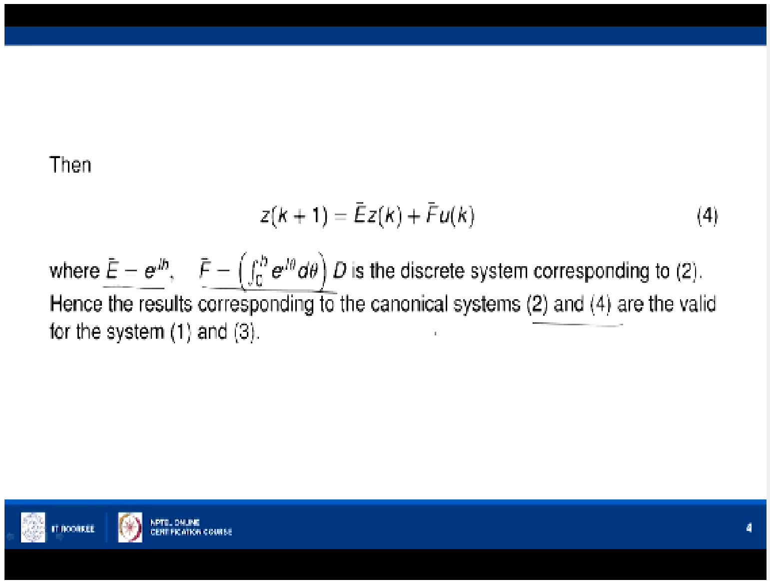

So now as we have seen that the given matrix and the corresponding Jordan canonical form will have the same behaviour for the controllability. So we can see that the result corresponding to the canonical system 2 and 4, if you prove controllability on the system 2, then the system 1 is also controllable. Similarly, if the system 4 is controllable, then the system 3 is also controllable. So that relation we can see using this result, the rank condition.

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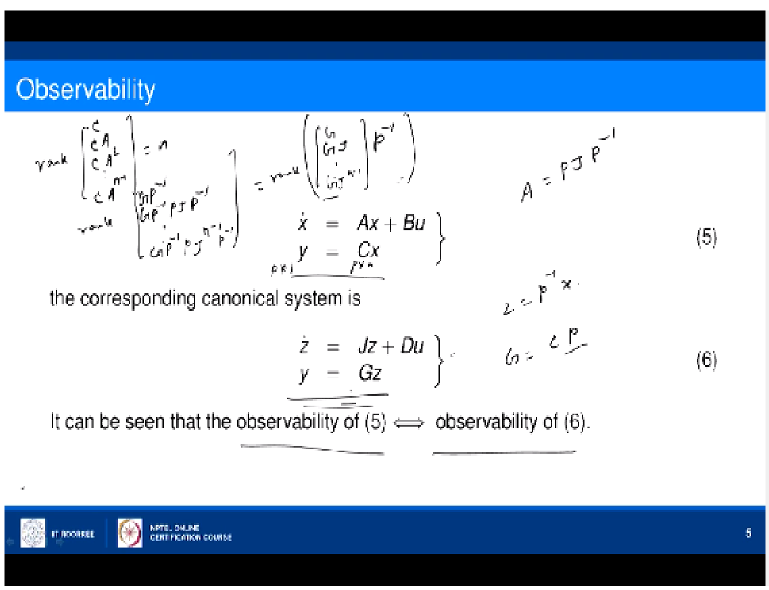

Now we consider the observability of the system x dot=Ax+Bu and the observation y=Cu and the matrix A and B sizes have been taken. Now let us say C is a P*n matrix and y is P*1 matrix which is the observation of the system. The corresponding canonical form is given by z dot=Jz+Du and y=Gz where G is given by, because z is nothing but P inverse x. So we will get G=C*P because y=Cx and x=Pz, so when we substitute, we will get y=Gz where G is given by the matrix CP.

So we can now show that the observability of the system 5 implies observability of the system 6 and vice versa. Observability of 6 implies observability of 5. So that can be easily seen by the rank condition. Observability of 5 means it is rank of the matrix C CA CA square CA power n-1. So the rank should be equal to n.

And we can obtain from this the matrix A. So here C is GP inverse and next is GP inverse and A is given by PJP inverse, etc. So GP inverse A power n is given by PJ power, A power n-1 is given by PJ power n-1P inverse. So we can get the rank of this matrix is same as rank of GP inverse and GJP inverse etc. GJn-1. So P inverse is common for all these entries here. So the rank of this matrix, but P inverse is a non-singular matrix.

So we get rank of this matrix is same as rank of GJ to the power n-1, that is same as the observability of the system 6 here. So we have shown that the observability of 5 implies observability of 6. And if you go in the reverse order, we get observability of 6 implies observability of the system 5, so both are equivalent.

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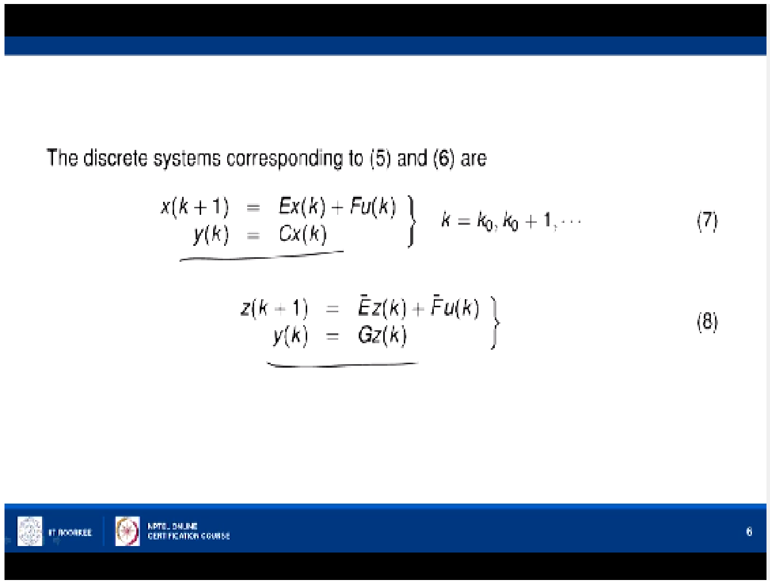

So if you prove controllability and observability for the canonical system, it is equivalent to proving the controllability and observability of the given system x dot=Ax+Bu and y=Cx type of thing. Now the discrete system corresponding to 5 and 6 are given by the equation 7 and 8 here. Where E and F were already defined and y of k is C*x of k is the same C matrix there and G is defined in the previous slide for the Jordan canonical system.

So now the controllability of the continuous system and controllability of the discrete system. How they are related, that we can see in the coming in this slide.

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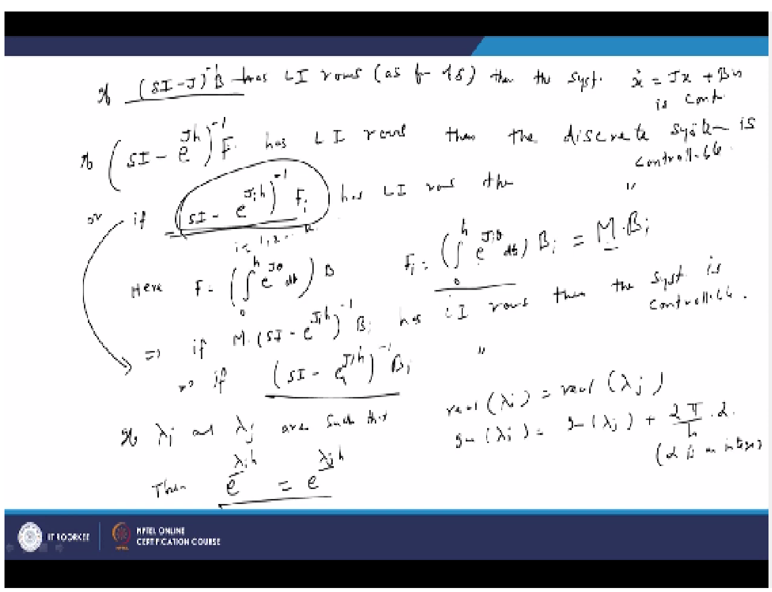

So we have shown that if SI-J inverse B has linearly independent rows as functions of S here, okay, then the system is controllable. x dot=; Bu is controllable. So if SI-J inverse B has linearly independent rows, then the system x dot=Jx+Bu is controllable and vice versa. Similarly, if SI-e to the power Jh inverse*F, okay, has linearly independent rows, then the discrete system is controllable, that is the discrete system given by this equation z of k+1 E bar z of k+F of uk is controllable.

So now we will show, we are interested in having the continuous system and the discrete system together to be controllable. Because in many practical problems when we have a continuous control system, if the model is a continuous control system and when we solve it using discretization, so we will get the corresponding discrete system. And if the original system is controllable and if the discrete system is not controllable, then the result may not be very useful.

Similarly, if the original system is observable and the discrete system is not observable, also it is not a very useful result. So whenever we want to handle a continuous system as a discrete system for numerical purpose, then the controllability, observability, stability, all the properties should be preserved in the discrete system. So we will see here under what condition the controllability, observability of the continuous system also implies the controllability, observability of the discrete system and vice versa.

So here we have already proved that if this resolvent SI-e to the power Jh inverse*F, if it has linearly independent rows as the function of S, then the system is, the discrete system is controllable. So using this, we will come to the following conclusion. So in other words, if SI-e power Jih inverse*F suffix i, okay has linearly independent rows, then the discrete system is controllable.

That is for each i, for i=1, 2, 3 up to k. The number of eigenvalues are k, that is lambda 1 lambda 2 lambda k are the distinct eigenvalue. For each eigenvalue, we have the Jordan block JI and correspondingly F has the blocks F1 F2 FK and the Fi block along with this. So if SI-e power Jih inverse Fi, so if this matrix has linearly independent rows, then the system is controllable. For each i, it should be valid.

So that is what we have seen. So we can easily see that this block Fi, the nature of the expression Fi is given by this. So F is given by integral 0 to the power A theta*d theta*B matrix. But we are using J matrix here. So F is nothing but integral 0 to the power J theta d theta*the B matrix. So if you assume like that here, so here F is integral 0 to the power J theta d theta*the matrix B, whatever matrix B we are considering.

Similarly, Fi means we have to replace it with J suffix i because of the Jordan canonical form. And B at the same time can be replaced by B suffix i. So it is integral 0 to the power Ji theta d theta*Bi matrix. So instead of showing SI-e power Jih inverse Fi, we can show that this thing. So if SI-e power Jih inverse*Bi has linearly independent rows, then the system is controllable. That is because you can see that Fi is made up of the matrix of the form integral 0 to the power Ji theta.

So e power Ji theta will be commuting with this matrix. SI-e power Jih inverse if we expand it, it will be in terms of e power Jih terms as a series expansion. So it will be always commuting with this integral e power Ji theta. So we can write Fi in the other side also. So instead of writing SI-e power Jh inverse Fi, we can write the; so if you write Fi as some matrix say M matrix*Bi where M is given by integral 0 to the power Ji theta d theta, that matrix is M and it is made up of all these e to the power Jh type of thing after putting the limits.

So it will be commuting with this matrix SI-e power Jih inverse. So we can write this M in this side. And we can also see that this M matrix is non-singular because it is coming from a non-singular matrix e to the power Ji theta and we are integrating from 0 to h. So the resulting matrix is non-singular. So the row of this matrix, if the rows of this matrix are linearly independent, then the rows of, without this M also will be linearly independent.

So it is enough if we check whether the rows of SI-e power Jih inverse*Bi are linearly independent for the controllability of the system. So for the controllability of the continuous system, we have to verify this one, SI-J inverse B, rows are linearly independent. And for each I, again we have to do. SI-J suffix i inverse*Bi should have linearly independent rows for the controllability for each i.

Similarly, this matrix should have linearly independent rows for each i for the discrete system. Now we will see that in some cases, the continuous system will be controllable; whereas, the discrete system will not be controllable, corresponding discrete system may not be controllable. Because if you see that if lambda i and lambda j are, let us say, 2 distinct eigenvalues, they are such that the real part of lambda i=the real part of lambda j.

But imaginary part of lambda i is imaginary part of lambda j + some 2pi/h*alpha, where alpha is an integer, 1, 2, 3, etc. So if we have this type of expression, then we can easily see that they will have terms like e to the power lambda ih. Similarly, e to the power J suffix jh will have terms like e to the power lambda suffix jh. And we can easily see that e power lambda i*h will be equal to e to the power lambda j*h.

So we will see that similar expressions will come as a function of S here. SI-e power lambda ih, e to the power Jih as well as SI-e to the power J suffix jh, both will have similar S functions. And if you have the linearly dependent rows of Bi matrix, then the theorem will not be valid for the discrete system. In other words, this will not happen, this, rows of SI-e to the power lambda ih inverse*Fi, they will not have linearly independent rows for all the values of i.

Because for i and J, if the eigenvalues are behaving like this, then the corresponding rows will be linearly dependent because of the nature of this expression. So in such cases, we will not get the controllability of the discrete system. Because the eigenvalues are distinct here, the continuous system is controllable but the discrete system will not be controllable for this case.

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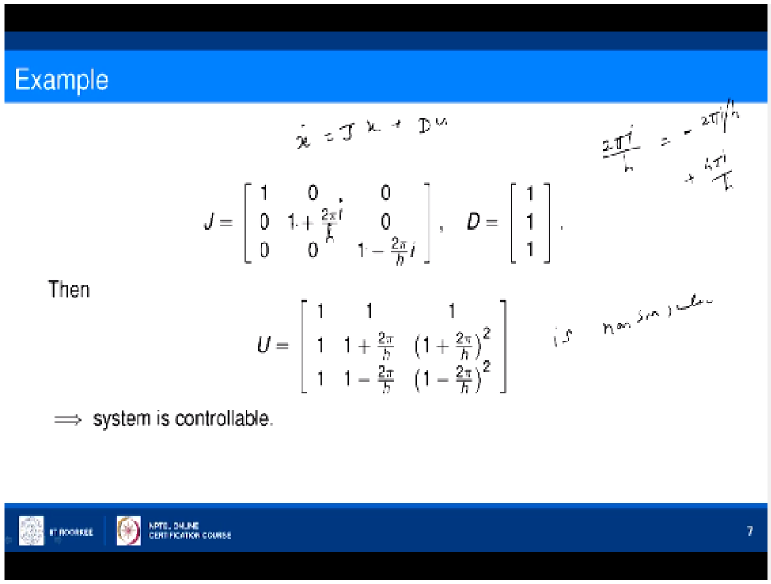

So for example, if you consider J to be like this 1 0 0 and 0, because it is a diagonal matrix, this 3 are the eigenvalues and the real parts are same, imaginary parts are like this 2pi I here, sorry, 2pi i/h and this is 1-2pi i/h. D is 1 1 1. Then the Kalman condition D J*D is the second column, J square*D is the third column. So we get the Kalman condition like this. This has rank 3. So it is non-singular. So this implies the system is controllable. The continuous system x dot=Jx + this Du if you take, so this system is controllable.

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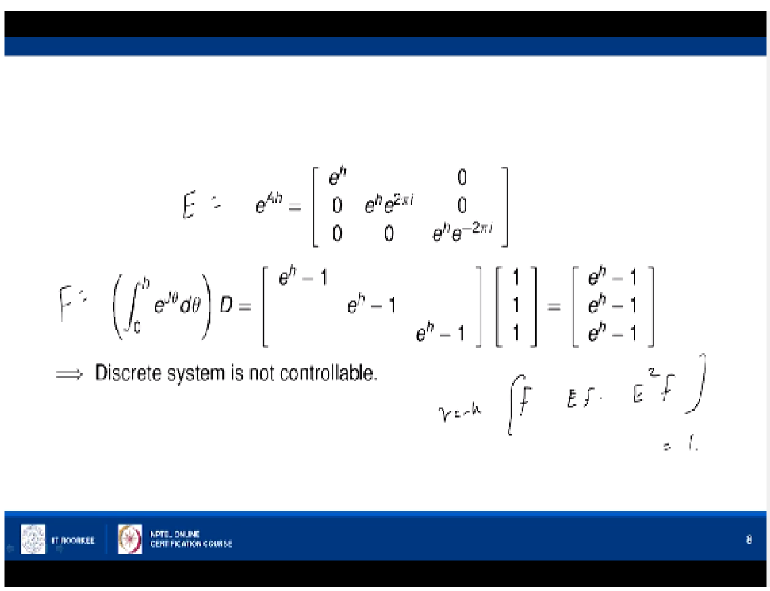

But the corresponding discrete system if you take, when we convert it into discrete system, we have to calculate e to the power Ah which is given by this. Here it is J. A is equal to J in this case. So e power Ah is given by these 3 eigenvalues that is e power h and e to the power h*e to the power 2pi i/h*h. So that will be 1. Similarly, here. So all the 3 will get only the terms like this. So e power 2pi i is 1, so we get all of them are e power h only.

Similarly, e to the power J theta*d theta*D for the control matrix, we get this expression. So now you can easily see that this is our e matrix and this is the F matrix. So if you calculate F and E*F and E square*F and the rank of this will be equal to 1 only. So the system is not controllable. Whereas the corresponding continuous system is controllable. So it depends on the h value or the eigenvalues should not behave in that particular manner.

Here we can see that the eigenvalues are like this. The real parts of this and this are same. Real part is equal and imaginary part is, the imaginary part of the second eigenvalue is 2pi i/h. And imaginary part of the third one is -2pi i/h. So their difference is, so if you take this one, their difference is this expression, +4pi i/h, so this expression. So according to the theorem, the imaginary part should not differ by this type of expression. So in that case, it will not be controllable, the discrete system is not controllable like this.

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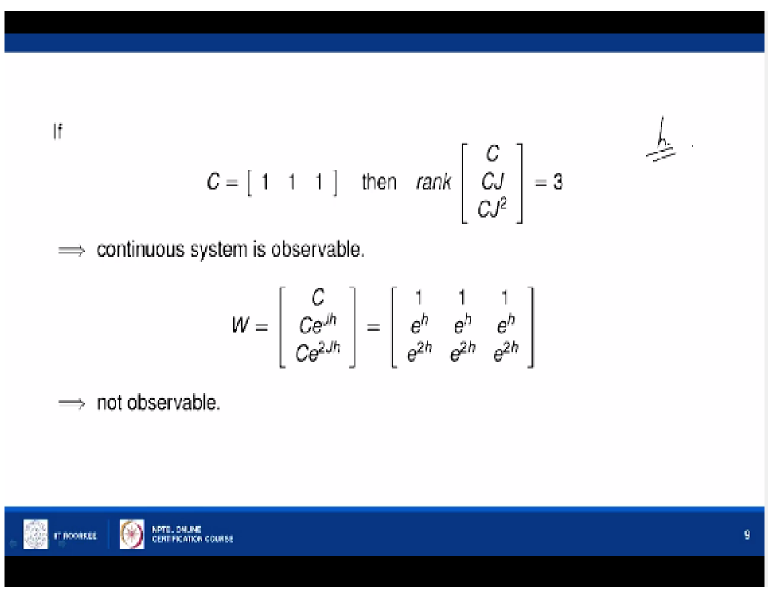

Similarly, we can do the, same theorem can be repeated for the observability. If the real parts of 2 eigenvalues are the same but imaginary parts are differing by 2pi/h*some integer, then the observability of the continuous system will not imply the observability of the discrete system. So we have to choose carefully the h value.

That is the conclusion that the time increment h should be chosen carefully so that this eigenvalues will not behave in this particular manner. So in this lecture, we have seen the relation between the continuous and the discrete system and their controllability, observability properties. Thank you.