Dynamical Systems and Control
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Lecture - 44
Stabilizability

Hello viewers. Welcome to the lecture on stabilizability of dynamical systems. In many practical control systems, it is desirable to find a control or design a control so that the system is stable at some desired equilibrium position. So in this lecture, we will see the procedure for stabilizing an unstable system.

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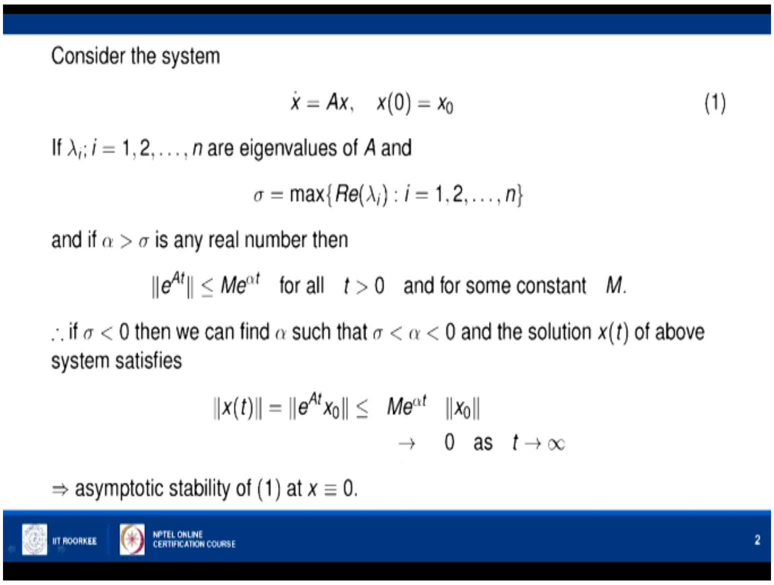

So first let us recall some simple results which we have seen in previous lectures. So consider the linear system x dot=Ax with initial condition x of 0=x0. So if the eigenvalues lambda i, i=1 to n or such that sigma is the maximum value of the real part of the lambda i, then we can show that the norm of the e to the power At the state transition matrix is<=M times e to the power alpha t where alpha is any real number>sigma where sigma is given here the maximum value for a suitable constant M.

So this result we have already shown in the stability lecture. So using this we can show that the solution of the system 1 that is x of t=e power At*x0 is a solution. So the norm of x of t is<=M times e to the power alpha t*the norm of the initial condition. So the initial position is always bounded, so norm of x0 is a bounded number and in case all the eigenvalues are negative or with negative real number then sigma has to be a negative number.

Then, we can find a number alpha which is also negative such that alpha is>sigma. So in this case norm of xt is<=M*e power alpha t that implies that norm of xt tends to 0 as it tends to infinity because alpha is negative number. So this shows that if all the eigenvalues of a matrix have negative real part, the system is asymptotically stable at the trivial solution x=0.

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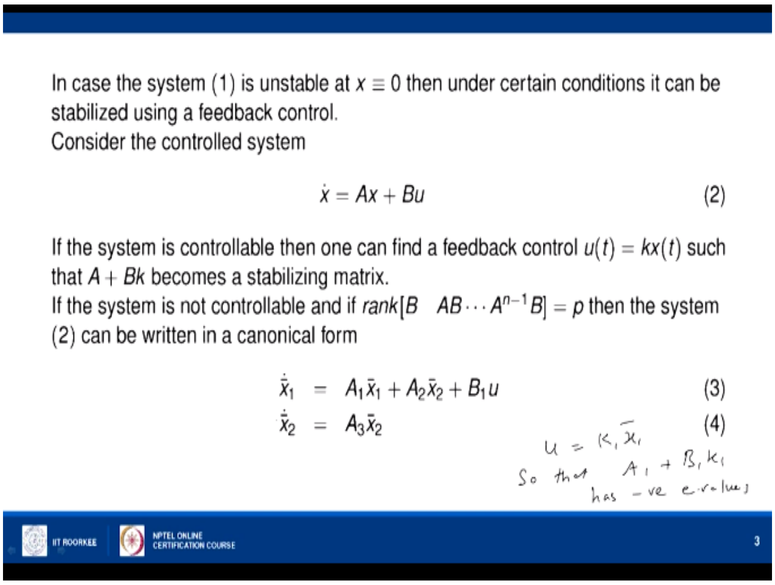

So in case the system is unstable at the trivial solution x=0, then we have also seen in previous lectures on feedback control that the system can be stabilized. We can find a feedback control u suitably so that the resulting system x dot=Ax+Bu becomes either stable or asymptotically stable depending on the requirement of the system. So this has been described in the feedback lecture with some examples also how to find the u of t=k*x of t.

And the procedure for finding the k matrix was shown earlier. So this is possible if the system is controllable, if the pair A and B are such that the rank of B, AB, A square B, etc that is equal to n. In that case, we are able to find a feedback control to stabilize the system but in case, the system is not controllable, A and B are in such a way that the rank is not full. Then, the rank of B, AB, A square B, A power n-1 B=p which is strictly<n.

So the system is not controllable. So in that situation how to stabilize this system, for that we have seen in our previous lecture on the system which is not controllable. So we will be able to find a canonical form for the system 2 in this particular form 3 and 4. That is we can split the system in two parts that is x1 cap and x2 cap represent the state of the system such that x1 cap dot=A1 x1 cap+A2 x2 cap+B1 u.

And x2 cap=A3 x2 cap, so this is possible that a procedure has been explained in a previous lecture.

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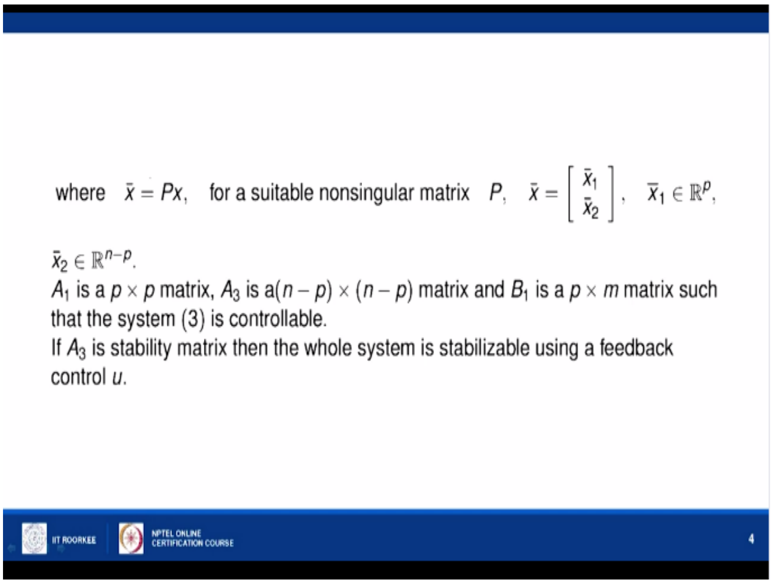

So where x cap is=p*x for a suitable nonsingular matrix p and so x cap=x1 cap x2 cap is the splitting of the vector x cap and the matrix A1 is a p x p matrix where this p is the rank of B, AB, etc and the matrix A3 is n-p x n-p square matrices. The system 2 has been reduced to the canonical form 3 and 4 and the first system the system 3 has the matrix A1 and B1 in such a way that it is controllable and the second system is uncontrollable portion of the given system.

So now how to stabilize this system? If the system 4 is a stable system that is the matrix A3 if it turns out to be a matrix with all the eigenvalues with negative real part, then system 4 is separately a stable system okay and then substitute that solution x2 in the equation 3. Then, we will arrive at a system x1 cap.dot=A1 x1 cap+B1 u+some term which is tending to 0 as t tends to infinity.

Because x2 cap will tend to 0 as t tends to infinity if you assume that A3 is a stable matrix. So we have a controllable system 3 and the extra term A2 x2 bar is tending to 0 as t tends to infinity. So now this equation 3, it can be stabilized using a suitable feedback control because the condition of controllability is valid for the system 3 and using the previous procedure we can find a K matrix such that this expression.

We can find u=so we can find u=some k1*x1 cap so that A1+B1 k1 has negative eigenvalues okay. So in that case, this system can be stabilized system 3 and system 4 is also stable. So x1 cap will tend to 0 as t tends to infinity and x2 cap also will tend to 0 as t tends to infinity. Ultimately, we can see that x is=p inverse x cap so that so x of t will also tend to 0 as t tends to infinity.

So the system original system itself is stabilizable okay under these conditions but if these conditions are not satisfied then we will not be able to guarantee any stabilizability of the system. If A3 is an unstable matrix, then the system will remain unstable in this particular using these terms okay. So now we will go for a different procedure.

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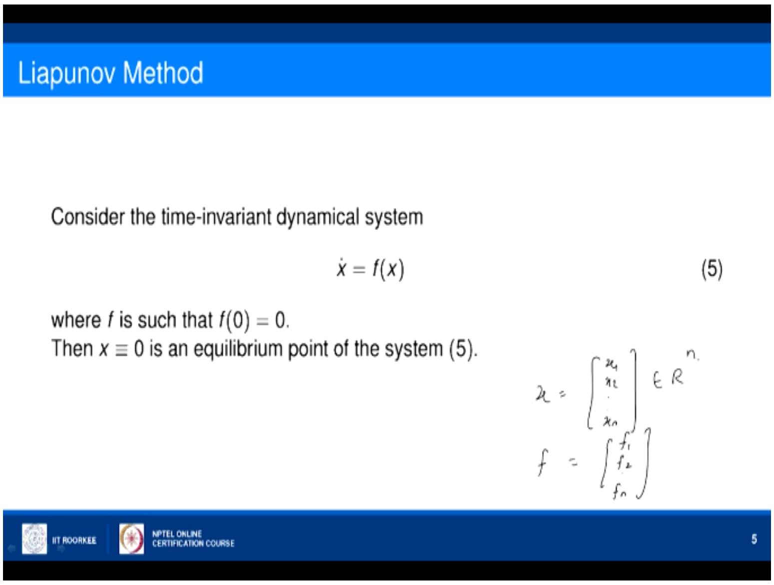

So in situations where the system cannot be stabilizable using this procedure, then we may try some Lyapunov method for stabilizing a system. So first we will see a system which is not having any control term, simply a nonlinear term and time invariant that is autonomous system x dot=f of x where f is either linear function or some nonlinear function such that f of 0 is 0. So that will imply that x=0 is the equilibrium point for the system 5.

So now we want to analyze the stability of the system at this equilibrium point. For that in some previous lecture, we defined the Lyapunov function and if you are able to find a Lyapunov function with this condition, then the system is we can make it a stable or asymptotically stable.

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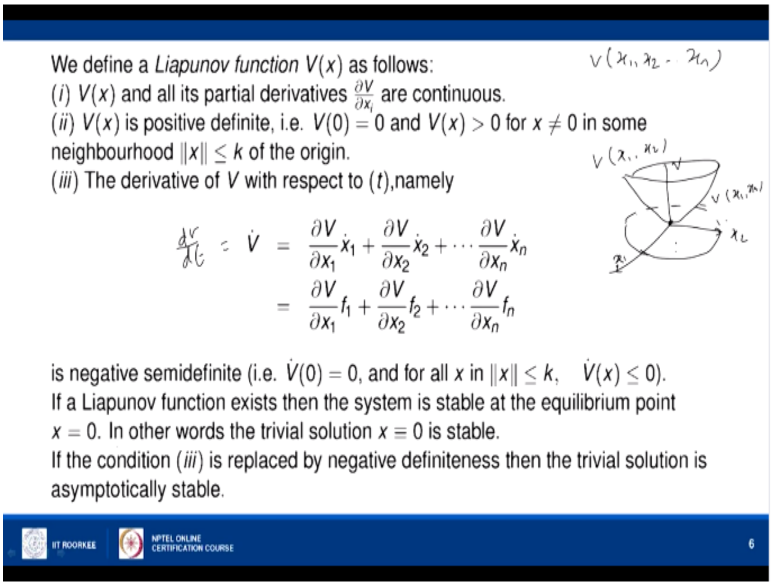

So if you are able to find a Lyapunov function V of x, here we note that x is a vector, this x is say x1, x2, xn belonging to belongs to Rn and f is also a vector, f of x means f1 of x, f2 of x, fn of x, so the nonlinear system of differential equations here. So when we talk about V of x, it means V of x1, x2, xn a function of n variable. So this function V of x and its partial derivatives are continuous function means continuous functions of this n variable.

So it is an n-dimensional function and the continuity of n-dimensional function should be applied here at various points of the n tuple and it is positive definite, that means of V of 0 is 0 and for all other non-zero V of x has to be strictly positive. So for example in a two-dimensional case V of x1, x2 then if we plot the graph of this function V of x1, x2 positive definite function, it means the x1 axis, x2 axis are here.

So V at 0, 0=0 and V at any point x1, x2 should be positive. This is the value of V, so the graph of this it is something like this. It is a surface V of x1, x2 always above the x1, x2 plane. So it need not be a parabola in type of a symmetric shape, it may be some shape surface which is just above the x1, x2 plane and 0 at the zero point. So that is given positive definite condition.

So it is not necessary that it should be infinitely like this. So if you are able to find a neighborhood that is norm of x<=k, let us say a circle of radius k we are taking, so within this region this V of x1, x2 if you define, it is positive definite, that much is sufficient for a Lyapunov function and the derivative of V with respect to t that is dV/dt=it is negative semi definite, so that is dV/dt is del V/del x1*x1 dot, etc.

So we have the system x1 dot=f1, x2 dot=f2, etc so that is given here. So dV/dt should negative semi definite the third condition. So that condition means V dot of 0 is 0 and it should be<=0 for all x such that in this region within the radius of k it satisfies this condition. So if we are able to find a Lyapunov function for a system given system then the system is stable.

So this proof has been done in a previous lecture on Lyapunov function and the stability of nonlinear systems.

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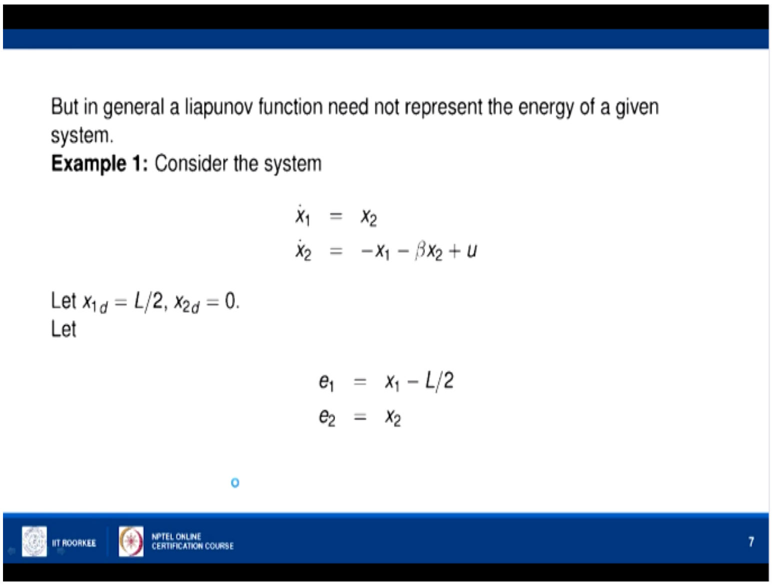

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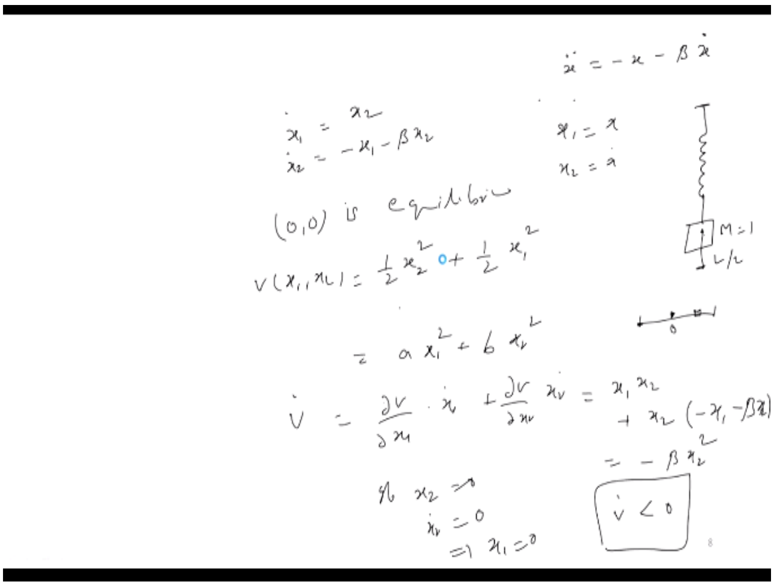

So now we consider a system like this. For example, x1 dot=x2, x2 dot=-x1-beta times x2. So this system it is nothing but a harmonic oscillator type of thing, x double dot=-x-beta times x dot. So this is either you can consider this as a mechanical system where the mass=1, mass*acceleration that is the left hand side that is the force and it is due to the spring constant. Here we are assuming the spring constant=1.

Otherwise, we can write –some k times x where k is a spring constant. For simplicity, let us assume that spring constant is 1, x double dot=-x the force is proportional to the negative of the distance from the equilibrium point that is the meaning minus the damping force. It is due to friction or air friction and other type of disturbances. So if you have such equation either it can be considered as a spring mass system or a harmonic oscillator which has the equilibrium position in the center and the mass is oscillating in both sides of the equilibrium point.

So this when we convert it into a system, it will be x1 dot=x2. If you write x1=x and x2=x dot we will get the system. For this system, it is obvious that the 0, 0 is equilibrium point if we equate the right hand side and then we can consider V x1, x2 it is=1/2 times mass is 1 and velocity is x2 square, 1/2 MV square that is the kinetic energy and 1/2 x this k is 1 here so 1*x1 square, so it is the potential energy.

So for any mechanical system of this type, the kinetic energy+potential energy it is the total energy of the system. So this will act as the Lyapunov function for this type of examples or instead of this type of energy function we can also consider energy functions x1 square+some B times x2 square, it is not exactly the kinetic energy+potential energy somewhat we can call it as resembling the energy function.

So this type of function normally will work as a Lyapunov function for simple linear systems of this type. So we can easily show that V dot by differentiating del V/del x1*x1 dot+del V/del x2*x2 dot which is equal to del V/del x1 that is x1*x1 dot is x2 and +del V/del x2 that is also x2*x2 dot that is -x1-beta times x2. So ultimately we will get x1, x2 cancels, we get -beta times x2 square.

So this is<0 if x2 is non-zero but if x2 is=0 and x1 is non-zero, it will not show that the system is negative definite, is not it? But we can easily see that if x2 is 0, x1 has to be 0 because if you substitute in the equation, if you put x2=0 in the, if x2=0 that will imply x2 dot is also=0. So from this equation, you can say from the second equation x2 dot is 0, x2 is 0 that will imply x1 is also=0.

So the V dot has to be negative semi definite, V dot is negative definite actually. So this implies that the system is asymptotically stable. That means it will stop, after certain oscillations it will come to a stop that is the meaning of asymptotic stability of the system but in general always the Lyapunov function need not represent the energy of a given system. So we can see several different examples in which the Lyapunov function is different from energy.

But only it should satisfy the conditions given here, the 3 conditions for stability or asymptotic stability of the system. So for example, now let us consider the controlled pendulum, sorry controlled harmonic oscillator or the spring mass system where other than the gravity or other than the natural pull of the system, we can introduce an extra control term, so that is given by this one.

So u is the control applied on the system. So in this case, we can have another equilibrium point also. It is not necessary that always the system should stop at the equilibrium point 0, 0. So we can introduce a desired equilibrium point. So for example in this spring mass system if there is no force extra force other than gravity let us say it will stop at a particular equilibrium point and then if you pull it down and leave it, it will oscillate and then again stop at the equilibrium point only.

But now if you are giving an extra force on the system, we can also say that the system should stop at this point. For example, from the equilibrium point let us say the distance is L. Let us say it is given here L/2 okay. The total amount of pull which we can make is because the spring has certain capacity, we cannot pull it throughout I think up to infinity. We can pull it up to certain level L let us say that is the maximum we can pull it.

So we want to stop the pendulum, sorry stop the system at the point L/2 from the equilibrium position okay. So if you want to do this, we have to apply an extra force other than the gravity, so that extra force is given by u in the second equation and the desired point in which we want to stop the spring mass system is the position is L/2 and because we are stopping it at that place, the velocity x2 desired should be 0.

So this is our desired state, desired position and desired velocity but actually when the spring is moving because it is changing with respect to time, so x1 and x2 are the actual value and the desired values are x1 desired and x2 desired. So the error in the current position and the desired position is x1-L/2 that is e1 and e2 is x2-the desired is 0, so it is x2 itself. So we call the error in the system as e1 and e2.

Now convert this given system of equation in terms of the error, so if you differentiate e1 that will be x1 dot-0. So you will get e1 dot is x1 dot itself and e2 dot is x2 dot itself.

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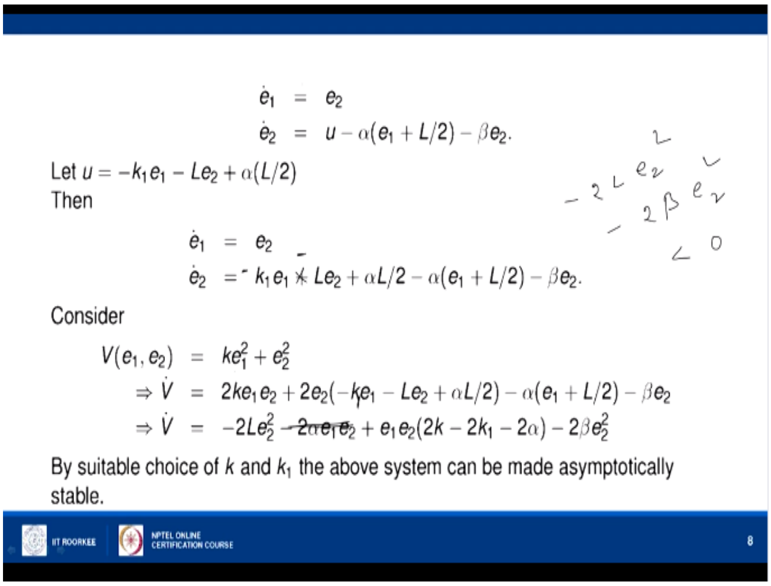

So we can write the system in this form e1 dot we denote it by e2 and e2 dot is given by from the equation we can write like this, u-alpha* in the place of x1 we will write e+L/2. So that is given here –beta times e2. So now we have to design a force, the control to be designed by ourself, for that how we will use the Lyapunov theory that is what we are demonstrating here. So let us select a control like this, u=-k1 times e1-L times e2+alpha times L/2 where alpha is the constant in the equation itself.

Alpha and beta are known values and here k1 and L we have to design ourself suitably for our particular goal of stopping the system at the desired position. So let the control u is given by –k1 e1-L e2+alpha times L/2. So if you substitute this control here, we will get e1 dot=e2 and e2 dot=the substitute u here. See the Lyapunov function, there is no standard procedure for finding a Lyapunov function.

It is somewhat a trial and error method only, so we will try with, normally we will try with this type of energy function which is resembling an energy function. So if it works well that is fine, otherwise we have to search for another Lyapunov function so that to test whether the system is stable or not but in this practical problem we know that the system can be made asymptotically stable because it is a spring mass system, we can definitely stop it at a particular place by applying a suitable force.

So we know that the system can be stabilized physically. So we can search definitely for a Lyapunov function to make it stable okay. So in a very random problem simply there is no physical meaning or anything just a mathematical model is given. So in that case we cannot guarantee anything. We do not know whether the system is stable or stabilizable, etc and there is no physical meaning also for that problem.

So in that situation, it may be very difficult to decide a Lyapunov function etc but in this type of problem, physical problem or real life problem we can always try to stabilize the system using this type of energy function if the system is a linear one because here this example the system is linear. So let us consider V e1 e2 is k e1 square+e2 square and then so it is obviously a positive definite function.

P of 0, 0 is 0 and for all other values it is strictly positive, so positive definite and then V dot is del V/del e1 that is 2 times k e1*e1 dot that is e2 so that is the first term+2 times e2*e2 dot. So we substitute the e2 dot from this equation that is –k times e1, etc. All the terms are there. Then, cancelling out certain terms and then we will get finally we will get -2L e2 square. So if L is a positive number, this -2L e2 square is strictly negative.

Then, we have the term -2 alpha e1 e2+e1 e2*2k-2k1-2 alpha. So these terms shows that I think this term has come extra, it is already added here, -2 alpha so this term is not there okay. So now we can see that by selecting this k because we have to select k1 and L, so by selecting this k1 and here also in the Lyapunov function also we can select a k value ourself, a positive value, so k1, k and alpha is already given in the equation itself, it is a known value.

Suitably, selecting k and k1 in such a way that this bracket becomes 0. What we see here is the remaining terms are simply -2L e2 square and -2 beta e2 square. These two terms are there, both of them becomes strictly<0 and so the system is asymptotically stable because we can argue in the similar manner.

If e2 is 0 by substituting in the equation, we can also show that e1 has to be=0. So both of them has to be simultaneously=0. Therefore, the system is asymptotically stable here. So this demonstrates that we can use the Lyapunov theory to design control in a practical problem like this.

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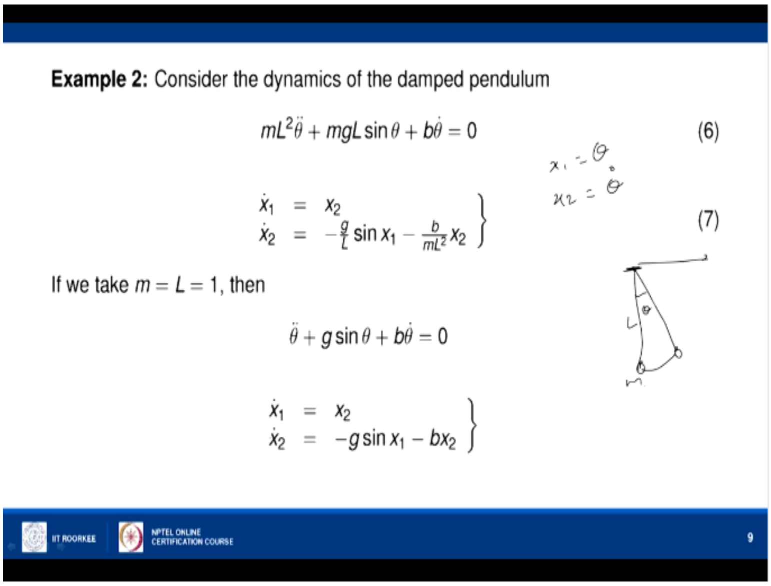

So now we will consider examples in which the Lyapunov function need not be in the form of an energy function. So let us consider the system mL square theta double dot+mg L sin theta+b theta dot=0. So this is a damped pendulum problem. So we have already seen this example. We have a pendulum of this type which is oscillating and this is the theta value. So we can write the equation where L is the length and m is the mass here and g represent the gravity.

So it is a standard equation of pendulum and the last term it represents the air resistance due to the air resistance it has stopped resisted. So the equation 6 represents the damped pendulum. When we convert it into the system form, we can write it is as x1 dot=x2 where x1 is theta and x2 is theta dot. Then, we can convert it in this form and for simplicity let us take mass is 1 and length is also 1 unit.

Then, it becomes a simple expression theta double dot=this or in the system form it is x1 dot is x2, x2 dot=-g times sin x1 and –b times x2. So it is a nonlinear system of equation. So in this case, if you try the energy type of function that is A times x1 square+B times x2 square that may not give any fruitful result as a Lyapunov function but we can try this Lyapunov function.

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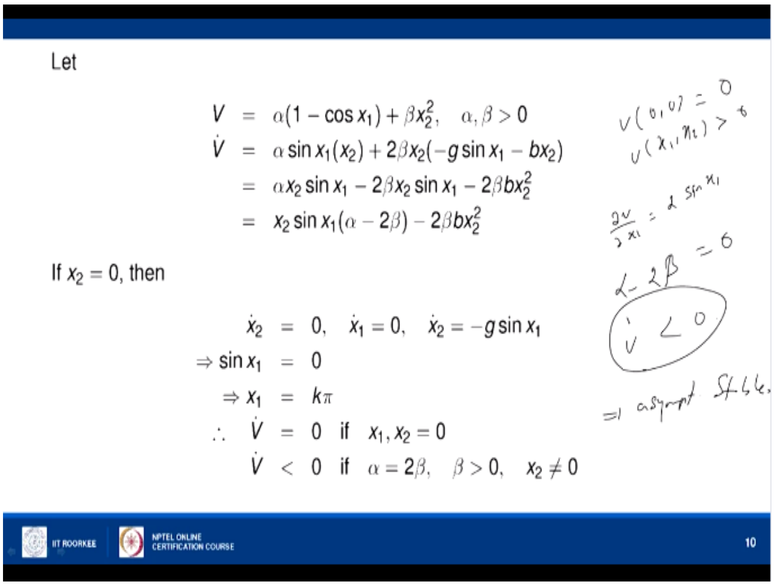

So again there is no fixed rule, it is also some kind of trial and error type of thing and it is arrived at this Lyapunov function will work for this particular system. There may be another Lyapunov function working for the same system but if we take this one, alpha times 1-cos x1+beta times x2 square. So the equilibrium point here is 0. The equilibrium point from the pendulum is it has to, after oscillation it will stop at the vertical position here.

And the velocity will be 0, so x1 that means theta is 0 and x2 that is the angular velocity also should be 0, so it will stop here is the equilibrium position. So if we try this Lyapunov function, you can see that it is always positive whether x1 is positive or negative, in all the cases it cannot exceed the value 1 so 1-cos x1 is always positive value and when x1=0 1-cos x1 is 0 and x2 is 0 implies the second term is also 0.

So it is 0 at the point 0, 0 V of 0, 0 is 0 and for all other values it can be easily seen that it is strictly positive, x1, x2 is strictly>0 for when both x1 and x2 are not 0. So it is a positive definite function. Now when we differentiate dV/dt and substitute the values del V/del x1, so that will give alpha times sin x1*x1 dot is x2 so that is given here and del V/del x2 is 2 times beta*x2 and x2 dot is substituted from the previous equation –g sin x1, etc.

So what we get is alpha times x2*sin x1-2 times beta x2*sin x1 and -2 times beta*small b*x2 square. So from this equation, we can see that if you select alpha-2 beta=0 or in other words alpha=2 beta by choosing these constants like this we get a suitable Lyapunov function in such a way that V dot becomes strictly<0, the same argument. If x2 is 0, x1 has to be 0 therefore it is negative definite function.

So this implies the system is asymptotically stable okay. So it will come to a stop after long oscillation. Here it is without control, it is a dynamical system, a natural system.

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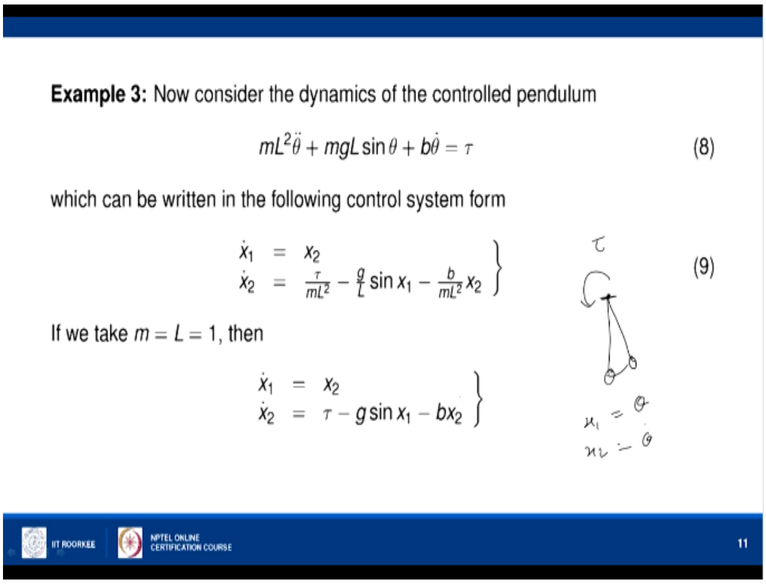

But if you want to stop this system at some other position let us say we want to apply a control on the pendulum. So the natural oscillation is due to gravity but if you apply a torque at this joint here, so that rotational force that is torque here can control the movement of the pendulum. So that is called the controlled pendulum and we can write the system equation as in the same manner x1 is theta and x2 is theta dot.

Then, the equation is converted into this particular form, x2 dot is – sorry tau/mL square etc. Now by taking the constants m and L as 1, the simple system is given by this expression. So it is the same system as we consider earlier except the forcing term tau or the control term.

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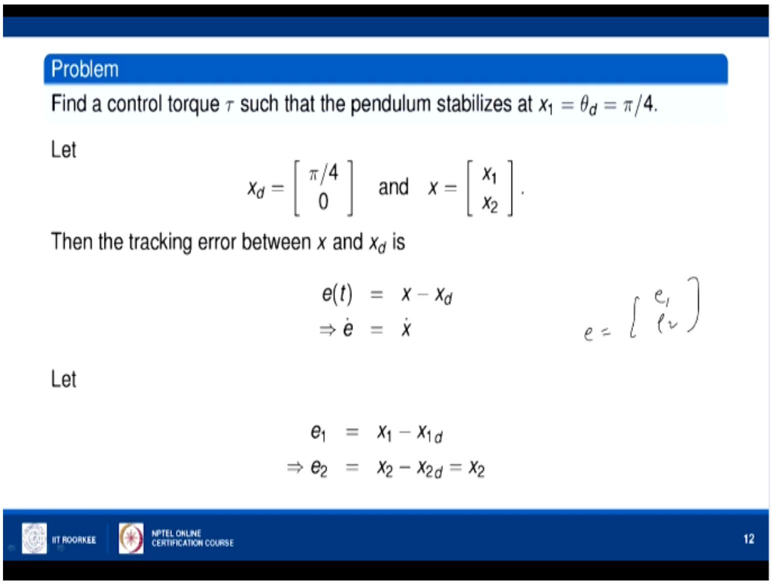

So now our aim is to find a control torque in such a way that the pendulum stops in this angle. We want that the pendulum stops at an angle pi/4 and the velocity here is 0 okay. Instead of stopping at the natural equilibrium point, we want an equilibrium point which is given by pi/4 and 0, the position and the velocity, so how much of torque we have to apply at the joint that is the problem here.

For that we consider the error, so when the pendulum is not at the equilibrium point, it will be at some other point x and x is x1, x2 and x desired is pi/4 and 0. So the error at every instant of time is x-x desired and if you differentiate this one, we will get the derivative of the error is simply x dot because x desired is a constant value here. So e dot=x dot only and if you write e as e1 and e2 if you split it as e1 and e2, so e1 is nothing but x1-x1 desire and e2 is this expression.

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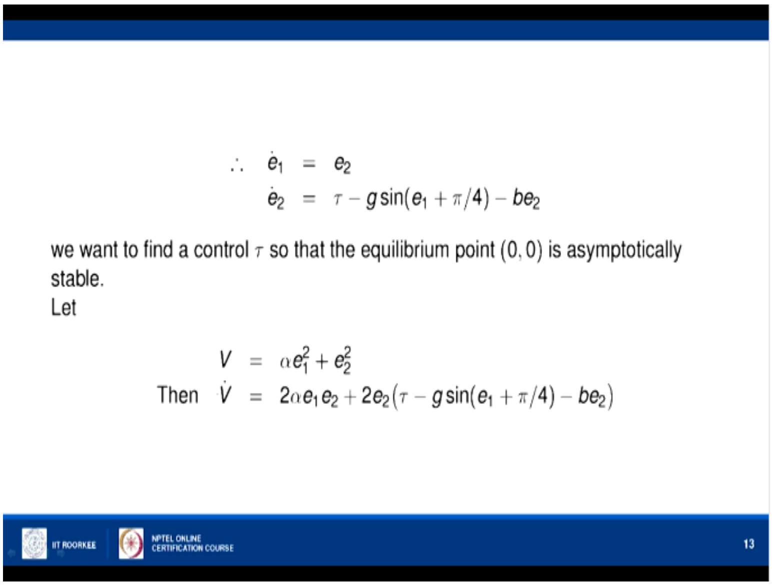

Now differentiating e1 dot and e2 dot separately and then substituting from the equation this equation we get this equation e1 dot=e2 and e2 dot is tau-this expression. If you try a Lyapunov function in this as in this previous problem because see these two problems are almost similar except that there is a control torque in the second equation, tau is added in x2 dot. So now if we try a Lyapunov function as here alpha times 1-cos x1 etc, this may not work for this thing, for finding a torque this may or may not work properly.

But instead let us try a Lyapunov function like this itself like the energy function. This may turn out to be a simple one. So let us say V=some alpha times e1 square+e2 square okay or beta times e2 square. Later we can select the alpha, beta suitably. If V is this one, the derivative dV/dt is given by 2 alpha e1 and e1 dot is e2+2 e2*the derivative e2 dot is in the right hand side, tau-g sin of e1+pi/4-b*e2. So this is the expression.

Now we want to make this V dot to be negative definite, so we have to select this control tau in such a way that V dot becomes negative definite. So that will work as a suitable controlled term.

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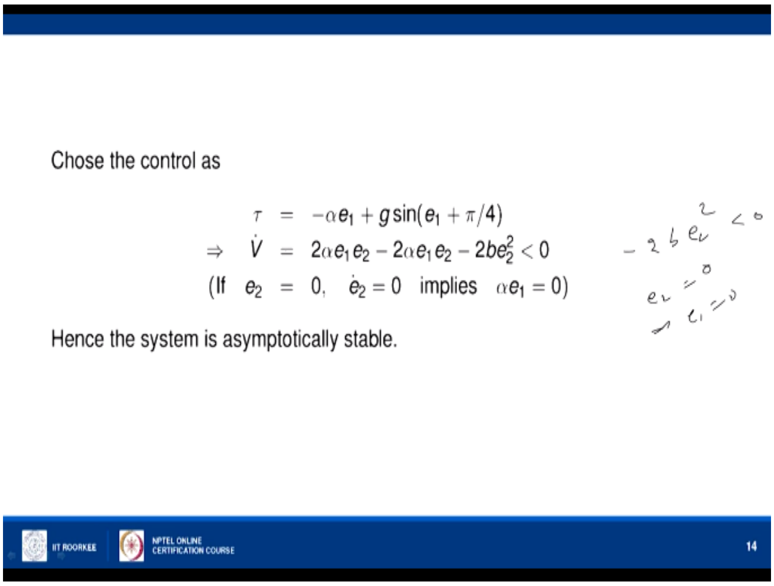

So we choose the control tau as actually we can remove this one, g sin of e1+pi/4 can be removed by taking tau as +g sin e1+pi/4, one term can be selected like that so that these two gets cancelled and the remaining term we want to cancel this also, 2 alpha e1 e2 because e1*e2 will not give any positive term is not it? So we want to make some squares only in the V dot expression.

So wherever we find uncomfortable terms like this, we just cancel it out by selecting the control in a suitable manner. So let us take tau as this the control as alpha times –alpha times e1+this term. If you substitute these two terms, we will see that V dot=that 2 alpha e1 e2-2 alpha e1 e2 and we will get this one 2 beta*e2 square. So these two gets cancelled, this will imply that this expression -2 b e2 square it will be<0.

And in case e2 is 0, it does not mean that V dot will be=0 because if e2 is 0 that will also imply that e1 is also=0 that we have seen in the previous also. So using that we can show that V dot is strictly negative definite. So this control will make the pendulum stop at the pi/4 angle with zero velocity. So we have seen here how to utilize the Lyapunov theory for designing the control for various control systems for a linear system as well as a nonlinear control system.

So in the forthcoming lectures, we will see some more examples, more complicated examples, how to make use of all the feedback controls and Lyapunov theory for designing controls of different systems okay. Thank you.