Numerical Methods: Finite Difference Approach

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Lecture – 17
Stability analysis of hyperbolic equations

Welcome to the lecture series on Numerical Methods Finite Difference Approach. In the last lecture we have discussed a hyperbolic equation and its solution procedures, based on like explicit method and the implicit method. In the present lecture we will just go for this like stability and convergence analysis of this hyperbolic equations.

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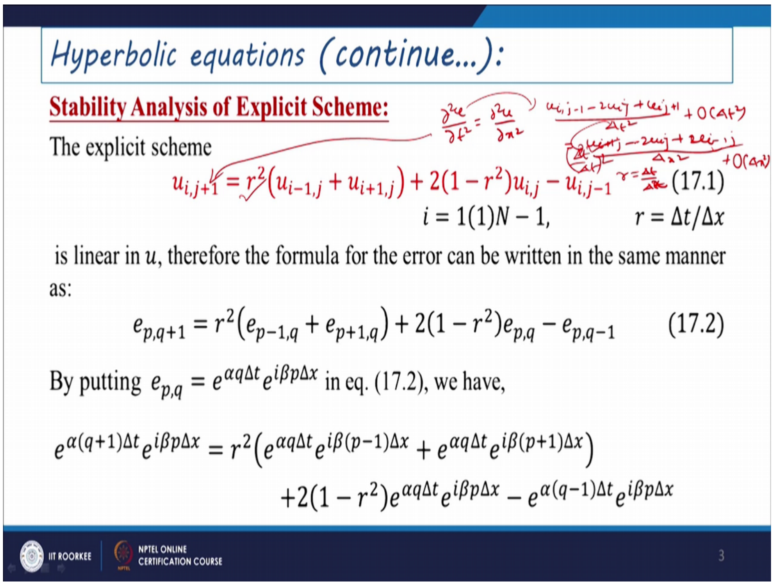

So, if you will just go for this stability analysis of based on this explicit scheme here. So, especially this explicit scheme for this hyperbolic equation that is just defined in the form of if the hyperbolic equation is the as del square u by del t square this equals to del square u by del x square with two initial conditions and two boundary conditions prescribed.

Then this discretization equation at the j plus 1th level it is just written in the form of u i j plus 1 this equals to r square u i minus 1 j plus u i plus 1 j plus 2 into 1 minus r square u i j minus u i j minus 1. Easily you can just find these equations in c, if you will just show discretized this equation that is that can be written in the form of u i j minus 1 minus 2 u i j plus u i j plus 1 by del t square plus order of del t square this equals to u i plus 1 j minus 2 u i j plus u i minus 1 j by del x square plus order of del x square.

If you will just consider r equals to del t by del x and neglecting the higher powers of x here then we can just write that one as u i j plus 1 this equals to; since it is just defined as del t by del x whole square here, so that is why r square it is just coming into the picture. So, u i minus 1 j plus u i plus 1 j this into r square since del t square it can be just come it up and del t by del x whole square if you will just write. So, we can just write this term clear itself as in this form here. That is why it is written as like a r square into u i minus 1 j. So, this is written as r square u i minus 1 j plus u i plus 1 j plus 2 into 1 minus r square u i j minus u i j minus 1, i is varying from one to n minus 1.

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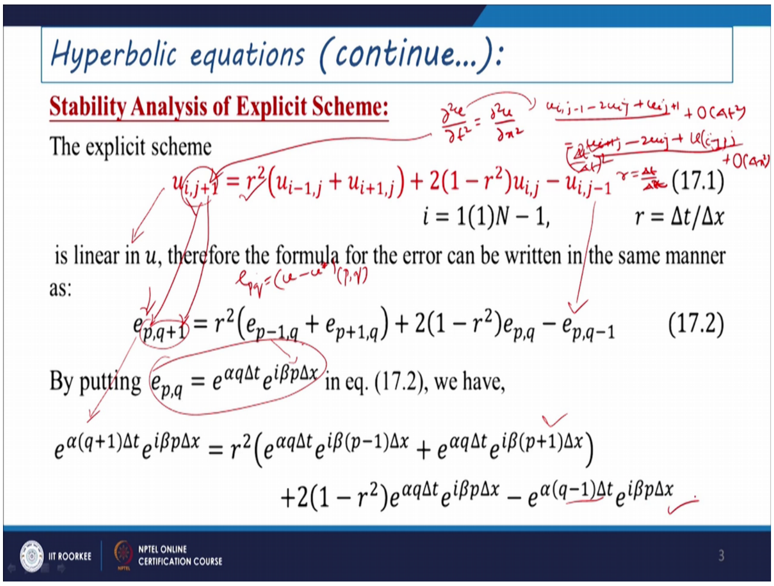

So, this equation is a linear equation in u therefore, the formula for the error can be written as in the same manner as we have defined in the earlier lectures for parabolic equation and elliptic equations. So, the error term it can be just written in the form of e p q plus 1, since a if you u is the numerical solution here and u star is the approximated solution. So, if you will just take this a approximate solution minus the true solution then especially it is just written as e of p q plus 1 based on this i j plus 1 points we especially we are just replacing this one as i as in the form of p here and j plus 1 as q plus 1 where e of p q it can be written as in the form of u minus u star at p q point. So, that is why it is written in this one here.

So, e of p q plus 1 this can be written as an r square. So, based on this like u minus u star at i minus 1 j point it can be written as e of p minus 1 q point plus u minus u star at i plus 1 j point it can be written as a e of p plus 1 q plus 2 into 1 minus r square and u minus u star at i j coordinate that can be written as e of p q there minus u minus u star at i j minus 1 point that can be written as e of p q minus 1 here. And especially we have just h replaced that one e of p q as in the form of e to the power alpha q del t e to the power I beta p delta x.

Already we have explained these things in the previous lecture that is for like parabolic equations and elliptic equations. So, that is why here directly we are just writing this error term in terms of exponential powers and if you just put this power terms as in the form of error term here as the coordinate p plus p q plus 1 here. So, this can be written as e of p q plus 1 as e of alpha into q can be replaced by q plus 1 into del t e to the power i beta. So, p is a their present, so p delta x this equals to r square. p minus 1 especially it can be written as a p minus 1 here. So, e to the power alpha q del t e to the power i beta into p minus 1 delta x plus e to the power alpha q since it is just written alpha q del t e to the power i beta into p plus 1 del x plus 2 into 1 minus r square and e to the power alpha q del t e to the power i beta p delta x minus e to the power alpha into q can be replaced by q minus 1 here. So, q minus 1 del t e to the power i beta p delta x.

So, if you just cancel some of these coefficient term. So, there itself, we can just get this equation as e to the power alpha del t this equals r square e to the power minus i beta delta x plus e to the power i beta delta x plus 2 into 1 minus r square minus e to the power minus alpha del t.

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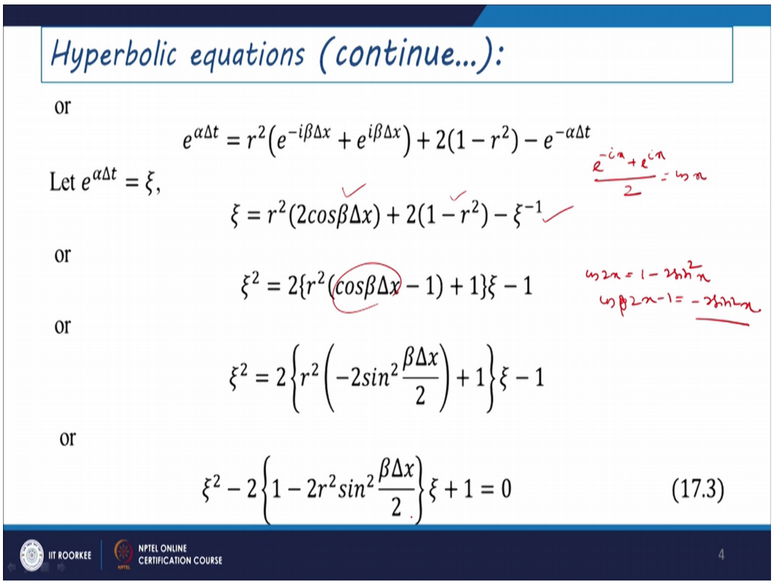

And if you will just put here e to the power alpha t goes to zeta here then theta can be written as r square into if you will just see here e to the power minus i x plus e to the power ix by 2 it can be written as like a cos x here. So, that is why is just written as if you will just see here 2 cos beta delta x. Since it is like my e to the power minus I beta delta x plus e to the power i beta delta x by 2. So, that is why two into cos beta delta x here plus 2 into 1 minus r square minus e to the power minus alpha del t. So, that is why zeta to the power minus 1 it is just (Refer Time: 07:04).

If you will just multiply zeta both the sides, then we can just often zeta square this equals to two into r square. So, if you will just see here. So, zeta is a multiplied means you can just consider this a 2 in 2, r square cos beta delta x minus 1 plus 1 into zeta minus 1. And if you will just linearize once more than j cos square this equals to 2 into r square, usually you have known that cos 2 x it can be written as 1 minus 2 sin square x.

So, that is why it can be written as like cos beta x sorry cos 2 x minus 1 this is like minus 2 sin square x. So, that is why if we will just replace here cos beta delta x. So, it can be written as like minus 2 sin square beta delta x by 1 plus 1 into zeta minus 1 there. So, this can be written as like zeta square minus 2 into 1 minus 2 r square sin square beta delta x by 2 whole into zeta plus 1 this equals to 0.

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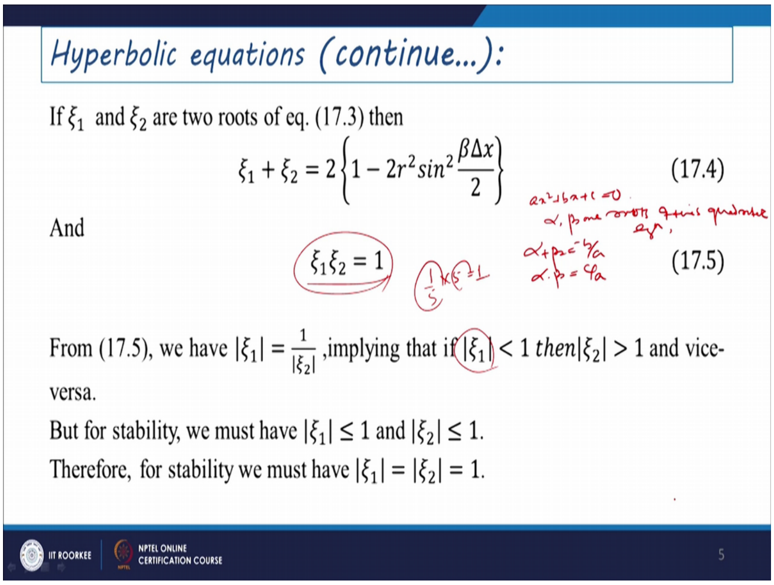

So, if we can just see here this equation is a quadratic equation. So, this contains two roots. Suppose these roots are in the form of zeta 1 zeta 2 and if zeta 1 and zeta 2 are the roots of this equation then we can just write zeta 1 plus zeta 2 that is nothing but you can just write minus b by a. So, if a equation in the form of ax square plus b x plus c equals to 0 usually if alpha and beta are roots of this quadratic equation, then we are just writing this alpha plus beta equals to minus b by a and alpha into beta this equals to c by a. So, that is why we can just write some of these roots as 2 into 1 minus 2 r square sin square beta delta x by 2 and zeta 1 into zeta 2 if you will just see this coefficient has 1, here this has coefficient 1. So, c by a means this is nothing, but 1 there itself.

So, from this equation if you will just see here since is zeta 1 in the zeta 2 equals to one we can just write absolute value of zeta 1 this can be written as a 1 by absolute value of zeta 2, implying that it must be absolute value of zeta 1 should be less than 1 then absolute value of zeta 2 is greater than 1. If you will just take in that form this means that 1 by 5 into 5 it should be 1 there. So, that why if 1 is less than 1 then other should be greater than 1 then we can just obtain this relationship and vice versa.

So, but for the stability we must have to get absolute value of zeta 1 should be less or equal to 1 and absolutely zeta 2 should be less or equal to 1. Therefore, stability we have to consider here absolute value of zeta 1 equals to absolute value of zeta 2 this equals to 1.

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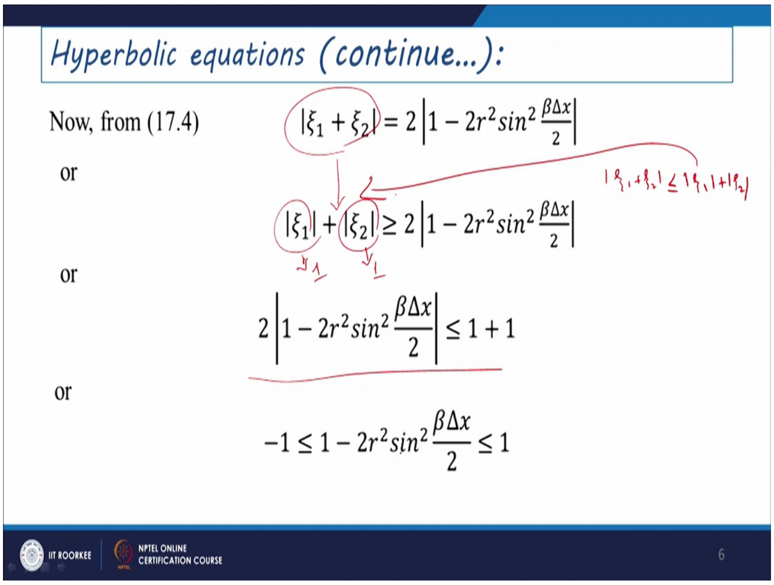

But if you will just put this relationship in the first relationship here then we can just get this relationship as absolute value of zeta 1 plus zeta 2 this equals to 2 into 1 minus 2 r square sin square beta delta x by 2. And if you will just see absolute value of zeta 1 plus zeta 2 always it is a greater equal to theta 1 plus theta 2 there itself. So, it is must be greater or equal to 2 into 1 minus 2 r square sin square beta delta x by 2 there. And obviously, we can just write this one as 2 into 1 minus 2 r square sin square beta delta x by 2 since zeta 1 is considered as 1 there and zeta 2 is considered as 1 there. So, that is why you can just write as 1 plus 1. So, I have just want to repeat this one since we are just considering absolute value of zeta 1 plus zeta 2 it is less or equal to 1 plus zeta 2 that is why this relationship establishes here.

So, if you will just take this inequality form here then we can just write this one since 2 can be divided here and we will have this inequality as minus 1 less or equal to 1 minus 2 r square sin square beta delta x by 2 it should be less or equal to 1 there. So, from the left inequality we can just write this one as r square if you will just see this 1 r square, it should be less or equal to 1 by sin square beta delta x by 2 and this means that r square should be less or equal to 1 since the maximum value of sin x would be lies between 0 and 1 there. So, that is why we can just consider in absolute form.

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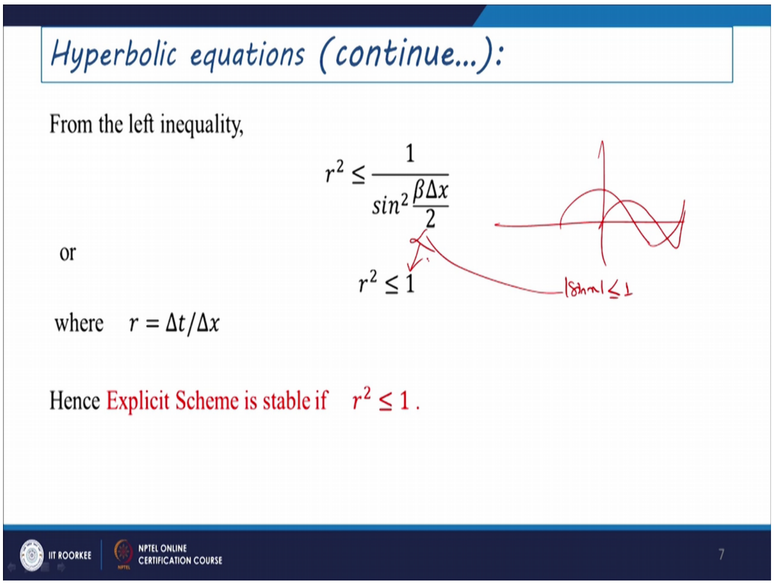

So, in this case we can just consider r square should be less or equal to 1 always for the convergence of explicit scheme. And especially if you will just see here that sin curve if you will just see. So, this is nothing but cos curve here. So, sign copy if we will just see this is just a lying between like 0 to 1. So, that is why sign of x is always less or equal to 1. So, that is why this relationship should be justified.

Hence we can just say that r is nothing but r del t by del x which is a stable if r square is a less or equal to 1 this means del t by del x whole square it should be always less or equal to 1 for the stability scheme to get the solution in explicit approach, or we can just say that whenever r square is less or equal to 1 we can just obtain the solution for hyperbolic equation using a explicit scheme.

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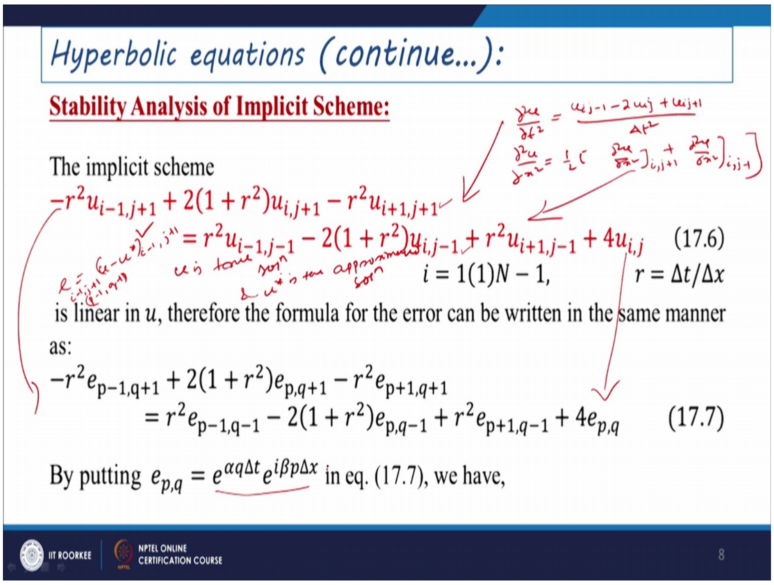

So, if you will just go for the stability analysis of a implicit scheme here the implicit scheme especially this is just written in the form of minus r square u i minus 1 j plus 1 plus 2 into 1 plus r square u i j plus 1 minus r square u i plus 1 j plus 1 this equals to r square u r minus 1 j minus 1 minus 2 into 1 plus r square u i j minus 1 plus r square u r plus 1 j minus 1 plus 4 u i j.

Especially what we are doing is in the implicit scheme we are just considering this is a central difference approximation for time derivative that is nothing but u i j minus 1 minus 2 u i j plus u i j plus 1 by del t square and for space derivative we are just considering this average of these derivatives at i j minus 1th level and i j plus 1 level. So, we are just considering this average of these two schemes there itself.

So, that is why we are just obtaining these schemes are in the form of if you will just see here j plus 1th level, one of these values we are just obtaining and j minus 1th some of these values we are just obtaining. So, that is why if you will just solve these two equations at like i j plus 1th level del square u by del x square plus del square u by del x square at i j minus 1th level. So, then we can just I have this relationship here, where i is varying from one to n minus 1 and j is varying from like your time scale since it is a varying we can just consider like 1 to m minus n minus 1 there.

So, especially if you will just see here this equations just gives you a linear representation of u therefore, the formula for the error can be written in the same manner as we have defined for this explicit scheme and this relationship can be written as a sincere. So, u is a solution and u star is the approximated solution, so that is why if you will just take e as u minus u star at the particular points then especially you can just write this one what i minus 1 j plus 1 point as p minus 1 and q plus 1 point which is defined at i minus 1 j plus 1 point there itself.

So, that is why it can be represented as a minus r square e of p minus 1 q plus 1 plus 2 into 1 plus r square e of p q plus 1 minus r square e of p plus 1 q plus 1 this equals to r square e of p minus 1 q minus 1 minus 2 into 1 plus r square e of p q minus 1 plus r square e of p plus 1 q minus 1 plus 4 e p q. Since i is a especially it is just replaced by the point p q there itself. And if u is the true solution and u star is the approximated solution in each of these points then we can just considered e as the error term at that points which is nothing, but u minus u star.

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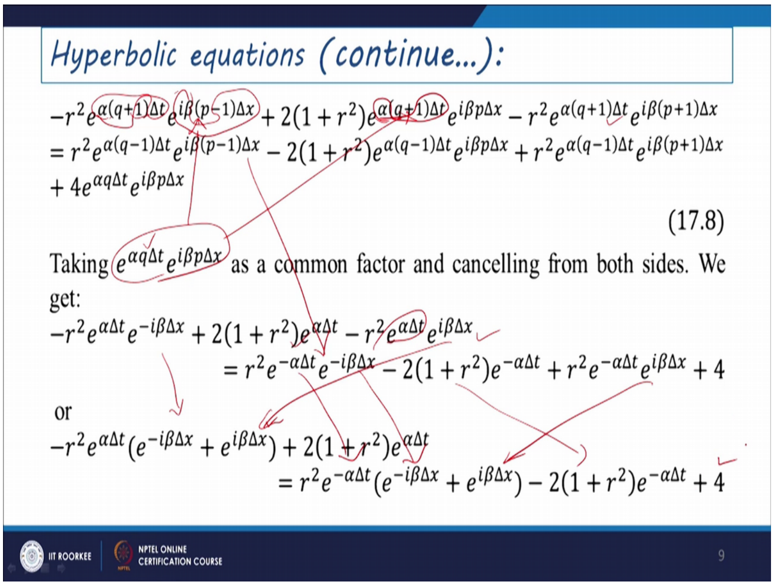

So, if you will just put here a of p q is equal to e to the power alpha q del t and e to the power i beta p delta x here then we will have this equation in the form like minus r square e to the power alpha into q plus 1 del t e to the power i beta p minus 1 delta x plus 1 minus 1 plus r square e to the power alpha into q plus 1 del e to the power i beta p delta x minus r square e to the power alpha q plus 1 del t e to the power i beta p plus 1 del x. This equals to r square e to the power alpha q minus 1 del e to the power i beta p minus 1 delta x minus 2 into 1 plus r square e to the power alpha into q minus 1 del t e to the power i beat p delta x plus r square e to the power alpha q minus 1 del t e to the power i beta p plus and delta x plus 4 e to the power alpha q del t e to power i p delta x; if you will just take common e to the power alpha q delta e to the power i beta p delta x from each of these storms here.

Then we can just get this equation is in the form of minus r square e to the power alpha del t since a alpha q delta t it has been taken common. So, that is why we will have here alpha del t only. And if you will just see the second term here since Ii beta p delta x it has taken common. So, we will have these terms like in this form here. So, e to the power minus i beta delta x plus 2 into 1 plus r square, similarly if you will just see here e to the power alpha del t it is just remaining over there and e to the power minus i beta delta x minus 2 into 1 plus r square e to the power. So, minus r square e to the power alpha del t i e to the power i beta delta x, if we will just see from this term here. And all other terms it will just followed in the same manner to get the compact form as we have just to consider in the earlier example.

Or if you will just take this one as minus r square e to the power alpha del t if you will just a common. So, this will be e to the power minus i beta delta x plus e to the power i beta delta x and plus 2 into 1 plus r square e to the power alpha delta t this is there. So, this term it has just taken since minus has alpha r square e to the power alpha del t is taken common. So, that is why this term is added it there, then the second side if you will just see here or the right hand side. So, it can be written as r square e to the power minus alpha del t. So, first term e to the power minus i beta delta x. So, it will be I beta delta x here. So, minus 2 into 1 plus r square e to the power minus alpha del t plus 4.

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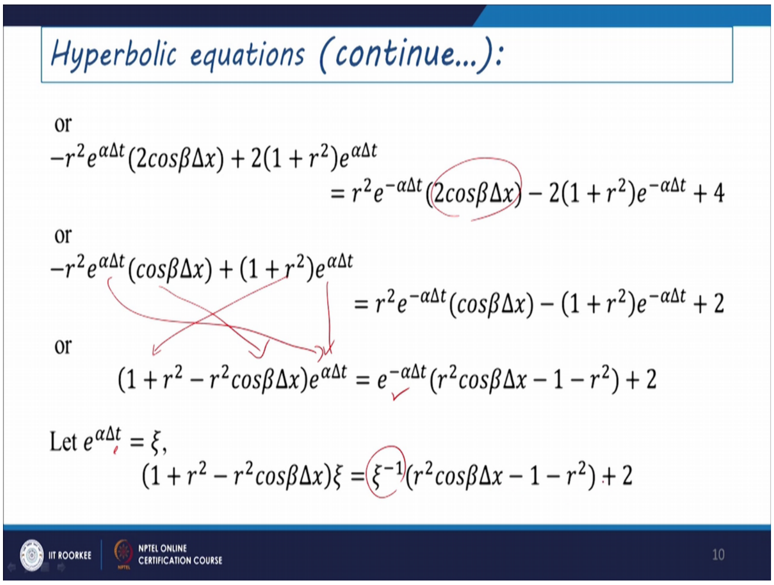

So, or if you will just calculate this one in a combined form here since e to the power minus i beta delta x plus e to the power i beta delta x it will just consider that will nothing, but it can just give you 2 cos beta delta x here. So, similarly we can write 2 cos beta delta x here also and remaining terms are as there itself.

And if you will just go for the further computation, 2 is common in each of this a terms. So, if you will just see here. So, that is why 2 can be taken out from both these sides and it can be cancelled and if you will just write this equation in a final form. So, this is just a taking as a 1 plus r square minus r square cos beta delta x e to the power alpha del t e to the power minus alpha del t r square cos beta del x minus 1 minus r square plus 2.

If you will just put here e to the power alpha del t equals 2 zeta here. We can just write this equation as 1 plus r square minus r square cos beta delta x into zeta this equals to zeta inverse since e to the power minus alpha del t it is there, so r square cos beta delta x minus 1 minus r square plus 2 here.

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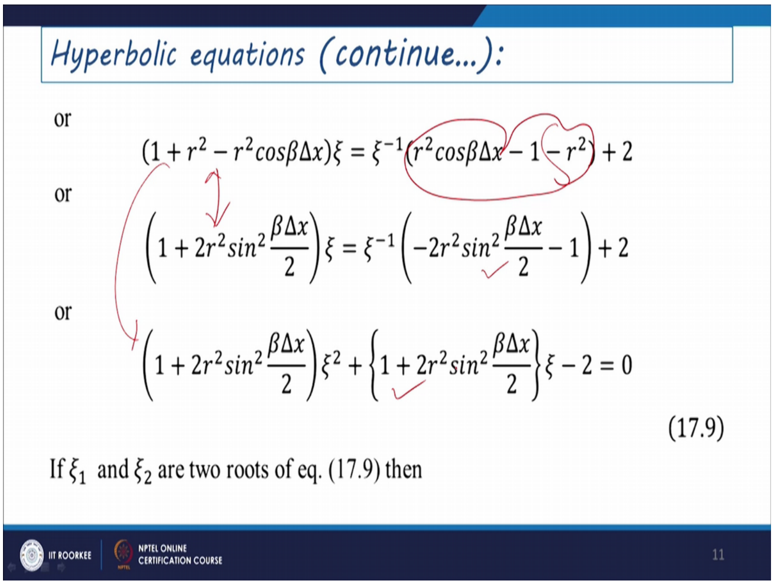

And if you will just multiply zeta at the both the sides we can just obtain this equation in a transport form as 1 plus 2 r square sin square beta delta x by 2 zeta square plus 1 plus 2 r square sin square beta delta x by 2 into zeta minus 2 equals to 0 here.

So, this is nothing but r square it will just take common here, 1 minus cos beta delta x that is why it is just taken as a 2 sin square beta delta x by 2 into r square. And here also same thing we have just on since you have just considering r square here. So, if you just consider this 1 minus r square common here or r square if you just take common here. So, this can be written as a minus 2 r square sin square beta delta x by 2 from the 2 terms here.

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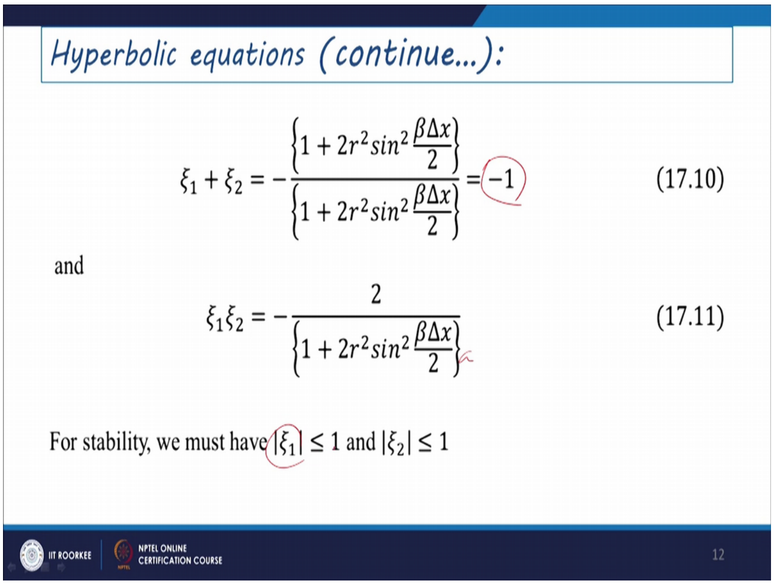

So, this represents a quadratic equation in a zeta if suppose 2 roots are there zeta 1 and zeta 2 then we can just write the sum of roots as minus b by a. So, that is it can be written as like minus of one plus 2 r square sin square beta delta x by 2 divided by 1 plus 2 r square sin square beta delta x by 2. This is nothing, but minus 1 here; and zeta 1 and zeta 2 this is nothing, but c by a. So, c is nothing, but here minus 2 by 1 plus 2 r square sin square beta x by 2 there is nothing, but a here.

So, for stability we have to consider zeta 1 absolute value it should be less or equal to 1 and zeta 2 absolute value it should be less or equal to 1. So, if you will just consider this condition there, we can just write absolute value of zeta 1 plus zeta 2 this equals to 1 since a absolute value you are just consider that are minus 1 can be 1 there implying that absolute value of zeta 1 plus zeta 2 it should be less equal to 1 here.

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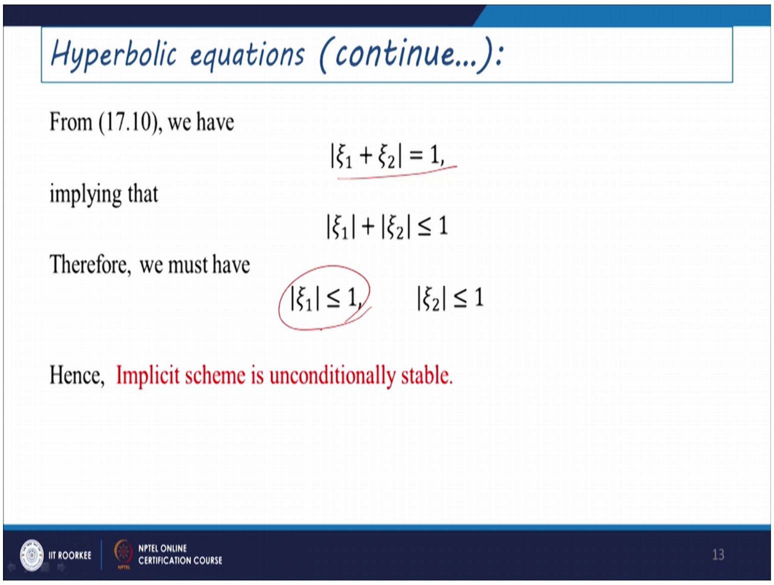

And if you will just go for like absolute value of zeta 1 it should be less or equal to 1 and absolute value of zeta 2 it should be less or equal to 1. If you will just put this condition in the second equation here that is a like zeta 1 plus zeta 2 that has a minus 1 there itself. So, we will have this equation that has a two conditions it should be satisfied absolute value of zeta 1 plus zeta 2 it is equals to 1, and zeta 1 zeta 2 it should also be like minus 2 by 2, 1 plus r 2 r square sin square beta delta x by 2. So, that is why we have like a absolute value of zeta 1 it should be less or equal to 1 and absolute value of zeta 2 it should be less or equal to 1. Hence the scheme is a unconditionally stable and especially we can just say that implicit scheme is a unconditionally stable for hyperbolic equations.

Thank you for listen this lecture.