Numerical Linear Algebra
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Lecture - 54
Householder QR factorization – II

Hello friends, welcome to this lecture. And here in this lecture we start we continue our study of QR decomposition. In if you recall in previous lecture we have discussed the QR decomposition of a square matrix.

It means that given a square matrix we need to find out we need to decompose a matrix a into Q into R where this Q is an orthogonal matrix and R is the upper triangular matrix. And we try to show in this lecture; we will show that, that such a representation of a square matrix is unique. And once and once the uniqueness is proven, then we are try to find out that how to find out the QR decomposition of a non square matrix that is a rectangular matrix.

So, that is the content of our this lecture. So, first let us consider the uniqueness problem. So, uniqueness problem means given a matrix A; which is a square matrix we have this representation Q into R.

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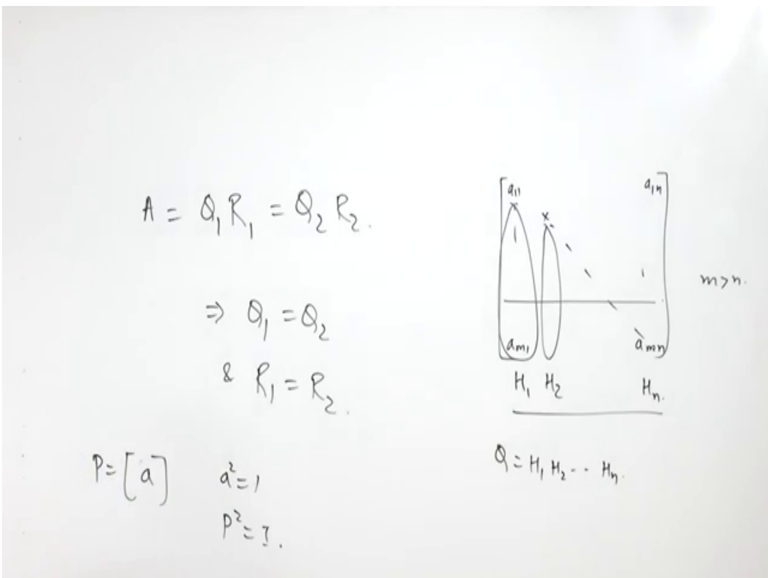

Now, we say that suppose we have two representation; Q 1 R 1 and Q 2 R 2. Then we want to show that this is possible only when, when Q 1 is equal to Q 2 and R 1 is equal to R 2. It means that Q 1 is same as Q 2 and R 1 is same as R 2. So, it means that QR decomposition of in square matrix is unique.

So, but before proving this uniqueness result; we need one result that is if a matrix is both upper triangular matrix and orthogonal matrix. Then P has the form. P square equal to I. So, that is our result which we are going to use to prove the uniqueness part.

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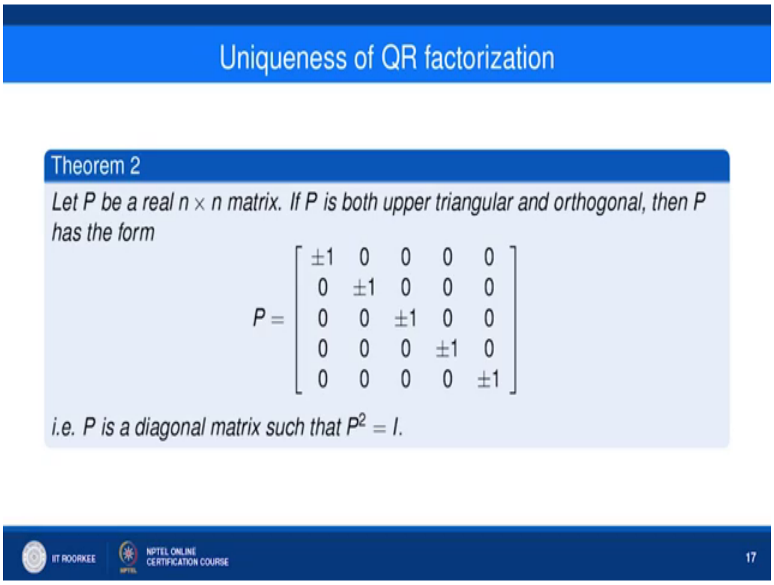

So, first let us prove this result; that if we have a; a let P be a real n cross n matrix; If P is both upper triangular and orthogonal matrix. Then P has a form this P is a scalar matrices a scalar matrix where a diagonal entries are plus minus 1 or you can say that P is a diagonal matrix such that P square is equal to I. And we try to prove this theorem with the help of mathematical induction, we can prove this without mathematical induction, but this is the best way to prove this theorem.

So, we prove this lemma by induction. So, if n equal to 1 this addition holds trevally. So, 1 means your P is nothing but this. P is equal to 1 here, if I consider this P as a some a here. Now, if P is upper triangle that is ok, but if it is orthogonal means your a square has to be 1. So, it means that P is an scalar matrix such that P square is your identity matrix.

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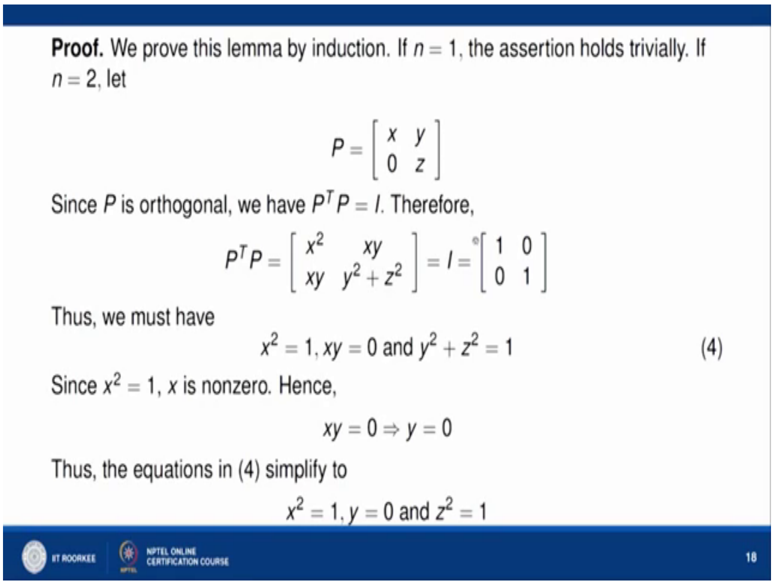

So, here for n equal to 1 this is correct. Now, let us says show that for n equal to 2. So, for n equal to 2, let P is equal to x y 0 z; because it has upper triangular matrix. So, that is why we have taken this form. And we want to show that this y entries is 0 and x and z is of the form that x square and z square is equal to 1.

So, here to show that this y is 0 or x and z the square root of 1. We use this result that P is an orthogonal matrix. So, P transpose P is given as I. So, let us calculate P transpose P. So, P transpose P is basically x square xy x y and y square z square. And we know that P transpose P is I that is 1 0 0 1. So, it means that x square is 1 x y is 0 and y square plus z square is equal to 1. So, we have these three relation that x square is equal to 1, x y is equal to 0 and y square plus z square is equal to 1.

Now, we have this x square is equal to 1. So, this implies that x cannot be a nonzero quantity. So, it means that if x is nonzero quantity; then x y equal to 0 possible only when, when we have y equal to 0. So, the first two relation gives you that x is the square root of 1 and y is equal to 0. So, once we have y equal to 0 then the last relation says that your z squared has to be 1. So, it means that this equation these three equation given as an equation number 4 is now, reduced to x square is equal to 1, y equal to 0 and z square equal to 1.

So, it means that the diagonal entries are either plus 1 or minus 1 and the off diagonal entries are simply 0. So, it means that for n equal to 2 we have shown that our result follows.

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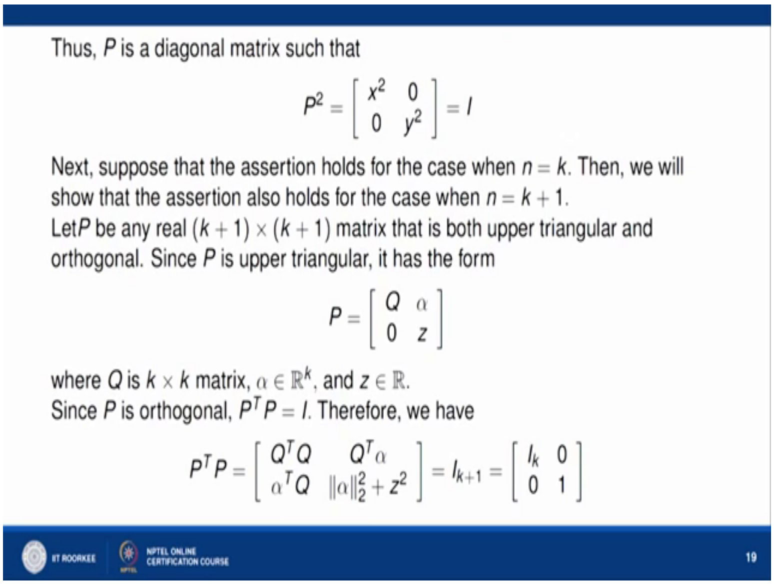

And let us assume our result is true for n equal to k. And we want to show that our assumption also hold for the k is when n equal to k plus 1.

So, using that if we have a matrix P which is both upper triangular matrix and a orthogonal matrix of size k cross k. Then P has to be a diagonal matrix such that P square equal to I; That we are assuming that it is true up to the size k cross k. Now, we want to show that this is also true for the size k plus 1 cross k plus 1.

So, let P be any real k plus 1 cross k plus 1 matrix that is both upper triangular and orthogonal matrix and we write down this P as Q alpha 0 z; where alpha is in vector in R k and z is just a real value and Q is k cross k matrix.

Now, since P is upper triangular it means that this Q is also an upper triangular matrix. Now, we want to see that this alpha is going to be a 0 matrix and this P can be written as a diagonal form. So, let us see how it is?

So, since P is orthogonal it means that P transpose P is equal to I. So, let us calculate P transpose P. So, P is given here, Q alpha 0 z, P transpose is given by Q transpose, alpha transpose 0 and z here because z is a real. So, as that transpose is same as z. Then we can calculate P transpose P and that is given by Q transpose Q, Q transpose alpha, alpha transpose Q and here we have alpha 2 square plus norm of alpha 2 square plus z square.

So, now we already know that P transpose P is at the matrix of size k plus 1. So, let me write it in this following form. Then if you compare this then Q transpose Q is nothing but I k and Q transpose alpha I is equal to 0 and 2 norm of alpha whole square plus z square is equal to 1. It means that we have these relations hold for as equation number 5.

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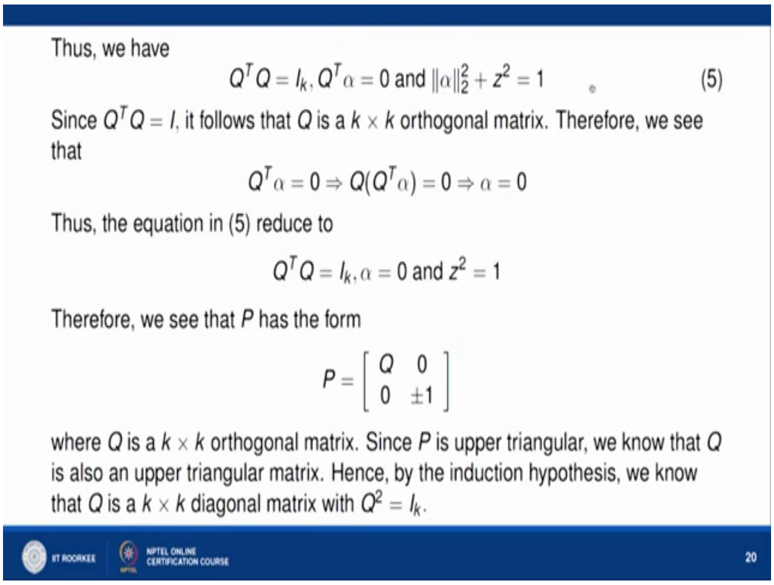

Now, since Q transpose Q is equal to I it follows at Q is k cross k orthogonal matrix. Therefore, we can say that once Q is orthogonal matrix it is nonsingular matrix also. So, then we can apply Q 1 Q transpose alpha equal to 0 and we can say that this is this will simply QQ transpose is I. So, we can say that this is nothing but alpha equal to 0.

So, using first second is reduced to alpha equal to 0. So, second will hold only when, when alpha is equal to 0. Then when alpha is equal to 0 and look at the last one and last one says that your z square has to be 1. So, it means that Q transpose Q is equal to I identity alpha equal to 0 and z square equal to 1.

So, now look at your P here. So, now, P will reduce to be reduce to Q 0 0 plus minus 1. Now, Q is what? Q is k cross k orthogonal matrix and since P is upper triangular matrix. So, Q has to be upper triangular matrix. So, it means that Q is a matrix of size k cross k. And it is both upper triangular matrix and orthogonal matrix. So, it means that by assumption Q has to has to satisfy this assumption that Q square equal to I k.

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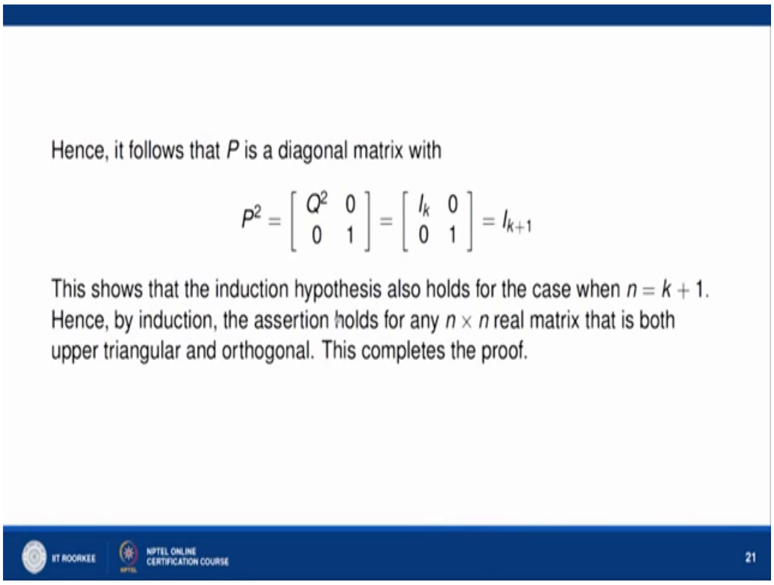

So, we can say that, P square if we calculate P square then it is nothing but Q square 0 0 1 and since Q square is equal to I k. So, we can write this as I k 0 0 1 and this is nothing but your I k plus 1. So, it means that our P is a diagonal matrix with the P square is equal to I k plus 1.

So, what we have shown here we have shown that our assumption which is which we have assumed is true only for k cross k matrices; Now, also valid for k plus 1 cross k plus 1. So, that shows that this our assumption is true for all values of n. It means that if we have a matrix of size n cross n and it is both upper triangular and say orthogonal matrix.

Then it is a it is a ortho it is a diagonal matrix; such that P square is equal to identity matrix and that proof that completes the proof of this result. And using this result now we want to prove the uniqueness of the QR decomposition of a given matrix. So, let A be a real n cross real nonsingular matrices nonsingular matrix and let A equal to Q R; where Q is orthogonal matrix and R is an upper triangular matrix.

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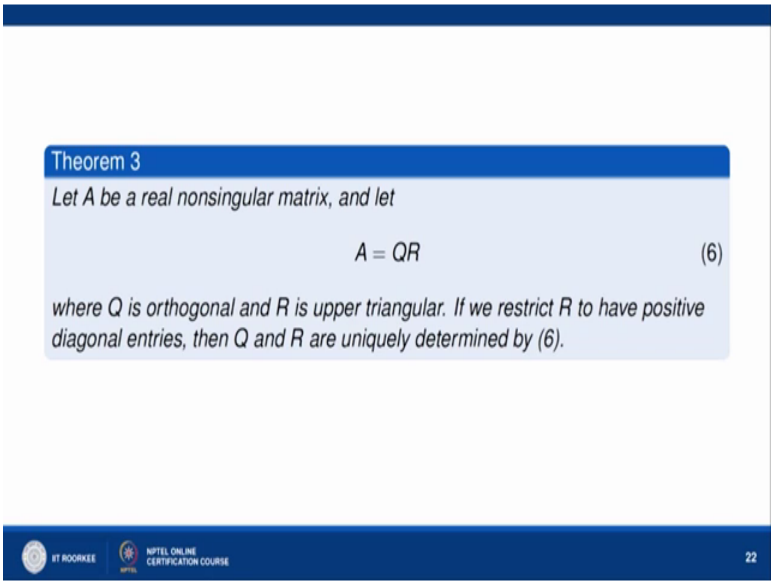

Now, if we restrict our R to have only positive diagonal entries, then Q and R are uniquely determined by 6; why we are restricting to positive diagonal entries. Because if you look at here we have P is a diagonal entries diagonal matrix such that P square is equal to y. So, it means the diagonal entries are either plus 1 or minus 1.

So, if we restrict it to only plus 1 values; then P is your nothing but identity matrix. So, we are taking this as assumption that R have only positive diagonal entries. Then we want to show that Q and R are uniquely determined by 6.

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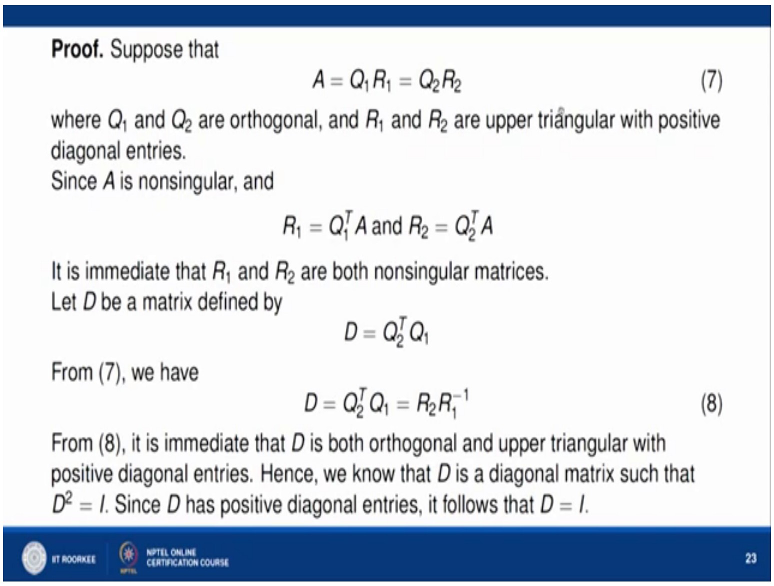

So, let us assume that we have two decomposition of A QR decomposition of A. So, A as Q 1 R 1 equal to Q 2 R 2.

So, now once we have Q 1 R 1 equal to Q 2 R 2 and both Q 1 and Q 2 are orthogonal matrix and R 1 and R 2 are upper triangular matrix. Now, we know that A is a nonsingular matrix. So, it means at R 1 and R 2 both are nonsingular matrices. So, I can write R 1 as Q 1 transpose A. Similarly, we can write R 2 as Q 2 transpose A.

And if we use the these two relation Q 1 R 1 equal to Q 2 R 2. Then we can write this as Q 2 you just multiply by Q transpose, then Q 2 transpose Q 1 equal to R 2 R 1 inverse. So, that, that D equal to Q 2 transpose Q 1 and then we have the representation D written as Q transpose Q 1 are equal to R 2 R 1 inverse.

So, using this if we define D as Q 2 transpose Q 1. Then it is also equal to R 2 R 1 inverse. Now, since Q Q 2 and Q 1 are orthogonal. So, it means that D is going to be orthogonal matrix. And since R 1 and R 2 are upper triangular matrices; So, it means that D is also an upper triangular matrix. So, now, D is both orthogonal and upper triangular matrices. And not only this it has only positive diagonal entries. Why? Because in R 2 and R 1 both are having positive diagonal entries; So, R 2 R 1 inverse will also have positive diagonal entries.

So, now; So, D is what? Now, D is having orthogonal D is orthogonal D is upper triangular and the entries in the diagonals are only positive entries. So, using the previous result, we know that D is a diagonal matrix. And not only it is a diagonal matrix; it is equal to I. Why? Because D square is equal to I and since it has only positive diagonal entries it has to be 1 only. So, it means that D has to be identity matrix.

So, if D is equal to identity matrix means Q transpose Q 1 is equal to I and R 2 R 1 inverse is equal to I. And it means that, Q 2 is equal to Q 1; if we use this relation and if you use this and this then this simply says that R 2 is equal to R 1.

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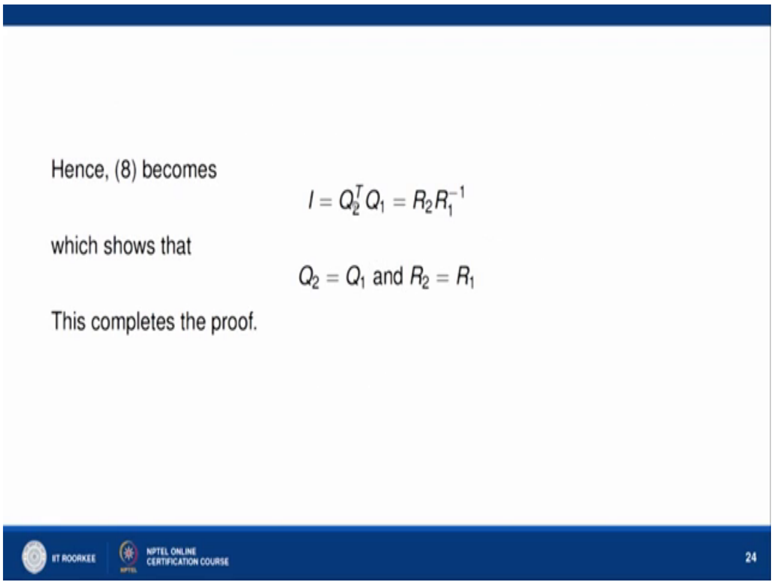

So, we have shown that, Q 2 is equal to Q 1 and R 2 equal to R 1. That proves the uniqueness of QR decomposition. So, it means that once we have this uniqueness result then we are show that whatever method we apply we must have only 1 QR decomposition method.

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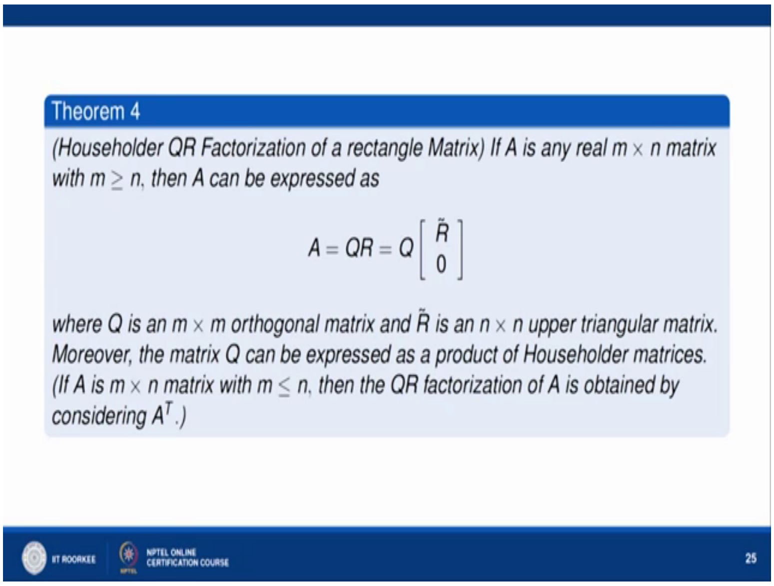

So, now we proceed further and we try to find out the QR factorization Householder, QR factorization of a rectangular matrices. So, here if a is any real m cross n matrix with the m greater than or equal to n, it means that we have less number of columns than the rows here. Then this a can be expressed as QR; where Q is the orthogonal matrix of size m cross m and this R is an n cross n upper triangular matrices upper triangular matrix of this kind.

Moreover the matrix Q can be expressed as a product of Householder matrices. And now suppose we here we started with the m cross n matrix, where m is greater than or equal to n. Then we have this representation, but if we have size of A as m cross n with m is less than or equal to n. Then we can consider this theorem for a A transpose.

So, let us consider this theorem 4 and once we are done with this theorem 4 then we can also deal with the problem of matrix where your m is less than or equal to n. It means a number of rows are strictly less than number of column. So, let us have the proof of this theorem for which is very short. In fact, it is similar to the proof of QR factorization of a square matrix. So, we can make it short.

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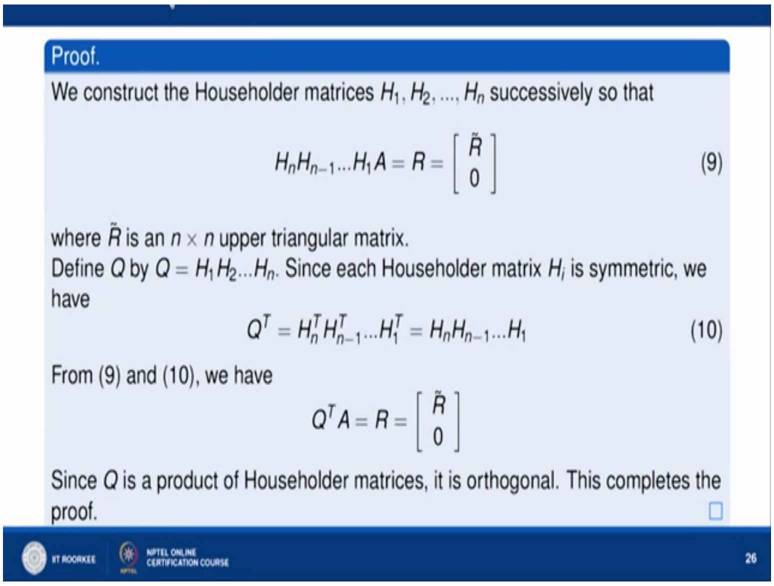

So, we construct the Householder matrix H 1 to H n successively. So, that H 1 H n H n minus 1 H 1 operate when we operate on A, then it is R. Now, here R we have R tilde 0. Now, if you look at let me write it here, here we have more number of rows. So, here we have m rows. So, here we have A 1 1 to A m 1 and here we have A 1 and here we have A m n. So, it means that m is greater than n.

So, if you recall when we try to find out this QR decomposition of a given matrix A. What we try to do here? we find out H 1, such that all the entries in the diagonal terms are simply vanish. And in H 2. What we do? we look at this term and all these entries we want to make it 0. And up to how many how much how many steps we need to go, here we go up to n step that is we need to go up to find out H 1 up to H n.

So, it means that here in this proof we need to go to n step. And this Q is going to be H 1 H 2 up to H n right. And here if you look at it may happen that some entries are zeros here and it still there are some something nonzero right.

So, here this R have this form R tilde zeroes and R tilde is an n cross n upper triangular matrix. And we can define Q by H 1 H 2 H n. And since each Householder matrix Hi is symmetric, we can say that Q transpose is nothing but Q transpose is given by H n transpose, H n minus 1 transpose, H 1 transpose.

And since each are symmetric matrix. So, Q transpose is also equal to Q. And Q trans since each 1 is orthogonal matrix. So, Q is also going be orthogonal matrix. So, we can write that Q transpose A as R this is nothing but Q transpose. So, Q transpose into A equal to R. So, we can write A as Q into R is it ok.

So, this completes the proof. So, it means that here we have nothing more to proof, here we simply follow the proof of the QR decomposition method. The only thing here is that here we can go only up to the n places, because we have only n columns and m is bigger than n. So, it means that it may happen that there are certain bottom rows are simply 0 rows.

So, here we define Q by H 1 H 2 H n. And since each Householder matrix Hi is symmetric, we can find out Q transpose as H n transpose, H n minus 1 transpose, H 1 transpose. And which is nothing but since H n transpose is H n, H n minus 1 transpose is H n minus 1. So, Q transpose is given by H n, H n minus 1 to H 1.

So, using this if you look at the equation number 9; then it is nothing but Q transpose A into R right. And since we know that each one each H 1 to H n are orthogonal matrix; because it is Householder matrices. So, it is orthogonal matrices also. So, Q is going to be orthogonal matrix.

So, it means that we can write this Q transpose A equal to R and since Q is orthogonal matrix. So, we can apply Q here, then I can write a as Q of R. So, that complete the proof of this theorem. And it is basically this that here we have m cross n matrix and since we have only n columns. So, we can find out only H 1 to H n. So, we Q we can write it H 1 to H n is it ok.

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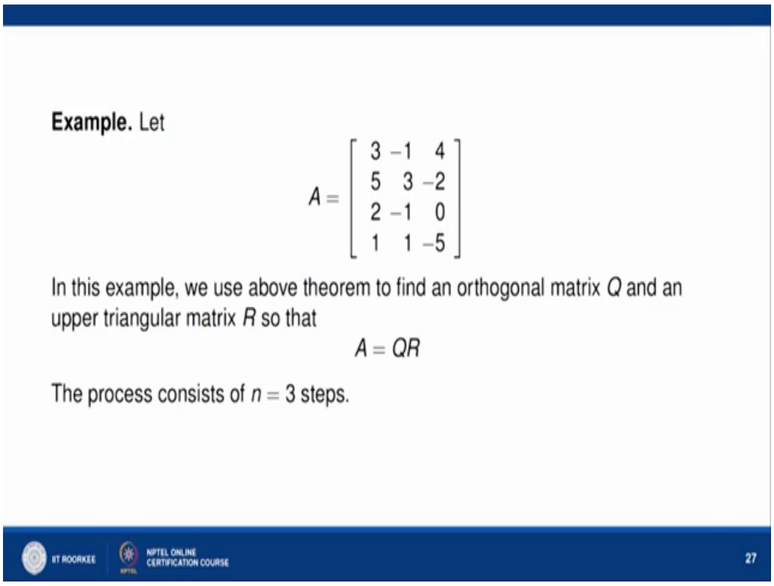

So, now, let us consider one example and try to see how to find out QR decomposition of this example. So, if we have let us find out the QR decomposition of this. Now, here a size is 4 cross 3. So, we can go up only up to the 3 steps; So, n equal to 3 steps.

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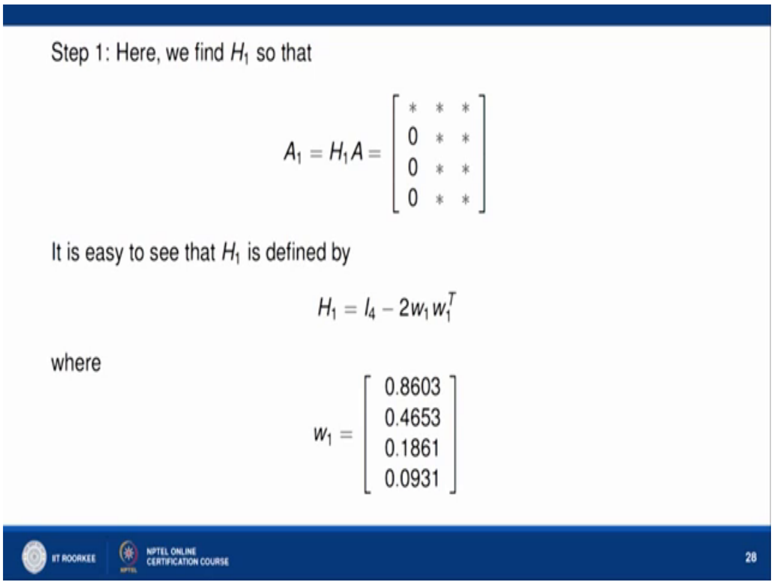

So, for that, first we need to find out H 1. Now, what is the work of H 1. That if we apply H 1 on A. Then in the first step; we should have in the first column all the entry is below the diagonal term has to be 0. And that we can achieve by using the algorithm. And that we can say that H 1 equal to I 4 minus 2 omega 1 omega 1 transpose. To find out this omega 1 we are using the algorithm and we say that w 1 omega 1 is given by this quantity.

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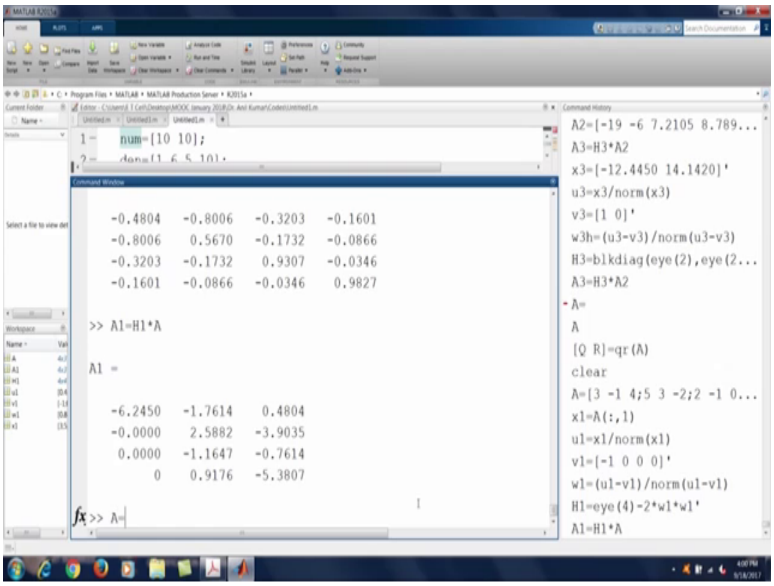

So, let us see what is going on. So, let me use MATLAB here. So, for that let us define your matrix A as this that is the matrix H A we are using 3 minus 1 4 5 3 minus 2. So, here we are using this 3 minus 1 4 5 3 minus 2. So, that is what we have written here.

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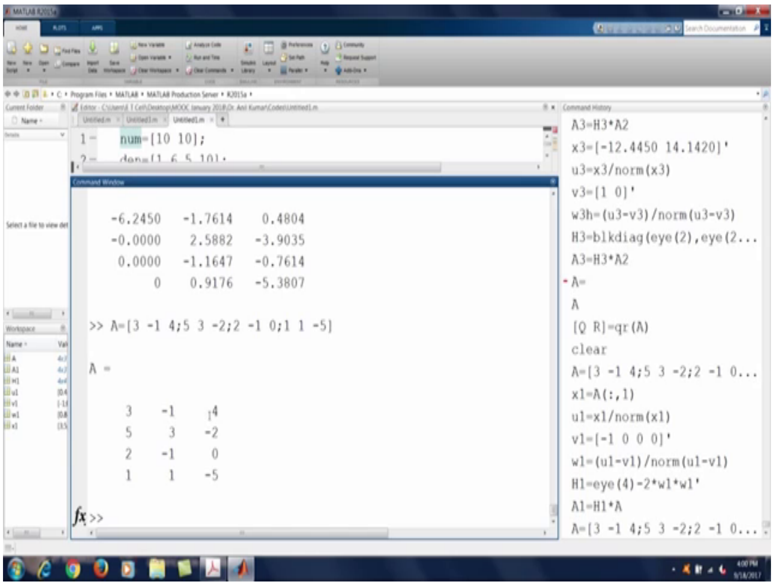

So, once we have A. Then we need to find out H 1 what is doing that it makes the first column as something which is minus 1 0 0 0 that we are going to consider. So, for that first write down what is x 1. So, let me write it x 1 here. So, x 1 is equal to we write the first column of matrix A; that is 3 5 2 1, but x 1 is a not a unit vector. So, we define a unit vector as u 1. So, let us call this as u 1 v u 1 is given by x 1 by norm of x 1. So, u 1 we can find out.

Similarly, for algorithm we need to find out v 1. So, v 1 is minus sign of the first term is here 3; So, minus sign of 3 that is 1. So, it means that v 1 is nothing but minus 1 0 0. So, let me write it here it is minus 1 0 0 0. So, v 1 is going to be this.

Now, u 1 we have v 1 we have and both are unit vector. Then algorithm says that w 1 we can define as u 1 minus v 1 sorry this w 1 is given by u 1 minus v 1 divided by norm of u 1 minus v 1. And if you calculate it is coming out to be 0.8603 0.4653 0.1861. And it is if you look at here we have w 1 as this.

So, here we are using algorithm to find out w 1. And once we have w 1 then H 1 is nothing but I 4 minus 2 omega 1 omega 1 transpose that also I am writing here. So, here we can write H 1 as I 4 this represent the identity matrix of size 4 cross 4 minus 2 into w 1 into w 1 transpose. So, if you calculate sorry it is not I 1 I H 1 is going to be this.

So, once we have H 1 then we can find out A 1 as H 1 into m and it is coming out to be this form. So, A 1 is minus 6.2450 and so on. And if you look at it is matching with this. So, A 1 is H 1 A which is minus 6 minus 6.2450 it is matching here.

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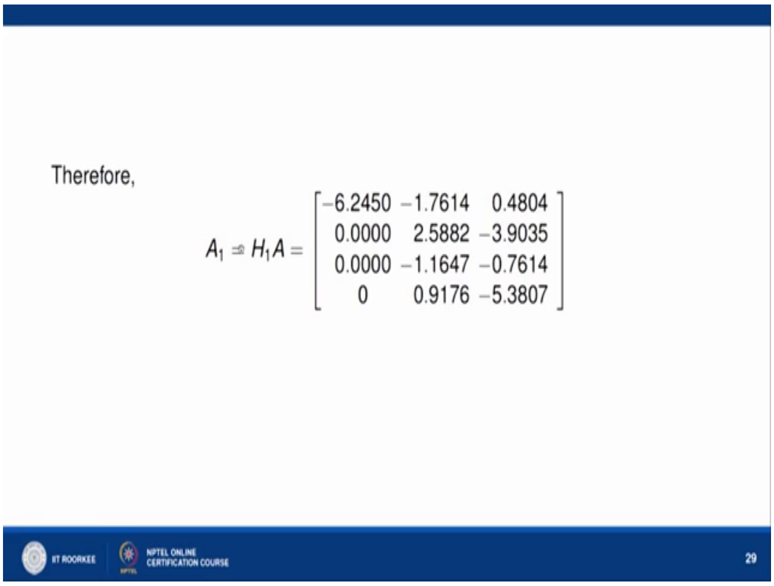

So, we can find out A 1 , but here if you look at here we have this entries in first column we have these entries below the diagonal term it is 0 , but if you look at the second column it is not 0; these two still are nonzero.

So, we need to go further we need to find out H 2. And to find out H 2 we write A 2 as H 2 into H 1; where A 2 which is given as H 2 into A 1; where H 2 is this, A 1 is given by this. It must be of this form. Here form is what that in first diagonal the first column entries below the diagonal term it is 0 and in second column entries below the diagonal term it is 0.

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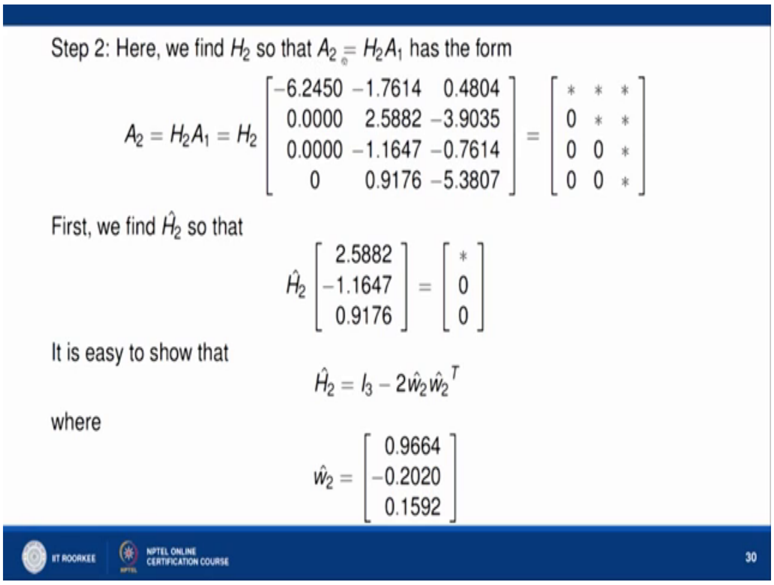

So, we need to find out H 2, such that it makes these entries this thing to this form right. So, here H 2 hat will first to define H 2 we need to define first of all what is H 2 hat? So, H 2 hat will give you what? look at the second column. This diagonal entries is 2.5882 let me write it here 2.5882 minus 1. So, here we have considered this. And we want to write this into this form.

Now, here what is this nonzero quantity. Since this entry is plus 1. So, this entry is going to be minus 1. So, minus 1 0 0; So, we need to find out H 2 hat which map this 2 minus 1 0 0. So, now, we again apply our algorithm and we can write H 2 hat as I 3 minus 2 omega 2 hat omega 2 hat transpose.

Now, how to find out this omega 2 omega 2 hat. Then we use algorithm this is our x, this is your y. So, now, to find out x we can define u 2 that is x 2 divided by norm of x 2 and v v 2 that is minus 1 0 0. And your w 2 is going to be u 2 minus v 2 divided by norm of u 2 minus v 2. And we can write this as W 2 hat. And here I am not doing this calculation you can do this. So, again W 2 hat is what? u 2 that is this divided by norm of this minus v 2 that is minus 1 0 0 divided by norm of u 2 minus v 2. So, W 2 hat is given to us.

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Thus, H_2 is defined as

$$H_2 = \begin{bmatrix} I_1 & 0 \\ 0 & \hat{H}_2 \end{bmatrix}$$

i.e. $H_2 = I_4 - 2w_2w_2^T$, where

$$w_2 = \begin{bmatrix} 0 \\ \hat{w}_2 \end{bmatrix} \in \mathbb{R}^4$$

Therefore, we have

$$A_2 = H_2A_1 = \begin{bmatrix} -6.2450 & -1.7614 & 0.4804 \\ 0.0000 & 2.99829 & 4.7451 \\ 0.0000 & 0.0000 & -2.5695 \\ 0.0000 & 0.0000 & -3.9561 \end{bmatrix}$$

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Now, once we have W 2 hat; we can define H 2 that is I 1 0 0 H 2 hat and we already know that H 2 is given by I 4 minus 2 omega 2 omega 2 transpose where omega 2 is this. So, it means that H 2 is going to be an Householder matrix here. Then if we operate H 2 on A 1 then A 2 will be of this form.

So, here if you look at here in the last column here let us this entry below the diagonal term is 0, here this is upper triangular matrix. So, here we can say that diagonal term is this minus 2.5695. So, here we have to find out another Householder matrix say H 3; which make these two term as nonzero term into and zero term.

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Step 3: Here, we find H_3 so that $A_3 = H_3A_2$ has the form

$$A_3 = H_3A_2 = H_3 \begin{bmatrix} -6.2450 & -1.7614 & 0.4804 \\ 0.0000 & -2.9829 & 4.7451 \\ 0.0000 & 0.0000 & -2.5695 \\ 0.0000 & 0.0000 & -3.9561 \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$$

First, we find $\hat{H}_3$ so that

$$\hat{H}_3 \begin{bmatrix} -2.5695 \\ -3.9561 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

It is easy to see that $\hat{H}_3 = I_2 - 2\hat{w}_3\hat{w}_3{}^T$, where

$$\hat{w}_3 = \begin{bmatrix} -0.8788 \\ -0.4771 \end{bmatrix}$$

Thus H_3 is defined by

$$H_3 = \begin{bmatrix} I_2 & 0 \\ 0 & \hat{H}_3 \end{bmatrix}$$

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So, again we go to find out H 3. So, H 3 we will find out such that A 3 which is given as H 3 into A 2 have the following form. A 3 is H 3 A 2 equal to H 3 A 2 is given by this. And this has the form that the entries in the third column below the diagonal term, is going to be 0.

So, how do you find out this H 3 hat look at the factor of H 3 hat. H 3 hat make these two term this two term into something nonzero into zero. Now, look at the something nonzero, here the sign of this is minus 1. So, it is going to be 1 0. So, H 3 hat will map this two 1 0.

Now, again since here we can use the algorithm. So, here it is some x 4 we can call this x 3 here; let us say x 3 here. So, it is a non unit vector. So, we define u 3 as this x 3 divided by norm of x 3, v 3 as 1 0 and w 3 hat as u 3 minus v 3 divided by norm of u 3 minus v 3. So, that gives you w 3 hat. Once we have W 3 hat your H 3 hat is given by I 2 minus 2 omega 3 hat omega 3 hat transpose.

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i.e. $H_3 = I_4 - 2w_3w_3^T$, where

$$w_3 = \begin{bmatrix} 0 \\ \hat{w}_3 \end{bmatrix} \in \mathbb{R}^4$$

Therefore, we have

$$R = A_3 = H_3A_2 = \begin{bmatrix} -6.2450 & -1.7614 & 0.4804 \\ 0.0000 & -2.9829 & 4.7451 \\ 0.0000 & 0.0000 & 4.7174 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

and

$$Q = H_1H_2H_3 = \begin{bmatrix} -0.4804 & 0.6189 & 0.2743 & 0.5576 \\ -0.8006 & -0.5330 & 0.1937 & -0.1935 \\ -0.3203 & 0.5244 & -0.4948 & -0.6145 \\ -0.1601 & -0.2407 & -0.8015 & 0.5235 \end{bmatrix}$$

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And then we can define W 3 omega 3 as 0 omega 3 hat. And if we calculate I 4 minus 2 omega 3 omega 3 hat then it is coming out to be H 3; where H 3 is of this form block diagonal matrix I 2 0 0 H 3 ha. So, it means that H 3 is since H 3 is written as I 4 minus 2 omega 3 omega 3 hat and omega 3 is of unit vector because omega 3 hat is going to be unit vector. So, it means that H 3 is a Householder matrix.

Now, if we apply H 3 on A 2, then you look at that here it will be a and upper triangular matrix form. And once we have if you look at the diagonal entries, in diagonal entries any entries below the diagonal terms is all 0. So, it is achieved here. So, it means that here we stop and we call this matrix as R. R is nothing but A 3. So, A 3 is H 3 A 2 and A 2 is H 2 A 1 and A 1 is H 1 A 1. So, we can write R as and this thing and Q is given by H 1 H 2 H 3 and it is like this here.

So, here we have achieved QR decomposition of a non square matrix that is a rectangular matrix. And if we have a rectangular matrix whose number of rows is less than number of columns, then we try to find out the QR decomposition for the transpose of that matrix. And again we can find out this Q and R using your MATLAB command; QR as QR of QR of A. So, that will give you your R that is minus 6.4250 minus 1 this is R we have achieved here. And we already know that it is unique. So, uniqueness is also followed from this is it ok.

So, here I will stop here our lecture. So, in this lecture what we have seen we have seen QO QR decomposition is a unique one. And we can extend this result to non square matrices. So, it means that QR decomposition of a non square matrix that is the rectangular matrix also we can find out with the help of the proof of the QR decomposition of a square matrix. So, in that we have done in this lecture.

So, in next lecture we will try to see some application of QR decomposition. So, here that is we stop and

Thank you for listening us, thank you.