Numerical Linear Algebra
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Lecture – 50
Least square solutions- II

Hello friends, welcome to this lecture. In this lecture, we will continue our study of finding the least square solution of the system A x equal to b and if you recall, we have discussed this that if we have a system A x equal to b where A is n cross n matrix and x is n cross 1 and b is and cross 1 matrix and if a has full rank, then we have a unique solution x as the given as x equal to A inverse b.

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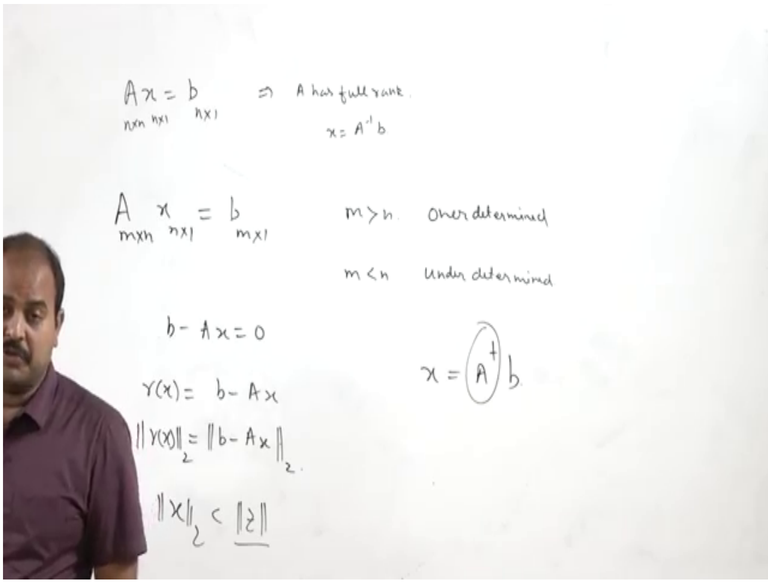

But the problem arises when we do not have say a square matrix or we can say that we have m cross n matrix x is n cross 1 and we have b where it is belonging to m plus 1. Now when m is greater than n, then we call this system as over determine system of linear equation, we write it over determined and if m is less than n, then we call it under determined or less determined.

And here, we can see that; this means that we have more number of equation; and then the number of unknowns so, in general, in the case of over determined system of linear equation, we generally do we; we do not have any solution and when m is less than n it means that we have less number of equation and unknown numbers are more and in this case, we see that we may have generally more than 1 solution or in both the cases, we can say that in general, we do not have any exact solution. So, in the absence of exact solution, we want to see that; how our solutions are say good.

So, in that case, we simply say that the solution means that b minus A x is equal to 0. So, exact solution means you are the vector x is said to be exact solution if b minus A x is equal to 0, but if it is not then we find out the residual b minus A x. So, for every x, we have a residual here and we try to see we want to find out the vector x such that this r x is minimum. So, we say that that has minimum a square solution.

So, now let us consider the two norm and we say that any vector which minimizes, this residual vector is known as least square solution; minimum least square solution. Now, it may have more than one least square solution on the same problem, then we say that we have a vector x is known as minimum norm least square solution, if norm of x is less than norm of all other least square solution of the same system.

So, we would in this lecture, we want to see that the minimum norm least square solution can be given as x as A dagger b and this A dagger is the pseudo inverse of a matrix A and in previous lecture, we have seen the properties which this A dagger satisfy and that property; those properties are known as moon point rows condition of on the pseudo inverse.

And so, this we have done in previous lecture. Now, let us consider the theorem which will help us to find out the properties of inverse pseudo pseudo inverse of A.

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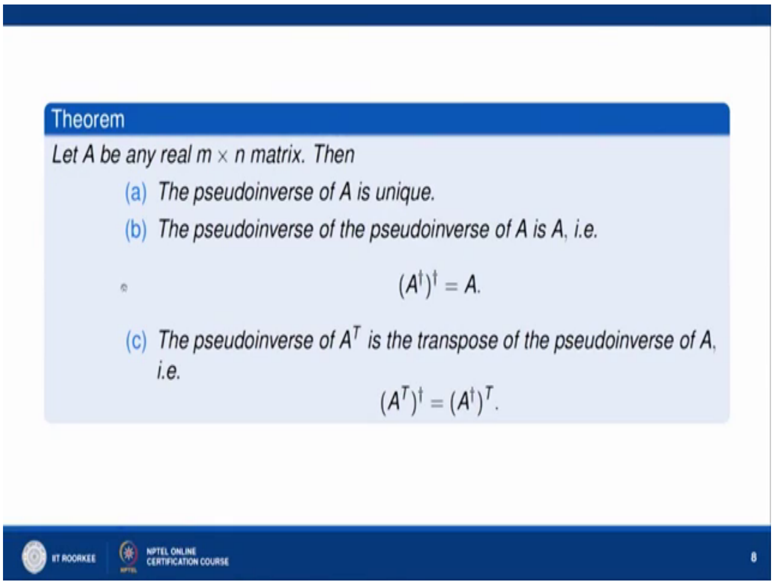

So, let A be any real m cross n matrix, then the pseudo inverse of is unique that is very very important thing that if we somehow We are able to find out some matrix satisfying these 4 properties and the fact that pseudo inverse of A is unique, we can say that that matrix is going to be.

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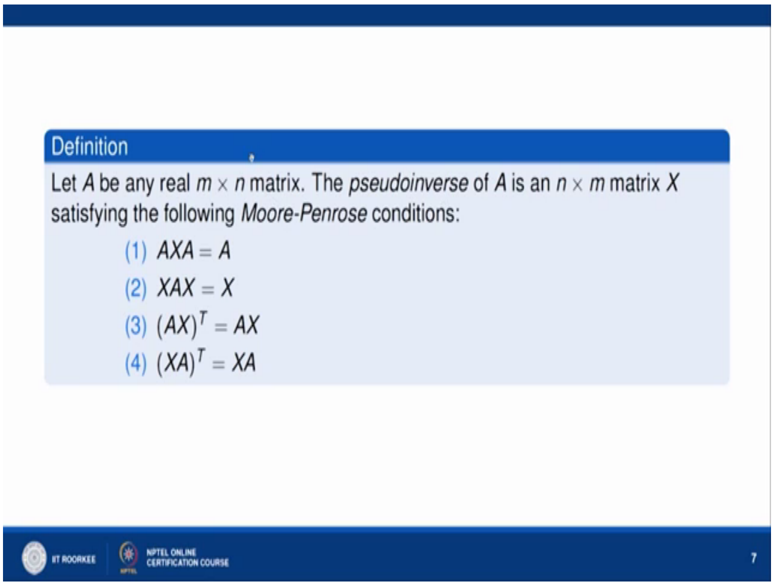

So, pseudo inverse of the given matrix and this b part the pseudo inverse of the pseudo inverse of A is A itself. So, it means that if you find out pseudo inverse of A dagger, then it is coming out to be A itself.

And the c part is that the pseudo inverse of A transpose is the transpose of the pseudo inverse of A and AA transpose dagger is equal to A dagger transpose and if you look at these properties are some properties which are similar to the properties of inverse in the case of square matrix. So, let us prove these this theorem these results.

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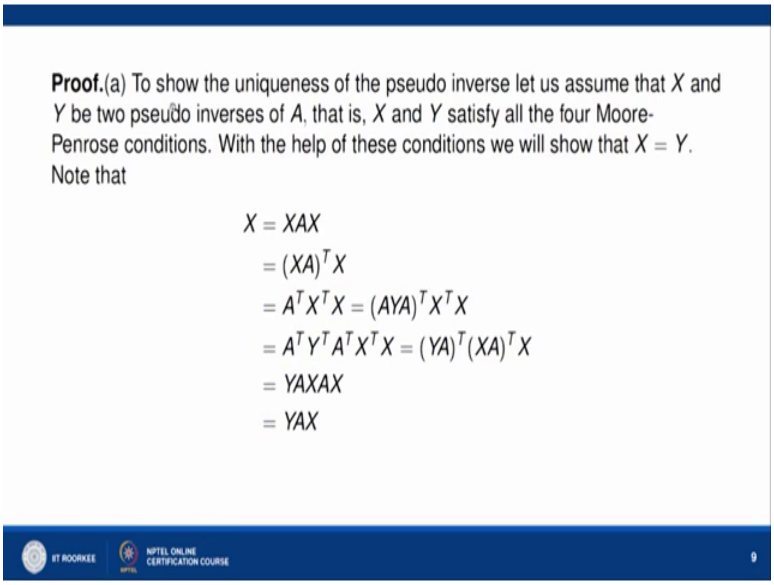

So, to show that unique uniqueness of the pseudo inverse, let us assume that X and Y be two pseudo universes of A; that is X and Y satisfy all the 4 properties listed as a more Moore-Penrose conditions. Now with the help of these conditions, we want to show that X is equal to Y. So, here we write X as XAX that is the second condition.

Now, now here we write XA here. Now, we know that this XA is a symmetric matrix. So, it means that XA can be written as XA transpose X. Now, we simplify this as A transpose X transpose X. Now here, we rewrite this matrix A with the help of this Moore-Penrose pseudo inverse Y. So, we can write A as in terms of Y as A as AYA. So, in place of A; we are writing AYA. So, this is this term is written as AYA transpose X transpose X and if you simplify, you will get A transpose Y transpose A transpose X transpose X.

Now, if you look at; we simplify this the first two terms can be written as YA transpose; the next two term can be written as XA transpose and we already know that since Y is A pseudo inverse, then YA is symmetric and XA is also symmetric by the same property that X is also a pseudo inverse. So, YA transpose can be written as YA and XA transpose can be written as XAX. Now here I can write AXA as A. So, YAXAX can be written as

YAX. So, it means that X can be written as Y of A X if Y and X; both are pseudo inverse of the matrix m. Similarly, here if you look at we have simplify we have change this XA as XA transpose, but rather than doing in place of XA, we are writing XA it transpose, if we look at this; A X as A X transpose.

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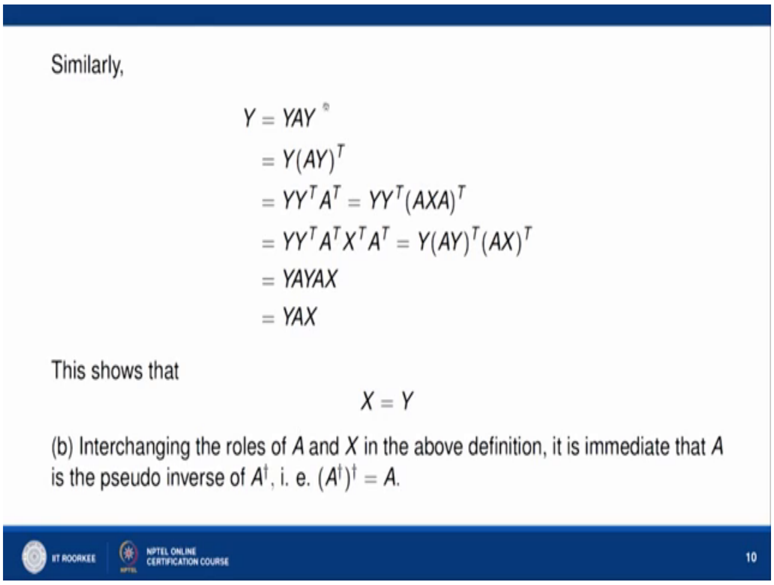

Because A X is also symmetric, then what we can get; so, move on to this Y and we write Y as YAY as we started with x as XAX. So, we write Y as YAY. Now, rather than looking this Y A, let us look at this A Y. So, since A Y is also symmetric, we write A Y transpose and if you simplify; we write YY transpose A transpose. Now, again as we replace A by A X AYA in previous step, here we replace this A by AXA transpose. So, if you simplify this it is YY transpose A transpose X transpose AA transpose and then you right; this second and third as the transpose of A Y and last two term as transpose of A X.

And we know that AYA symmetric A X is symmetric. So, we can write this as YAYAX A YA Y X. now the middle 3 terms can be written as a by using the property that Y is pseudo inverse of the matrix A. So, we can say that Y can be written as YAX and we have already shown that X can be written as YAX. So, if you compare these two things, we can say that your X is same as Y. So, it means that if we have 2 matrices satisfying all the 4 condition of Moore more Moore-Penrose conditions, then both the matrix has to be same, it means that pseudo inverse of a given matrix is going to be unique.

Now, if you looking at the proof of B here, this can be easily observed that interchanging the roles of A and X in the above condition, it is immediate that is A pseudo inverse of A dash. So, it means that when we have defined pseudo inverse with the help of these conditions, if we replaced by say A by A dagger and X by any matrix say X tilde, then we can easily observed that the pseudo inverse of A dagger is coming out to be a itself.

So, that we are living as an exercise.

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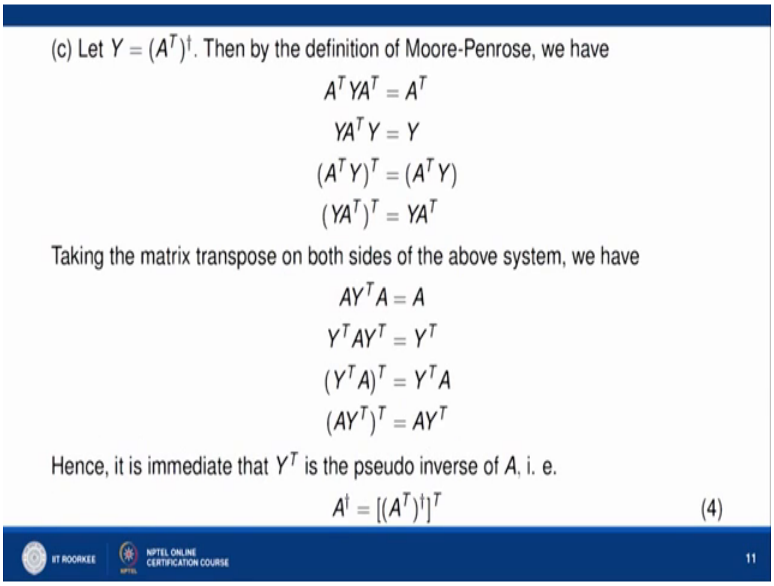

And looking at the last part that we want to find out say pseudo inverse of A transpose. So, let us write Y as A transpose dagger and we want to find out this Y, then we know that this Y satisfy the following conditions that we are just replacing A by A transpose in the definition of pseudo inverse. So, first condition is that A transpose YA transpose is A transpose YA transpose Y as YA transpose Y is symmetric and YA transpose is symmetric. So, we have written all the 4 condition satisfied by say pseudo inverse of A transpose.

Now, we simply we transpose all the 4 conditions and we can write this as A Y transpose A as A Y transpose A Y transpose is Y transpose Y transpose A transpose is Y transpose A. So, what we have done; we have just taken the transpose of the above 4 condition and we have these 4 condition if you look at the these 4 condition and the and the conditions given as a pseudo inverse, then we can say that this Y transpose is going to be the pseudo inverse of A, but we know that the pseudo inverse of A is going to be A dagger. So, and pseudo inverse is unique.

So, A dagger has to be equal to Y transpose. So, Y transposes is what Y is given as A transpose dagger. So, A transpose dagger transpose is your a diagram. Now looking at the taking the transpose; here we say that A dagger transpose is nothing, but A transpose dagger. So, pseudo inverse of A transpose is nothing, but the transpose of pseudo inverse of A that proves the last condition last property.

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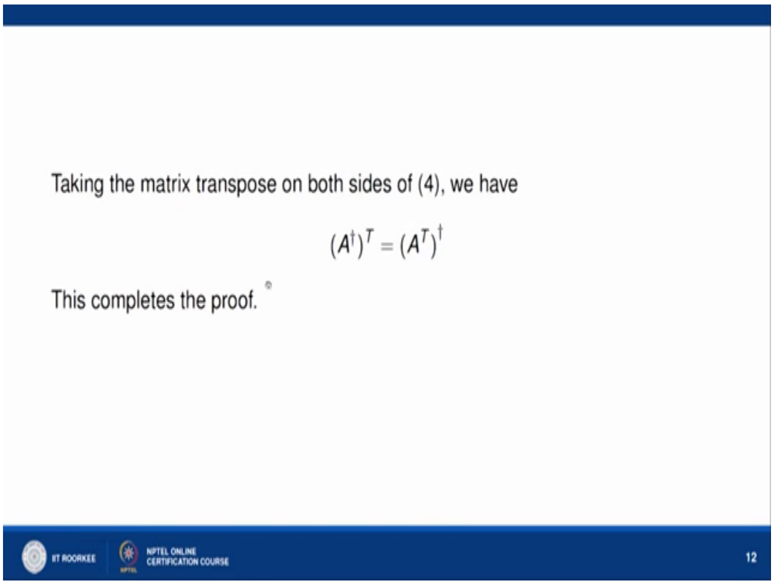

Now, so, this is what we have done that taking the matrix transpose on both the sides of 4; we have transpose of the A dagger is equal to A transpose dagger and which complete the proof of this theorem.

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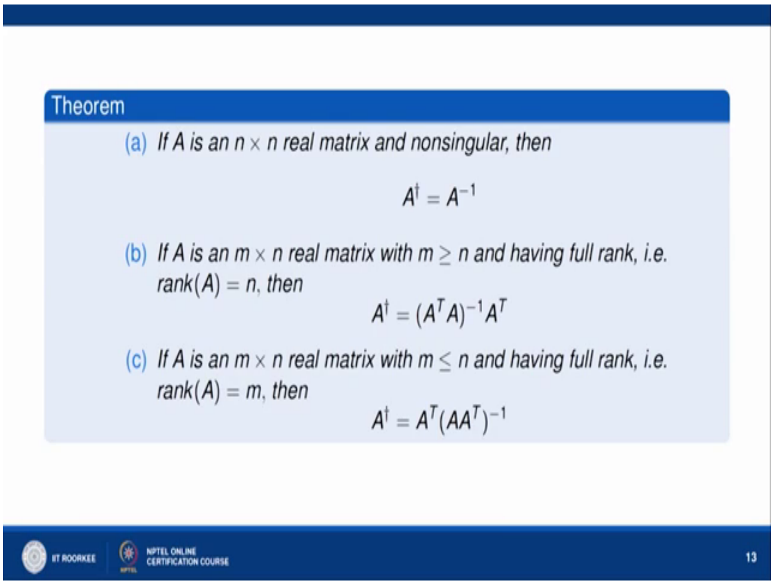

Now, moving on next theorem which gives us the tool that; how to find out A dagger; what we have seen just now in we have seen 4 condition to be satisfied by pseudo inverse and in next theorem; in the previous theorem, we have seen the result which is in resemblance of A inverse in the case of square matrix.

So, and now in this theorem; we try to find out how to find out pseudo inverse; so, if A is any n cross n real matrix and non singular, then A dagger is nothing, but A inverse. So, in the case of a square matrix and non singular matrix, we have A dagger matching with your inverse, but if A is not a square matrix and m is greater than equal to n and having full rank, then full rank means rank of A is equal to n; then your A dagger can be written as A transpose A inverse A transpose.

So, this will give you the formula for A dagger for any m cross n real matrix with m greater than 10, but in the case when m is less than or equal to 1 n and having full rank that is rank of A is equal to m, then A dagger can be given as A transpose AA transpose minus 1 it is this; once you once you know how to deal with this b you can easily handle this c so, to show.

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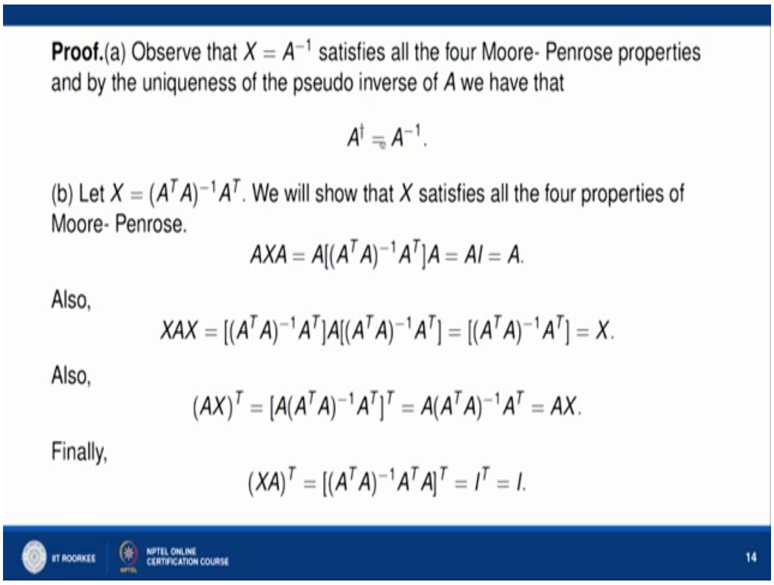

The first part that in the case of square non singular matrix your A dagger is nothing, but A inverse, you simply observe that this A inverse satisfy all the condition given more as given as Moore upon Penrose properties and we can see that since a pseudo inverse is unique and A inverse satisfy all the condition given Moore-Penrose conditions. If you look at and these are not very difficult to verify here, it is AXA equal to A. So, if is A inverse.

Then of course, this is true and here also f X is A inverse than it is A inverse A into A inverse; that is going to be A inverse and A A inverse is identity matrix. So, identity matrix is a symmetric and similarly A A inverse is again A symmetric matrix. So, A inverse satisfy all the 4 condition listed here. So, it means that A inverse is going to be pseudo inverse of A matrix A when A is invertible. So, in in the case of invertible matrices your A dagger is same as A inverse.

Now, so, that proves the first part of this theorem. Now moving on to second part, we want to show that when a is any rectangular matrix where m is greater than or equal to n, then we can find out the pseudo inverse as A transpose A inverse A transpose.

So, we need to show only this thing that this if we assume this X as A transpose A inverse A transpose, then this X will satisfy all the 4 properties of Moore-Penrose. So, first property is that AXA equal to A. So, you just calculate what is A X A. So, AXA a x is this A transpose is A inverse A transpose if you simplify then it is what A transpose A inverse and it is A transpose. So, A; so, this will give you simply i. So, it is nothing, but a. So, first property is quite trivial.

And if you look at XAX; so, x is this A transpose A inverse A transpose AA transpose A inverse A inverse A transpose. So, if you simplify again you will get A transpose A inverse A transpose and that is nothing, but X and we can verify that A X transpose which is given by this is nothing, but A X and finally, we can verify in a similar way that XA transpose is nothing, but X m.

So, finally, we calculate we want to show that XA symmetric matrix. So, if you calculate XA writing X has A transpose A inverse A transpose into A. Now, if you look at this is nothing, but A transpose A inverse into A transpose is that will give you an identity matrix. So, it is nothing, but identity.

So, XA transpose is identity, but if you look at this XA is also same as I this is X and if you multiply a here then XA is going to be identity. So, it means that XA transpose is equal to I is equal to XA and this will also indicate one; one very important thing that your X is A left inverse of the matrix A. So, it means that XA is going to be identity matrix here.

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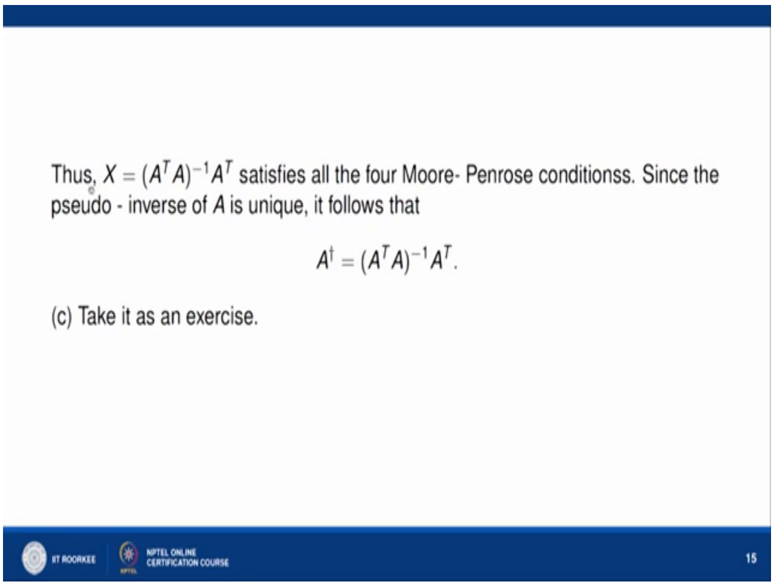

Similarly, so, it means that X is equal to A transpose A inverse AA transpose satisfy all the 4 Moore-Penrose condition and we already know that pseudo inverse of A is unique. So, it implies that A dagger, it can be written as A transpose A inverse A transpose.

And c part I am leaving it as an exercise we already know that if a matrix A is of size m cross n where m is greater than m is less than n, then if you look at the transpose of this matrix, then A transpose is a matrix of size n cross m and n is bigger than m and using the proof given in 2, we can easily find out that what is the say pseudo inverse of a here in this particular case; when m is less than n. So, I am giving this as an exercise; this is not very difficult so, use this fact that.

If A has a A is of size m cross n m is less than n, then A transpose have more number of rows than columns and the pseudo inverse of any matrix is unique and pseudo inverse of A transpose is the transpose of pseudo inverse. So, using these three condition, you can easily verify, you can easily prove the last proof; now moving on the next result which will help us to find out pseudo inverse with the help of singular value decomposition of a theorem. So, let of a matrix.

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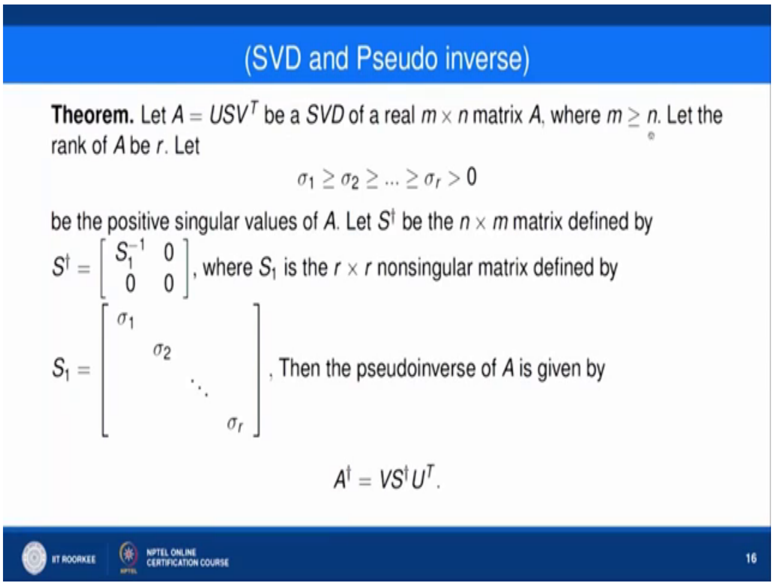

So, let A is USV transpose be a singular value decomposition of a real m cross n matrix A where m is greater than equal to n and let the rank of A be r and we arrange our singular values in this order be the positive singular values of A sigma 1 2 sigma r be the positive singular values of A and let S dagger be the n cross m matrix defined by S in S 1 inverse 0 0 0 where S 1 is the r cross or non singular matrix defined by this. So, S 1 is the diagonal matrix of size r cross r and it contains the all the positive singular values of A.

So, with the help of this S 1, we define S dagger which is a block matrix where this is r cross r and remaining r of appropriate size. So, that we have S dagger and once we have S dagger with us, then the pseudo inverse of A is given by A dagger as V S dagger U transpose. So, if A as USV transpose, then A dagger can be written as VS dagger U transpose where S dagger can also be defined with the help of S and that is nothing, but S 1 inverse 0 0 or if you can say that your; if you want to define S dagger in terms of S, then S dagger is nothing, but same as S the only thing is in in the diagonal, we have 1 upon sigma I as singular values. So, S dagger has a singular values reciprocal singular values of A S. So, this is the only difference between this thing.

So, here you can find out S dagger as A the same as S the only thing is your diagonal entries are replaced by reciprocal values. So, A dagger is written as VS dagger U transfer and we want to prove this.

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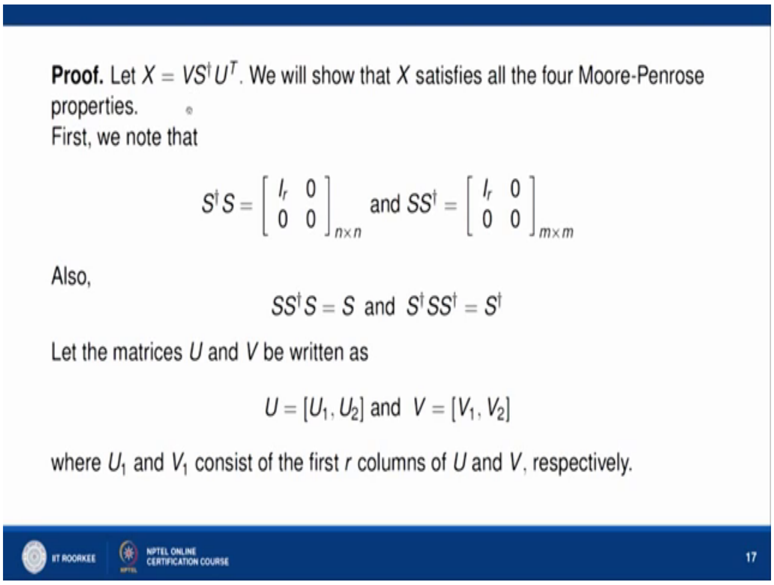

So, again to show that it is a pseudo inverse, we will simply show that this satisfy all the 4 properties of listed there as more Penrose properties. So, we let us take X as VS dagger U transpose and we show that X satisfy all the all the 4 Moore-Penrose properties. So, first we know that S dagger S is what.

So, S dagger S is we are finding out S dagger SS I r 0 0 where this is the identity matrix of size r cross r and rest are appropriate size 0 matrices and similarly, we can calculate SS dagger and it is also I r 0 0 look at S dagger as it is of square matrix of n cross n and SS dagger is a square matrix of size m cross n. So, that is why I am saying that these 0 matrices are appropriate matrices appropriate 0 matrices of size that we have to see.

Now, also with the help of this we can also calculate SS dagger S which is given as S and S dagger SS dagger S S dagger that is that I think you can write it here I am just writing one here and then you can simplify. So, we have shown that S dagger S is this and SS dagger is identity matrix of this and we can easily verify that SS dagger S is equal to S and S dagger S S dagger S equal to S dagger.

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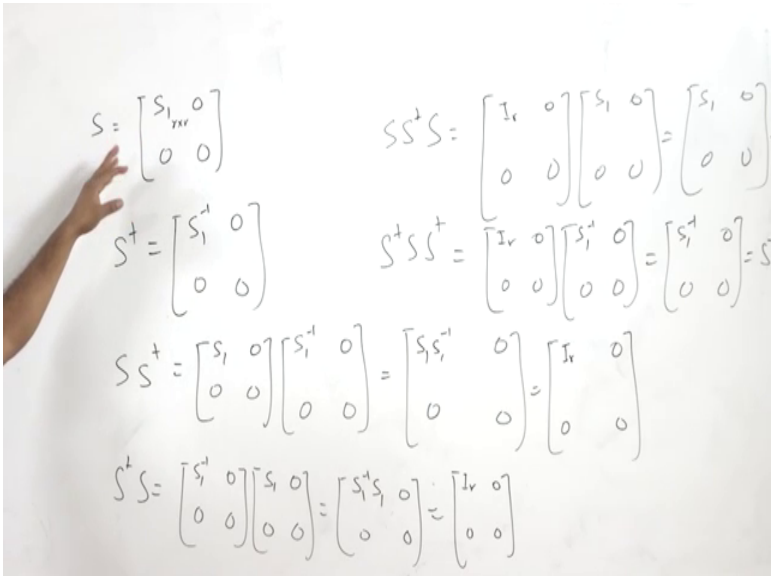

In fact, we can write it like this S as this partition matrix diagonal matrix S 1 0 0 and S dagger is S 1 inverse is 0 0 and we can easily calculate SS dagger as this and if you multiply these block diagonal matrices then it is coming out to be I r 0 0.

Similarly, you can calculate S dagger S and it is also you can obtain by multiplying the block diagonal matrices and it is given as I r 0 0 and we can calculate. Now SS dagger S as SS dagger is we have already calculated as this and into S that is S 1 0 0 and if you calculate we have this X 1 0 0 and this is nothing, but your S. Similarly, we can calculate S dagger S into S dagger. So, S dagger S we have calculated and we can write write down the expression for S dagger that is given here and if you multiply we will get S 1 inverse 0 0 which is nothing, but S dagger.

So, we can see that these this can be is verify that SS dagger S is S and S dagger; S dagger is equal to S dagger. Now, we also partition our matrices U and V as this U as U 1, U 2 and V as V 1, V 2 where U 1 consists first our columns of U and V 1 consists the first our columns of matrix V. Now calculate the AXA and A is USV transpose and X is VS transpose U transpose USV and A as USV transpose. So, if you multiply you will see that here V transpose V is identity because V is orthogonal matrix.

And U transpose U is again orthogonal matrix. So, if this can be simplified as USS dagger SV transpose.

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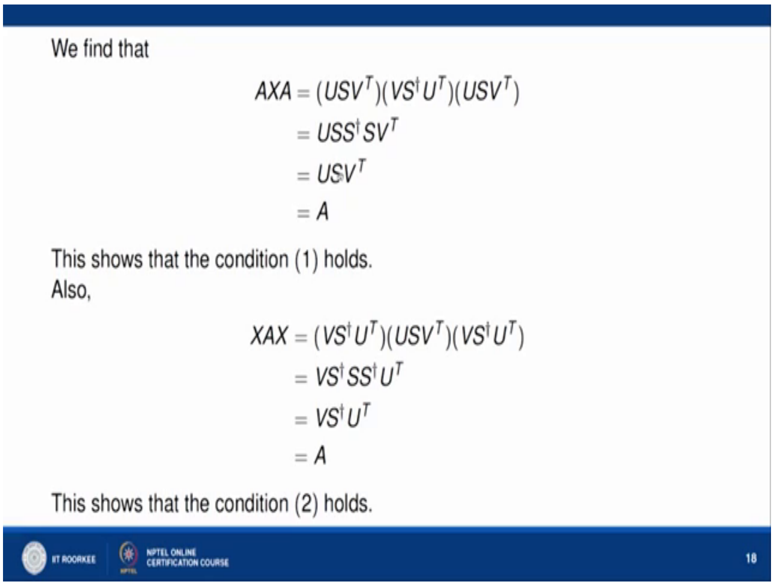

And we already know that SS dagger S is going to be S, just now, we have already obtained and this is nothing, but matrix a. So, AXA is equal to A and which shows that the Moore-Penrose condition 1 is satisfied that AXA equal to A, similarly we want to find out XAX as A. So, write down XAX and if you simplify U transpose U is identity and V transpose V is identity.

So, we have X dagger SS dagger U transpose now this we have calculated as S dagger. So, it is nothing, but VS dagger U transpose and this is nothing, but your X here. So, we have shown that XAX is equal to x and this shows that the condition two will also hold. So, this prove the second condition that XAX equal to X.

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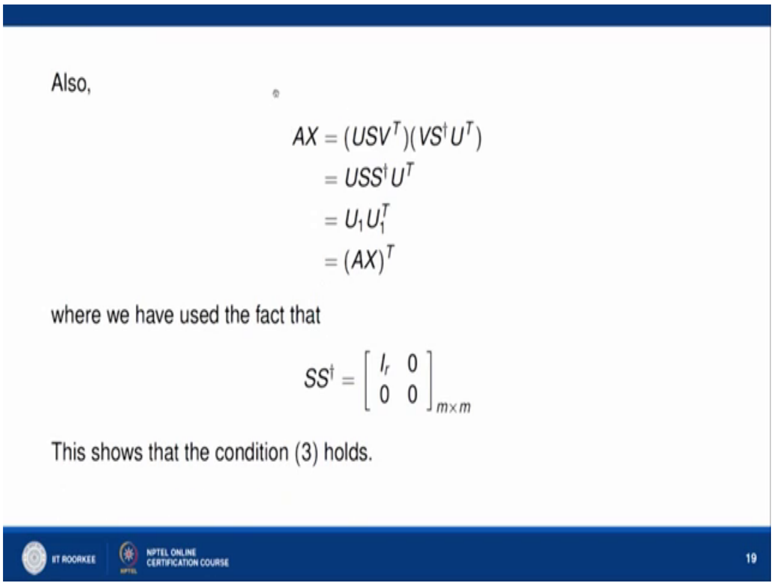

Now, moving on the next property that A X is symmetric matrix; so calculate AXA is USV transpose X is VS dagger U transpose and V transpose V is identity. So, we have U SS dagger U transpose.

Now, if you calculate this SS dragger SS dagger is basically what it is the identity metric I r 0 0 and if you simplify that will give you U 1 U 1 transpose and U 1 U 1 transpose is nothing, but A X transpose. So, here we have proved that A X equal to A X transpose and here we have used the fact that SS dagger is this block diagonal matrix of size m cross m and this shows that the condition number three will also hold that A X is a symmetric matrix.

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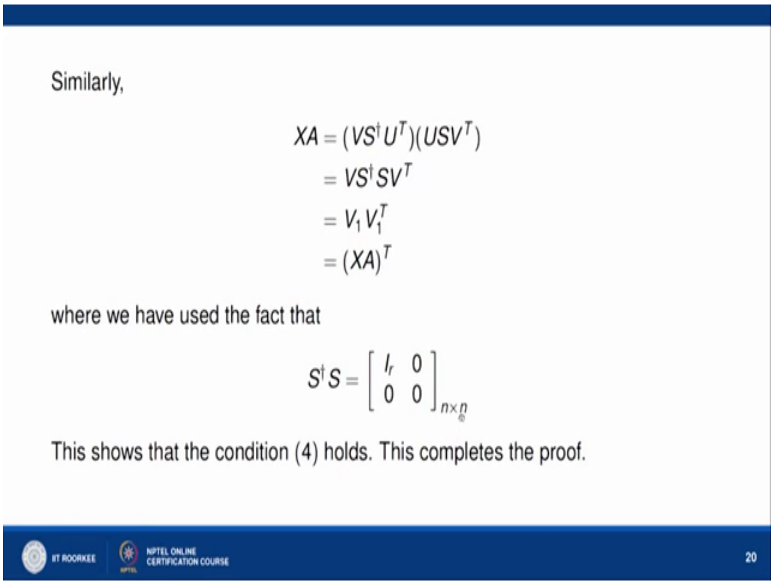

Similarly, we want to show that XA is also a symmetric matrix. So, calculated XA and VS dagger U transpose USV transpose and UU U transpose U is identity. So, we have V S dagger S V transpose.

Now, we have already know that S dagger S is an identity matrix of size n cross n, then this VS dagger as V transpose is written as V 1 V one transpose and V 1; V 1 transpose can be written as XA transpose and this shows that condition for holds. So, it means that this x which is known as VS dagger U transpose satisfies all the properties listed as pseudo inverse of the matrix A. So, it means that this X which is known as VS dagger U transpose is the pseudo inverse of the matrix A whose singular value decomposition is given by us V transpose. So, A dagger is given as V S VS dagger U transpose it is a pseudo inverse of a matrix A where A can be written as USV transpose.

So, it means that if we know pseudo a singular value decomposition of a matrix A, then we can find out the pseudo inverse of a matrix A by this formula and where S dagger is nothing, but the matrix S whose diagonal entries are replaced by reciprocal of the singular values of sigma S.

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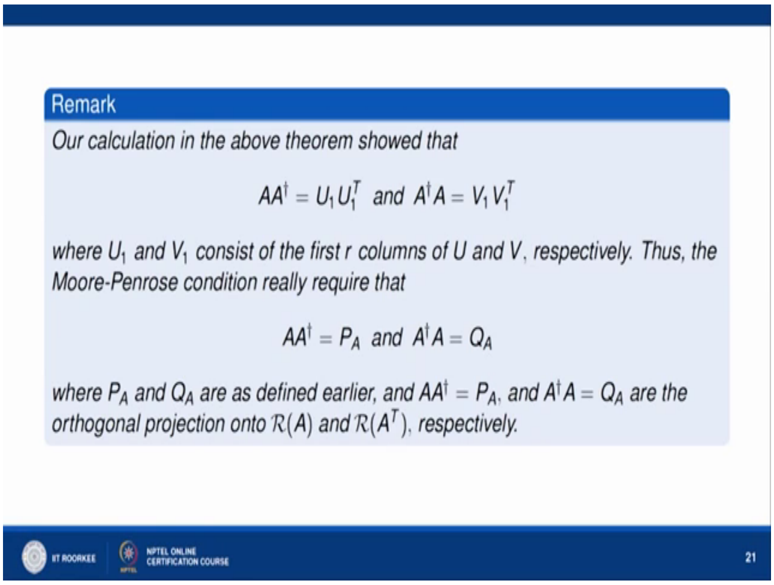

So, once we have proved this, then we can easily see one very important remark, we our calculation in the above theorem showed that AA transpose as A A dagger is equal to U 1 U 1 transpose.

And A dagger A as 1 V 1 transpose, if you recall this is what we have written A x equal to U 1 U 1 transpose what is X? X is a dagger. So, A A dagger is U 1 U 1 transpose and A dagger A is V 1 V 1 transpose that is what we have list here and here U 1 and V 1 consists of the first are columns of U and respectively and we have already seen these matrices and we know that this U 1 U 1 transpose m gives you the orthogonal projection of on the range space of m.

And similarly, V 1 V 1 transpose gives the orthogonal projection on ranges range space of A transpose. So, this implies that A A dagger is orthogonal projection on given as P A and A dagger A is the orthogonal projection given as Q of A and we already know that orthogonal projections are unique. So, it means that we can also find with the help of this the that A dagger is orthogonal projection on onto range a space of A and A dagger A is an orthogonal projection on range space of A transpose.

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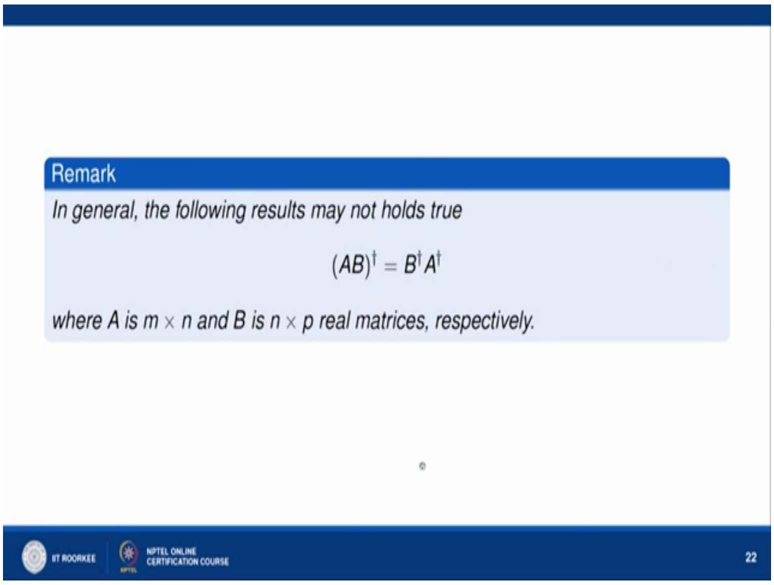

So, now we have seen several properties in which your inverse and pseudo inverse behave in a similar way, but now we consider one properties where they may differ. So, in general we know that if we have inverse of a A product is equal to B like if we want to find out inverse of A B, then it is given as B inverse A inverse, but the singular formula for the case of a pseudo inverse may not holds true.

So, we want to show that in general the following result may not hold true that is A B dagger is not equal to B dagger A A dagger where A and B are matrices which are compatible to each other such that that we can write down this formula so, to look at that this may not hold true.

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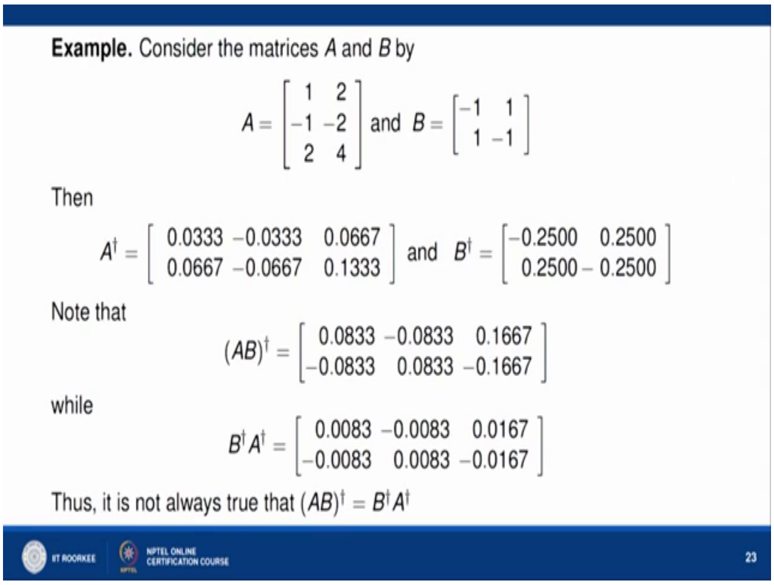

Let us consider the following example here; A can be written as 1 2 minus 1 minus 2 2 4 and B as written as minus one one and one minus one and we can find out the in pseudo inverse of A and pseudo inverse of B and we can calculate the pseudo inverse of A B also and we can say that pseudo inverse of the A B is the following matrix and pseudo inverse product of pseudo inverses is like this.

And we can see that these two are not same. So, it means that this is one instance where they are not following in a same way and if you look at here, this is the instance where we have a matrices not of full rank. So, this kind of example, you can find out in the case of rank deficient matrices. So, we can verify in with the help of MATLAB also let me use MATLAB to verify this. So, we want to verify the following example through MATLAB. So, here we have.

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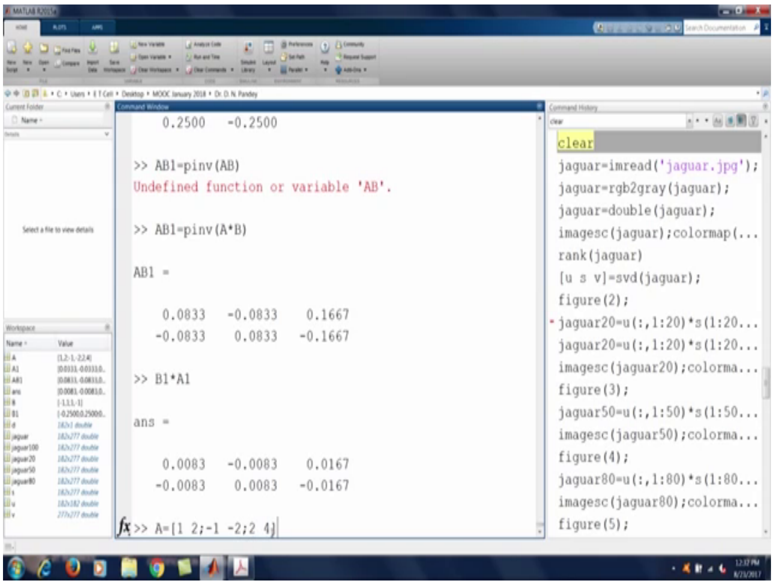

First find out what is a. So, A is listed as 1 2 first row is 1 2 second row is minus 1 minus 2.

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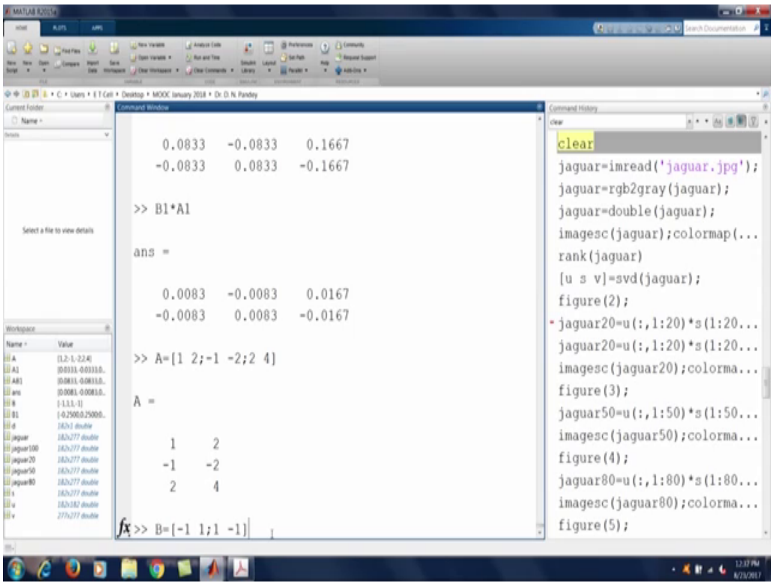

Third row is 2 4 and if you write it is given as this.

Similarly, if you want to write down B where B is this minus first row is minus 1 1 and second row is 1 minus 1 that is given here.

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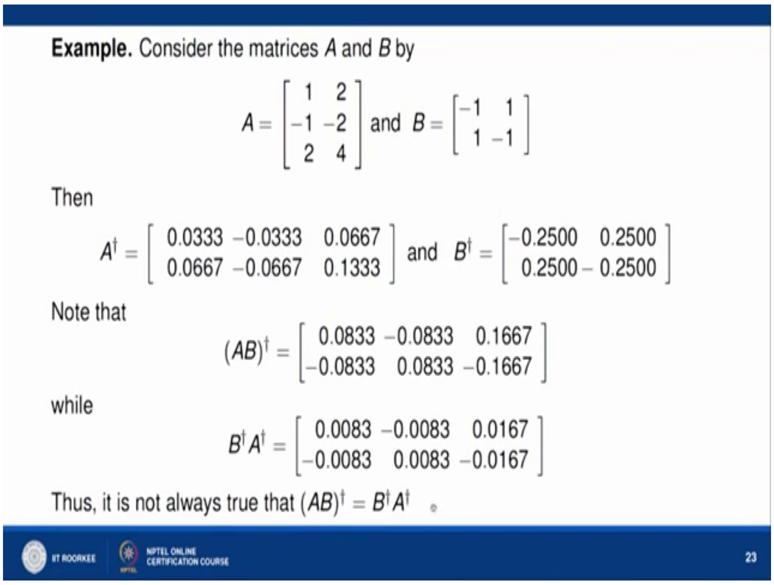

Then you can write B as this. Now we can find out pseudo inverse by the following command; let us write B 1 as P in V.

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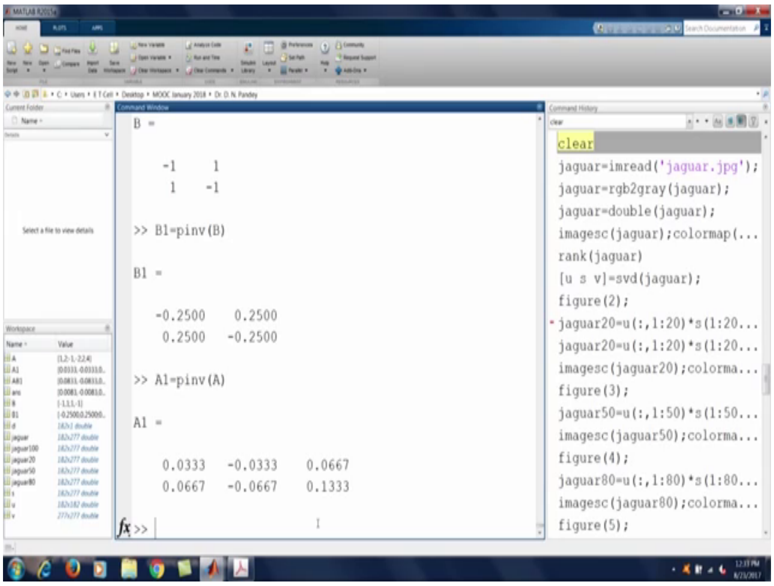

So, I and V gives you inverse, but if you write p inv, then it will give you this pseudo inverse. So, B 1; let us call pseudo inverse of B and it is given as this and if you verify, it is given as B dagger is given by this and here it is matching.

So, similarly we can write down a one as P in off A. So, that will give you inverse A dagger A dagger of A dagger here. So, A dagger is given by this. So, similarly we want to show that A B dagger is not equal to B dagger A dagger.

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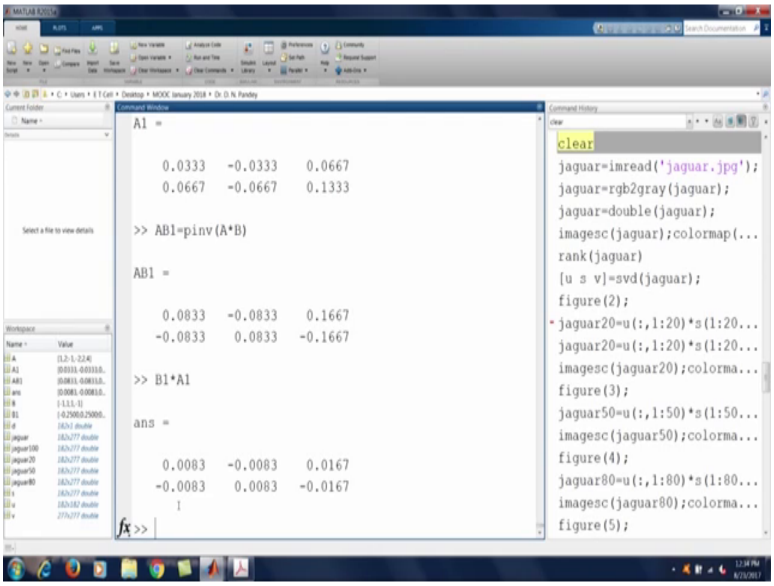

So, let me write it A B A B 1 which gives you pseudo inverse of any this is this quantity and if you look at this is A B dagger and A B whole dagger is given by this and it is matching with this.

Similarly, we find out now we want to show b dagger into A dagger now let us see this is and it is you can see that these two are not same. So, here we have just verified our example through MATLAB; now once we are done with how to find out pseudo inverse of a given matrix.

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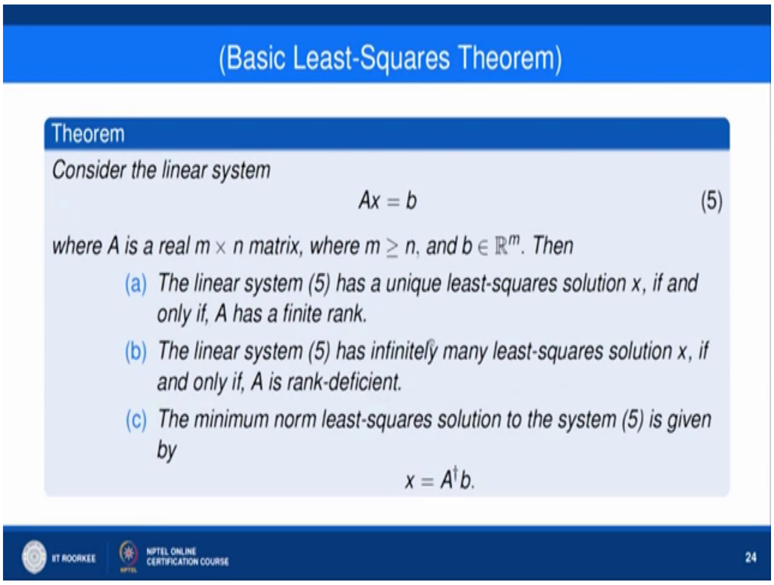

Now, let us utilize it to find out the minimum norm least square solution for the system of linear equation. So, now, consider the basic least square theorem which says that consider the linear system A x equal to b where a is a real m cross n matrix where m is greater than equal to n and b belongs to R m.

Then this is the linear system, 5 has a unique least square solution if and only if a has finite rank finite and full rank the linear system five has infinitely many least square solution x if and only if in a is rank deficient. So, if a has full rank, then we have a unique solution uniquely c square solution if is a rank deficient, then then it has infinitely many solution and the minimum norm least square solution to the system 5 is given by A dagger b. So, that we are going to prove here.

So, first let us try to understand how we can find out the least square solution x.

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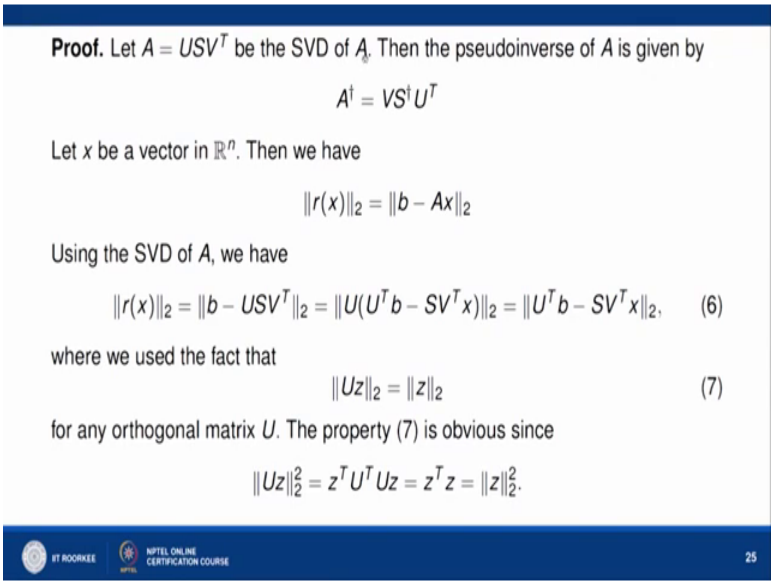

So, let A is USV transpose be the singular value decomposition of A, then the pseudo inverse of A is given by A dagger as VS dagger U transpose that we have already seen. So, now, let us find out that let x be A any vector in r n, then we can find out the residual vector as this are x equal to b minus x. So, take the 2 norm of r x and it is given as 2 norm of b minus A x now using the singular value decomposition of a we can write a as USV transpose x.

So, this we can write b minus us V transpose x to norm of this and if you simplify you take U outside, then it is U as U transpose b minus SV transpose x now this can be written as U transpose b minus SV transpose x 2 norm of that here, what we have done here we have utilises fact that two norm of U z is same as to norm of z if U a done orthogonal matrix and this is; So, here you of z call this as z, then U 2 norm of U z is same as 2 norm of z here. So, this can be easily verified because two norm of U z can be written as z transpose U transpose U z that now U transpose U is orthogonal.

So, U transpose U is an identity matrix; so, z transpose z and this is nothing, but 2 norm of z. So, using this property that 2 norm of z is same as 2 norm of U z where U is an orthogonal matrix, we can write the 2 norm of r x as two norm of U transpose b minus SV transpose x.

Now, the good thing about this system six is this that U is an orthogonal matrix and S is a kind of a triangular matrix or diagonal matrix and if we call this U transpose b as some other matrix and V transpose x as some new vector then this system this resolve then this system A x equal to b is reduced to a very nice system.

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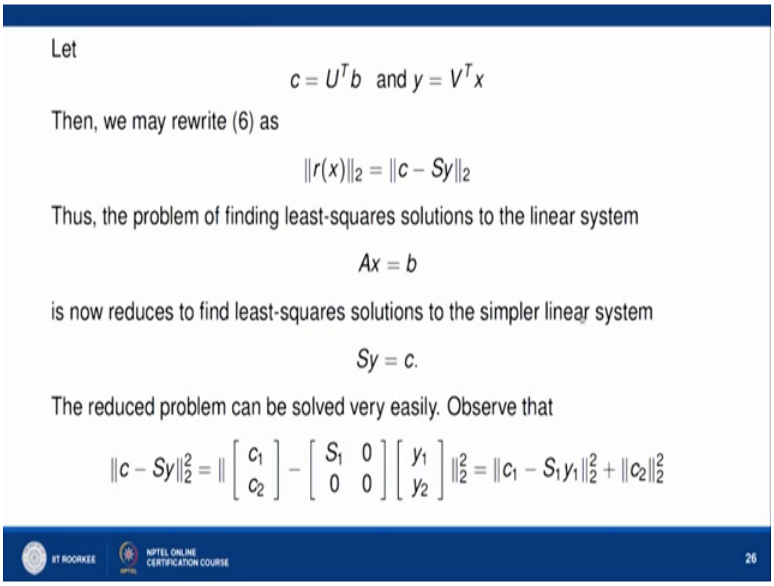

Let us see how it is. So, let see as U transpose b and y as we transpose x then residual vector r x 2 norm of residual vector r x can be written as two norm of c minus sy thus the problem of finding least square solution to the linear system A x equal to b is now reduced to find out least square solution of this simpler linear system as y equal to c where this is a a diagonal system.

And this is a say triangular system with any systems of linear equations and we know how to solve this simpler linear system the reduced problem can be solved very easily and we can observe that c minus S y 2 of this square can be written as seek I am writing as c 1 c 2 and as as S 1 0 where S 1 is r cross r matrix and y 1 y 2 and if you simplify, you will get this as a c 1 minus S 1 y and c 2 and if we take the 2 norm of this and take the square of that we will get this 2 norm of c 1 minus S 1 by whole square plus 2 norm of c 2 whole square and if you look at this carefully then we can find out the solution of this simpler linear system.

Sy equal to c using this thing so, here if you look at this.

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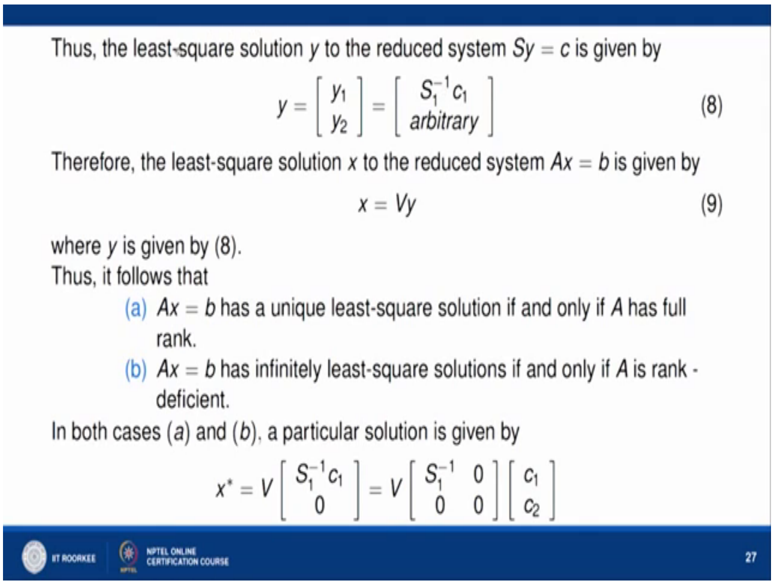

Then we can say that the least square solution y to the reduced system S y equal to c is given by y which have these component y 1 and y 2. So, y 1 consists the first r component and y 2 is the remaining n minus r component.

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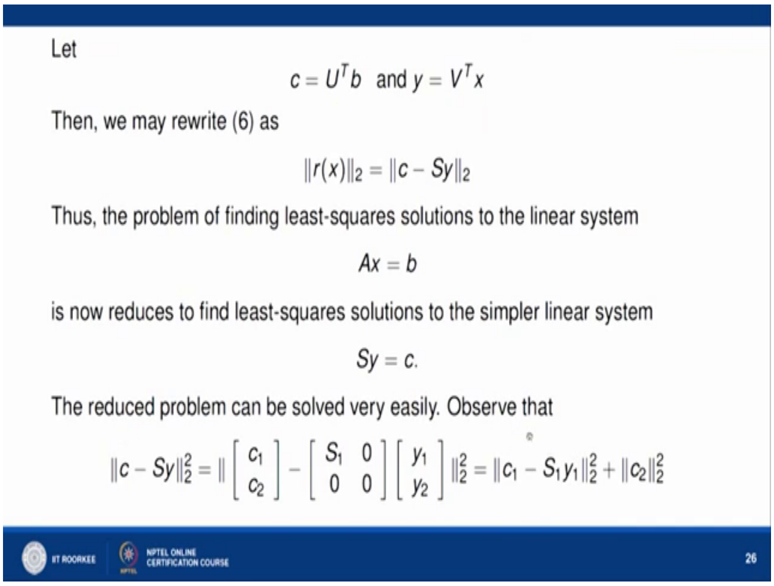

So, y 1 is what if you look at if you want to minimize this then this c 2, I cannot this, we do not have any control on c 2, but if you look at if you want to minimize it, we want to make it a 0. So, we can say that S 1 y 1 is equal to c 1 or we can say that y one can be taken as S 1 inverse c 1.

So, we can write that y one as S 1 inverse c 1 and y 2, we do not have any control. So, we can take any arbitrary value for this y 2. So, least square solution y to the reduced system S y equal to c is now the of this form that the first our component is S inverse c 1 S S 1 inverse c 1 and the remaining components are arbitrary components.

So, therefore, the least square solution x 2 the reduced system A x equal to b is given by x equal to V y where y is given by this now this proof the first two first two result of the theorem and which is this that if A x equal to b has a unique least square solution if and only if a has full rank if we have unique solution, then then we have unique x and we have unique y and we do not have any arbitrary component here. So, in that case, we can say that we have S 1 will consist of the entire or full rank matrix or it will consist the n singular values of the matrix a.

But and if A has full rank then we can say that we do not have any arbitrary vector y 2 here and this S 1 inverse c one gives you the vector y. So, this proves the first thing that A x equal to b has a unique least square solution if and only if a has full rank.

And if A has rank deficient, then there is a arbitrary component available here and that makes the solution as non unique or you can say that that makes the solution as infinitely many solutions. So, in this case when A is rank deficient we have infinitely many least square solutions. So, that proves the first two part now in both the part the solution is given by this once we have y we can operate V on y and we will get the solution x m. So, x star is equal to V S 1 inverse c 1 0. Now we are taking a particular solution.

So, we are we can assume we may assume that y 2 is nothing, but 0 since y 2 is arbitrary we can take it 0. So, let us say that x star is equal to V S 1 inverse c 1 as 0. Now, if you cooperate V we can write it V S 1 inverse 0 0 and c c 1, I can write it c 1 c 2. So, our solution particular solution is given by this.

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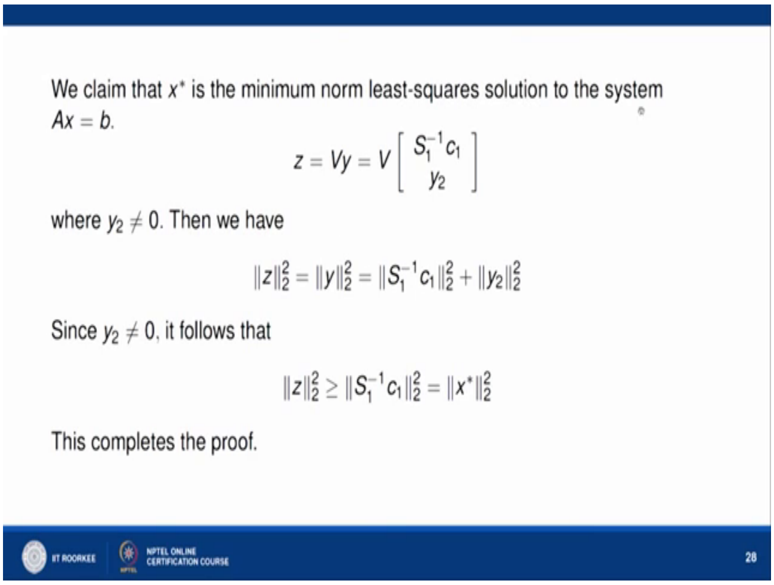

Now, we claim that this particular solution which is given by this is a minimum norm least square solution to the system A x equal to b. So, it means that this is the system having the minimum norm it is a least square solution, but also it is having the minimum norm.

So, so, consider another minimum norm least square solution that is z as y where V YY is this S 1 inverse c 1 and the y 2 component is arbitrary. So, let us say that y 2 is non zero this also minimize the minimizes r x here is that. So, this is a least square solution, but we want to show that x star is the minimum norm least square solution. So, z is any least square solution of the system A x equal to b. Now here we consider that this y 2 is nonzero and if you calculate the 2 norm of z whole square.

Then it is nothing, but two norm of y whole square because V is an orthogonal matrix. So, this can be written as S 1 inverse c 1 two norm of S inverse c 1 square plus two norm of y 2 square now if y 2 is nonzero then of course, this is bigger than two norm of S 1 inverse c 1 square and this is nothing, but the two norm of x star inverse. So, we can say that if we take any least square solution of the system A x equal to b that is certainly bigger than this x star and x star is.

So, we can say that this x star is known as minimum norm least square solution of the system A x equal to b and this complete the proof. So, to show the last part that the minimum norm least square solution of the system five is given by x as A dagger b to show this let us observe this.

So, your we have just shown that this x star is the minimum norm least square solution. So, it means that if we have any least square solution of the system A x equal to b then among all those solution x star having the least norm now our claim is that x dagger is nothing, but S star is nothing, but A dagger b. So, if you look at x star has this representation V S 1 inverse 0 0 and c 1 c 2. Now what is this c 1 c 2 if you look at c 1 c 2 is given by the matrix c and which is nothing, but U transpose V.

So, I can write this as V and this is what this is a representation for S dagger. So, V S dagger and this c can be written as U transpose V. So, we can say that V S dagger U transpose operating one V or operating one V. So, V as dagger U transpose is known as A dagger. So, x star can be written as A dagger operating on V let me write it here.

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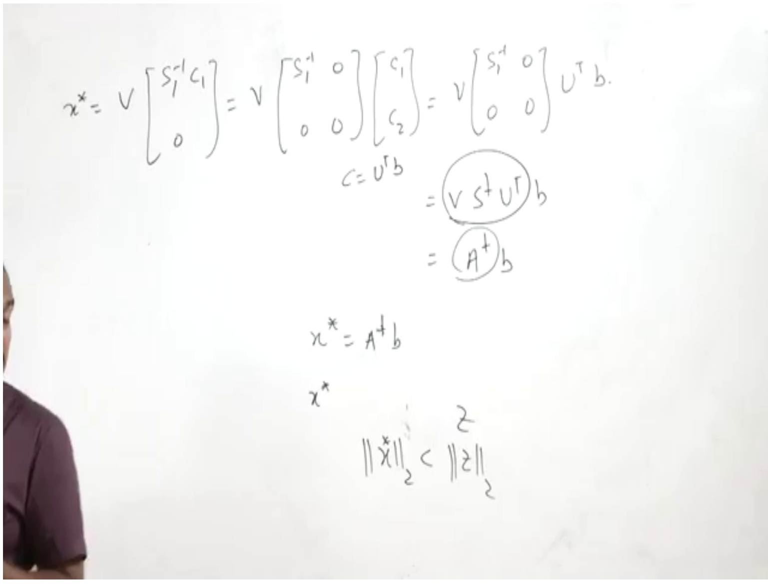

So, that is what we are just here x star is written as V into S 1 inverse c 1 0 and it is nothing, but we can write it V as this product I can write it as product of these two matrices here we have S 1 inverse 0 0 and operating on c 1 c 2 and here if you know the c 1 c 2 is basically this c and cv we already know that it is U transpose b that we have already assume.

So, in place of c we are writing U transpose V. So, we write V S 1 inverse 0 0 U transpose operating on V. Now this is the representation for S dagger. So, we write V S dagger U transpose operating on b and if you look at this is the representation for the pseudo inverse of matrix a. So, this we can say that x star is nothing, but A dagger b and we have already shown that x star is minimum norm least square solution it means that if you have any other least square solution then x star x star two norm of x star is less than 2 norm of this least square solution z is that.

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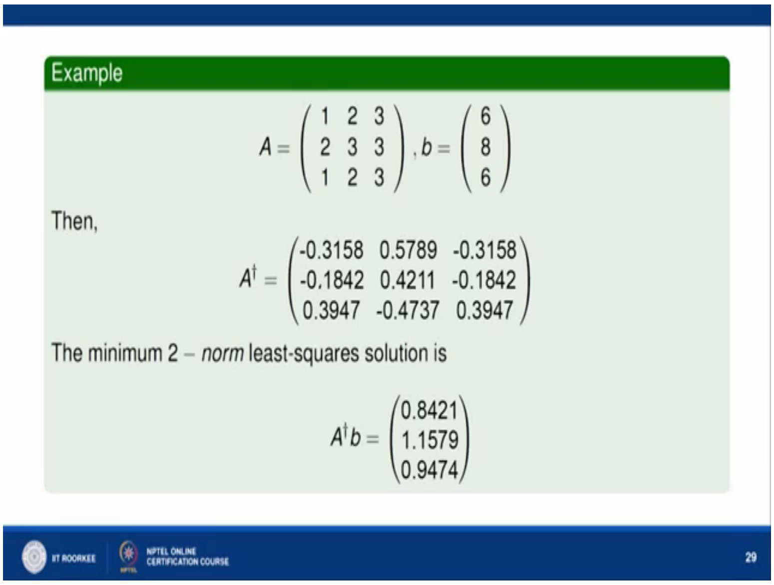

So, if we take A as 1 2 3 2 3 3, A 1 2 3 and b as 6 8 6, we first we can calculate A dagger and the minimum two norm least square solution we can calculate by A dagger into b and the same we can verify here here we write our matrix A as this.

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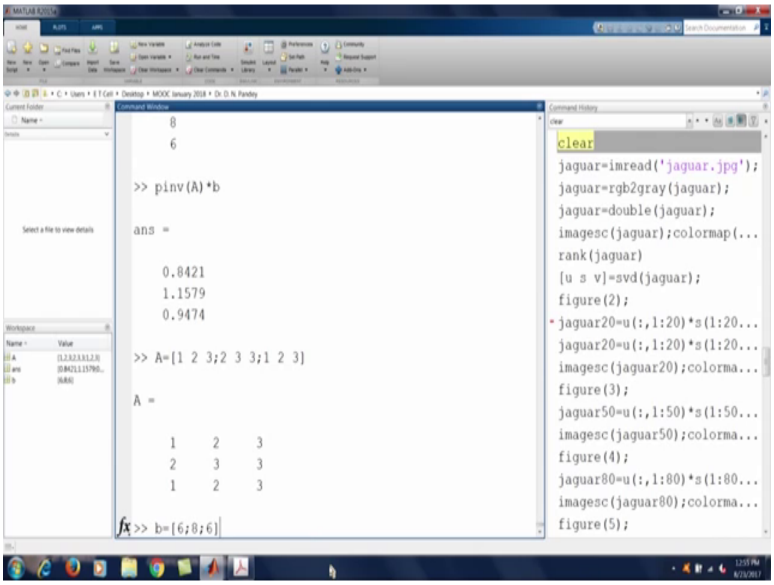

So, this is our matrix a one two three first row is 1 2 3 second row is 2 3 3 and last row is one two three and if you calculate b b is coming out to be 6 8 6.

And then we can calculate our solution as A dagger b and it is coming out to be this value. So, it means that we can find out say minimum two norm least square solution by finding A dagger b so, with this we close our lecture.

So, in today's lecture, we have seen that how this singular value decomposition theorem singular value decomposition of a matrix helps us to find out least square solution of a system A x equal to b and we have also seen that minimum two norm least square solution can be obtained by writing A dagger b where A dagger can be obtained with the help of singular value decomposition of a matrix A.

So, we will stop here and we will continue our study in next lecture thank you very much for listening us.

Thank you.