Numerical Linear Algebra
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Lecture - 27
Stability of Numerical Algorithms- I

Hello friends. I welcome you to my lecture on Stability of Numerical Algorithms. There will be 2 lectures on this topic, this is the first lecture on stability of numerical algorithms.

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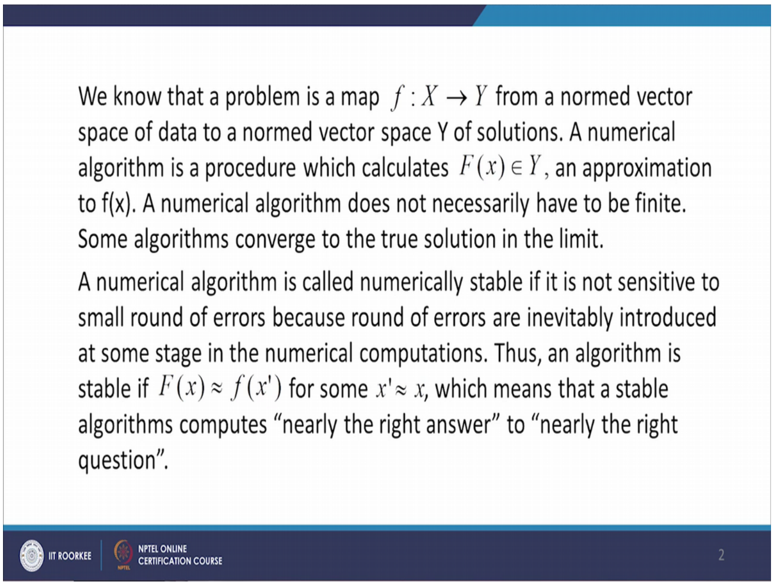

We have seen that a problem is a function f from X into Y, where X and Y are normed vector spaces X is the normed space normed vector space of data and Y is the normed vector space of solutions. A numerical algorithm is a procedure which calculates F x, capital F x belonging to by f is the function from X into Y, where X and Y are non vector spaces and numerical algorithm is a procedure which calculates capital F x belonging to y, capital F x belonging to y which is an approximation to the actual value F x.

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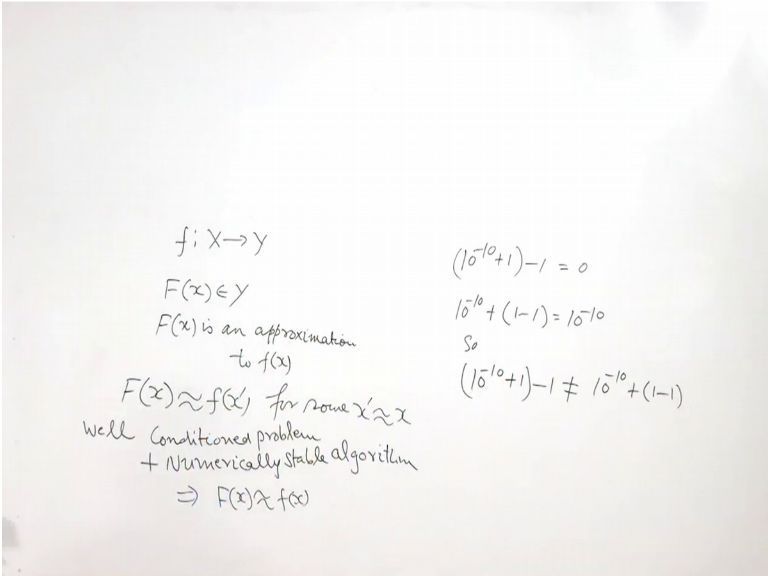

So, this F x is then approximation to f x. A numerical algorithm does not necessarily have to be finite. Some algorithms converge to the 2 solution in the limiting case. A numerical algorithm is called numerically stable if it is not sensitive to small round of errors because round of errors are inevitably introduced at some stage in the numerical computations. Thus an algorithm is said to be stable if f x is approximately equal to F x is approximately equal to, F x dash for some x dash approximately equal to x, which means that a stable algorithm computes nearly the right answer to nearly the right question.

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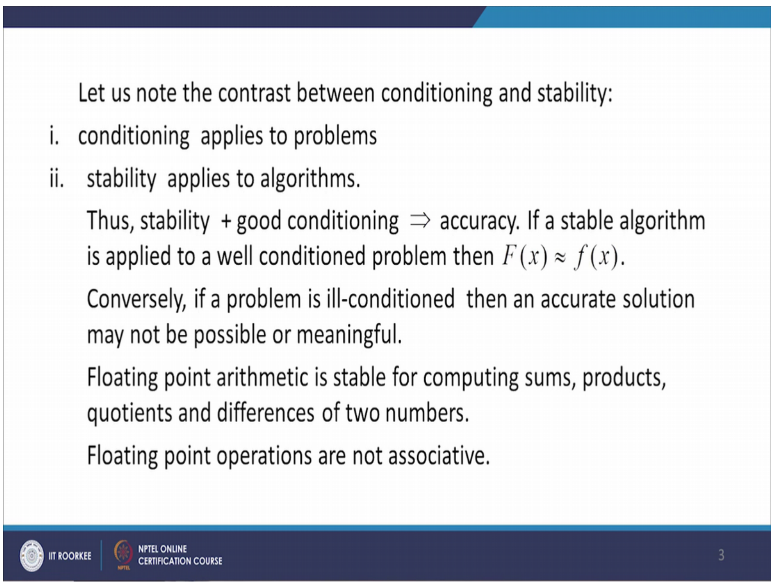

Let us note the contrast between the conditioning and stability. Conditioning as we have seen applies to problems and stability applies to algorithms. Thus we can say that stability plus good conditioning implies the accuracy.

If a stable algorithm is applied to a well conditioned problem then capital F x is approximately equal to. So, conditioning well conditioned problem if we have well conditioned problem if we have a well conditional, well conditional problem, then well condition problem plus numerically stable algorithm implies that F x is approximately equal to f x. Now, conversely if a problem is ill conditioned then an accurate solution may not be possible are meaningful.

Floating point arithmetic is stable for computing sums, products, quotients and differences of 2 numbers. Floating point operations however are not associative as we can see from this example.

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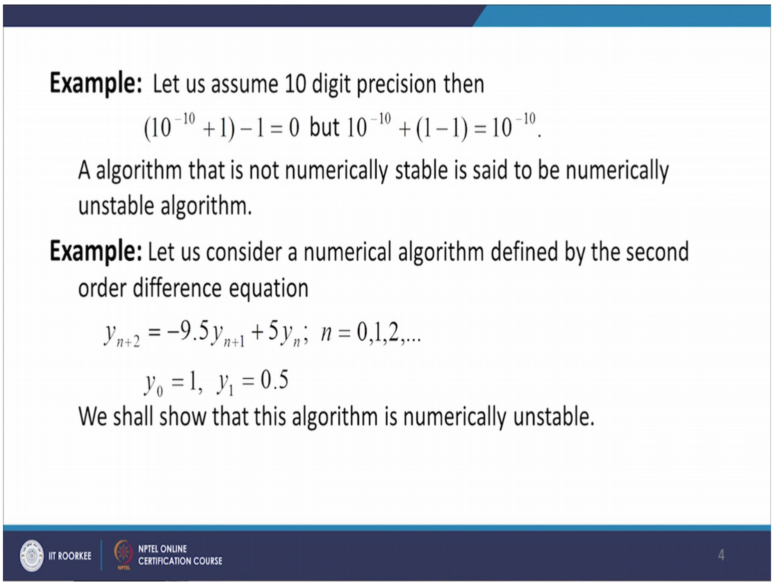

Let us assume 10 digit precision then 10 to the power minus 10 plus 1 minus 1 is equal to 0 in the 10 digit precision. But 10 to the power minus 10 plus 1 minus one is equal to 10 to the power minus 10 so that means, that the floating point arithmetic is not associative, 10 to the power minus 10 plus 1 minus 1 this equal to 0 in 1 digit consider 10 digit precision, but 10 to the power minus 10 plus 1 minus 1 is equal to 10 to the power minus 10. So, this is not equal to and therefore, the floating point arithmetic is not associative.

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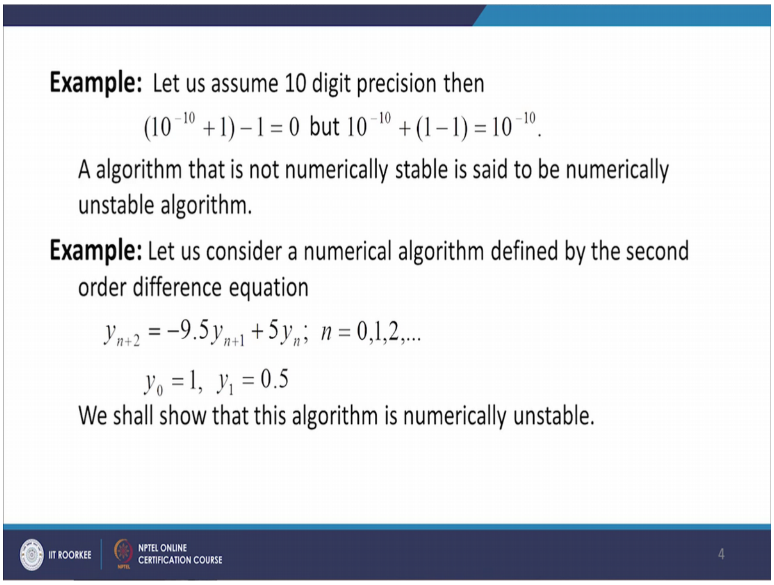

An algorithm that is not numerically stable is said to be numerically unstable algorithm. Let us consider a numerical algorithm defined by the second order different equation. The difference equation we have considered is y n plus 2 equal to minus 9.5 y n plus 1 plus 5 y n where n x value 0, 1, 2, 3 and so on and we are given that y naught is equal to 1 and y one is equal to 0.5 we shall see that this algorithm is numerically unstable. So, y n plus 2 is equal to minus 9.5 y n plus 1 plus 5 y n, n is equal to 0, 1, 2 and so on; y naught is equal to 1, y 1 equal to 0.5.

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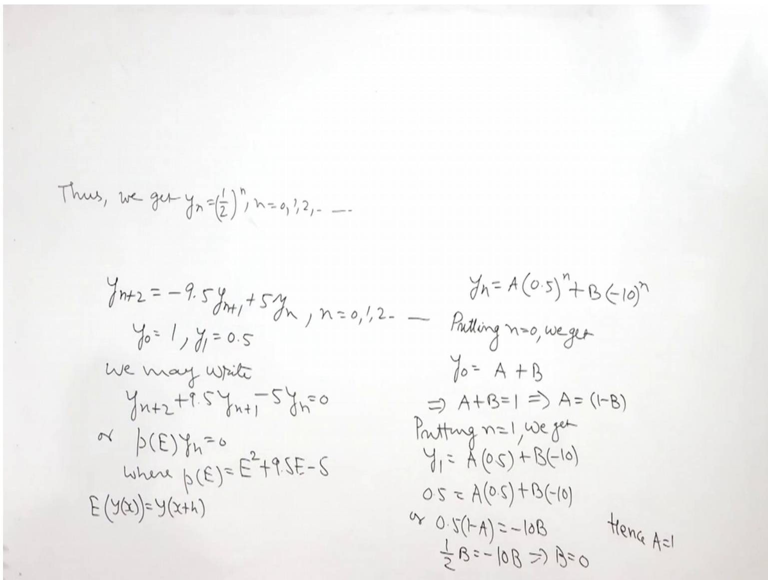

Now, let us see how it is numerically unstable let us consider a machine where beta is equal to 10s so that means, the decimal system we are working and 3 digit rounding arithmetic; that means, t is equal to 3. The given equation can be rewritten as y n plus 2 plus we may write the given equation as y n plus 2 plus 9.5, y n plus 1 minus 5 y n equal to 0 or equivalently we can write it as p E y n equal to 0, p E, where p E equal to E square plus 9.5 E minus 5 and y E v mean where y E v mean it is the shifting operator. So, E y x, E y x is equal to y x plus h. So, E is the shifting operator here.

Now, when we want to solve a difference equation we write the corresponding auxiliary equation. So, here p is equal to E square plus 9.5 E minus 5. So, we have the auxiliary equation as p lambda equal to 0, that is 2 lambda square plus 19 lambda minus 10 equal to 0 when you factorize this equation, the factors are 2 lambda minus 1 into lambda plus 10 equal to 0 which gives us the, which give us the values of lambda as 0.5 n minus 10.

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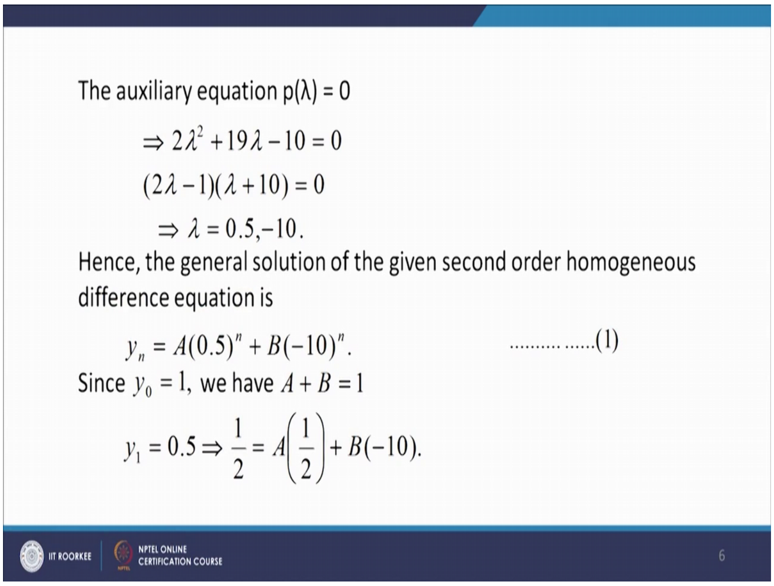

Hence the general solution of the given second order homogeneous difference equation is equal to y n equal to A into 0.5 to the power n plus B into minus 10 to the power n. So, we have now in order to calculate the constants A and B we will make use of the given conditions y naught equal to 1, y 1 equal to 0.5. So, putting n equal to 0 we get y naught equal to A plus B and y naught is equal to 1, which gives you A plus B equal to 1.

Now, let us put n equal to 1. So, putting n equal to 1 we get y 1 equal to A into 0.5 plus v into minus 10, y 1 is given to be 0.5, so 0.5 equal to A times 0.5 plus B into minus 10.

Now, A plus B equal to A gives you B equal to 1 minus A or you can say A equal to 1 minus B, A equal to 1 minus B. Now, what do we notice here? We can write it as 0.5 times 1 minus A equal to minus 10 B, 1 minus A, 1 minus A from here is equal to B. So, we write half B equal to minus 10 B which implies that B is equal to 0 and when B is equal to 0 A is equal to 1, hence A is equal to 1. And thus we get y n equal to 1 by 2 raise to the power n which is the general solution of the given second order homogeneous difference equation.

Now, from here we can note the following. From the sequence y n we can see that as n increases y n decreases y naught is equal to 1, y 1 is equal to 1 by 2, y 2 equal to 1 by 2 square y 3 equal to 1 by 3, 1 by 2 to the power 3 and so on.

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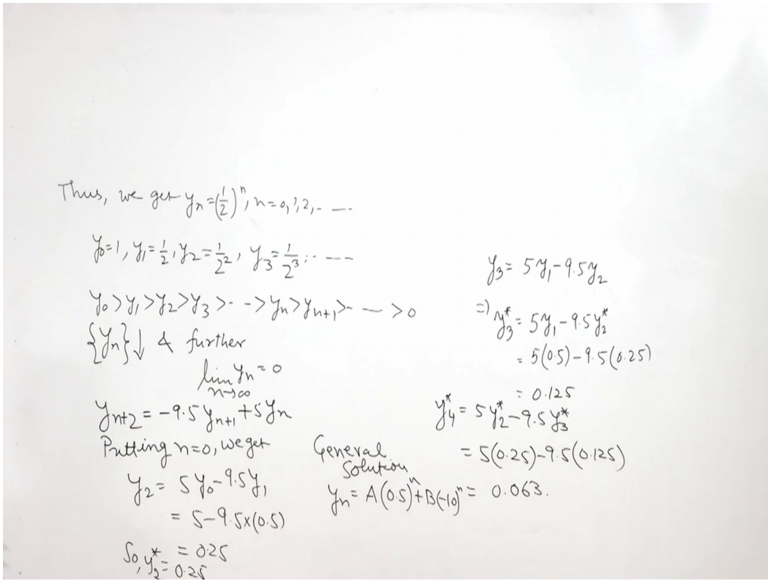

So, y naught is greater than y 1, y 1 is greater than y 2 y n is greater than y n plus 1 and so on. All these values are greater than 0.

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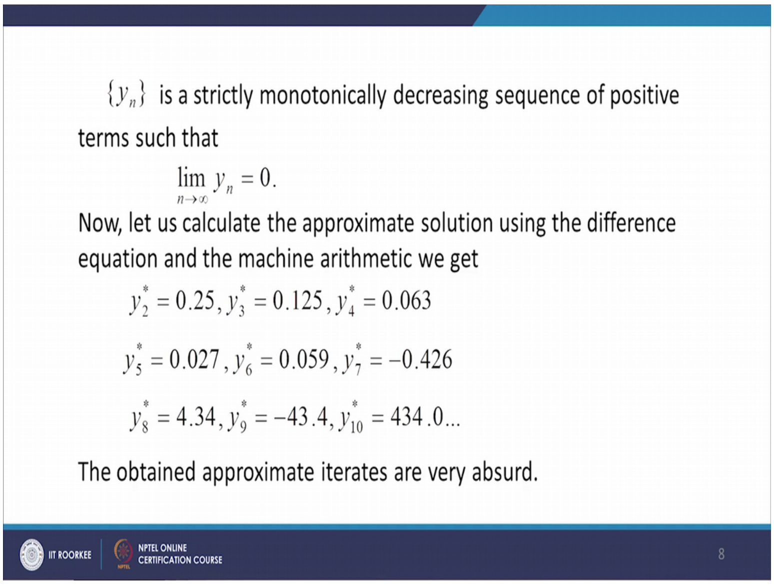

Now, so, y n is a strictly monotonically decreasing sequence of positive terms and also we see that, y n is strictly monotonically decreasing sequence and further limit n tends to infinity y n is equal to 0, 1 by 2 raise to the power n as n goes to infinity is equal to 0. Now, let us calculate the approximate solution using the difference equation and the machine arithmetic where we have assume that beta equal to 10 and t equal to 3, decimal system having 3 digit rounding arithmetic.

So, our scheme is y n plus 2 is equal to minus 9.5 y n plus 1 plus 5 y n. So, these our scheme y n plus 2 equal to 5 y n minus 9.5 y n plus 1; y naught and y 1 we are given, so we can start from y 2. If you put n equal to 0 in this scheme putting n equal to 0 what do we get? y 2 equal to 5 y naught minus 9.5 y 1. Now, y naught is given to be equal to 1. So, we get 5 minus 9.5 into y one is given to be 0.5.

So, when you calculate this it comes out to be 0.25, we denote it by y 2 star. So, this, so we have computed values the values which are computed from the given difference equation we denoted by y n star. So, y 2 star is 0.25. Again we put n equal to 1 here we get y 3 equal to minus 9.5 in to y 2 plus 5 y 1. So, y 3 which is equal to 5 y 1 minus 9.5 y 2 we are writing the computed values as y n star. So, this gives you y 3 star is equal to 5 y 1 minus 9.5 y 2 star and 5 y 1, 5 y 1 is equal to 5 into 0.5 minus 9.5 into 0.25. So, when you compute this y 3 star comes out to be 0.125.

In this way we continue find y 4 star, y 4 star is equal to 5 y 2; when we put n equal to 2 here. So, y 2 y 2 will become y 2 star minus 9.5 y 3 star. So, putting the value of y 2 star as 0.25 we get 5 into 0.25 minus 9.5 into 0.125 which comes out to be 0.063. So, like this we continue. And then we arrive at y 5 star, y 5 star is 0.027 y 6 star is 0.059, y 7 star is minus 0.426.

You can see now the there is fluctuation of sine, earlier it has positive now it has become negative then we get y at star as 4 minus 3 4; 4.34 y 9 star is minus 43.4. So, the values of y n star have started fluctuating starting with y 6 star and then y 10 star we can find y 10 star is very big 434.0. So, the obtained approximate iterates are very absurd from the values of y n star we notice that they are absurd there is the values are absurd.

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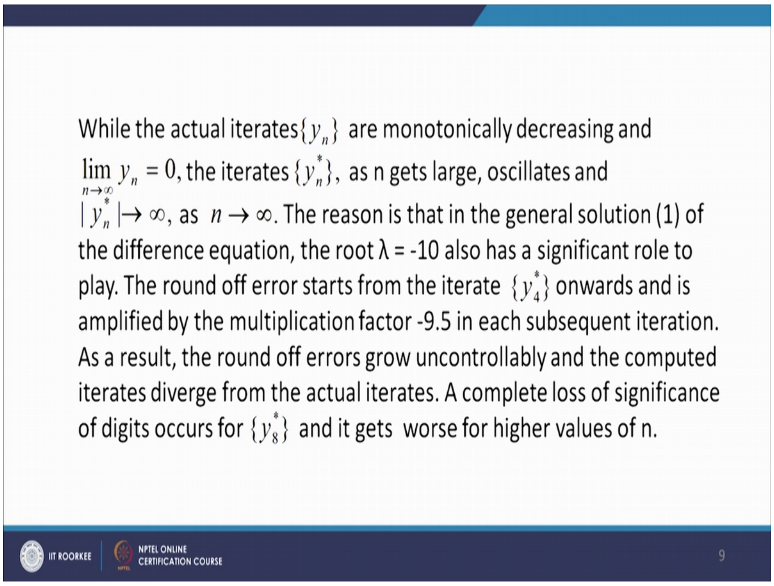

The actual iterates y n we have seen are monotonically decreasing, y n is equal to 1 by 2 to the power n they are monotonically decreasing and y n tends to 0 as n goes to infinity, while the iterates y n star as n gets large oscillates as we have seen and modulus of y n star tends to infinity. We can see that because the value of y 10 star has become 434 when you calculate y 11 star it will be much larger. So, n be the negative sign because they are fluctuating, but then the modulus of y n star, so numerically the y n star is tending to infinity when we take the modulus of y n star. So, it tends to infinity as n goes to infinity.

The reason is that in the general solution we have seen the general solution of the different equations general solution was y n equal to A into 0.5 to the power n plus B into minus 10 to the power n. So, this the root lambda equal to minus 10 has a significant role to play, but when what we do here. So, because of this minus 10 to the power n when we calculate y n it comes out to be 1 by 2 to the power n then we get a monotonically decreasing sequence which can decreases 0. So, this minus 10 has a significant role. The round of error is starts from the iterate y 4 star onwards you can see y 4 star is 0.063 and then y 5 star is 0.027.

So, y 0.25 that is y 2 star y 3 star 0.125 then y 4 star is 0.063, here itself because of beta equal to 3 the round of error is starts taking place. And so, it gets, it in the next iterate y 5 star and so on it is amplified, because of the multiplication factor minus 9.5. When you calculate y 5 star you will multiply by 4 star by minus 9.5. So, it gets amplified because of this factor in each subsequent iteration as a result the round off errors grow uncontrollably and the computed iterates diverge from the actual iterates. So, what will happens is that. So, as a result of this multiplication of minus 9.5 the computed iterates y n star they grow uncontrollably, they grow uncontrollably and the computed iterates and diverge star and diverge from the actual iterates.

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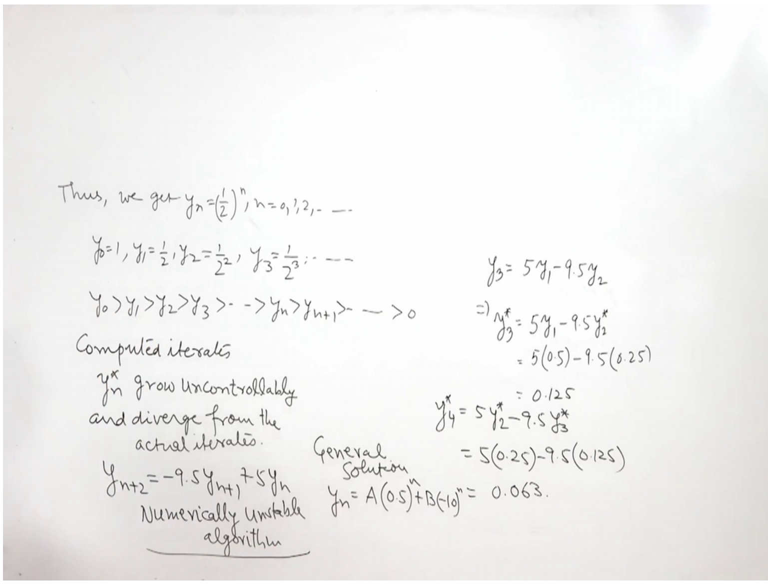

Now, we can notice one more thing, a complete loss of significance of digits occurs for y 8 star let us go to y 8 star, y 8 star you can see is 4.34. Earlier if you look at the values of the iterates they are 0.25, then 0.125, then 0.063, then 0.027, then 0.059, then minus 0.426. So, we had 0 point some value. Now here it y 8 star becomes 4.34. So, the there is a complete loss of significance of digits for y 8 star and it gets worse for higher values of n because then we are multiplying them by minus 9.5 that that factor and so the error gets amplified the round of error gets amplified and the computed iterates get worse for higher values of n.

So, this is we can say this scheme are this numerical algorithm is n unstable algorithm. So, we have y n plus 1, y n plus 2 equal to minus 9.5 y n plus 1 plus 5 times y n it is unstable, numerically unstable algorithm. With this I would like to conclude my lecture.

Thank you very much for your attention.