Numerical Linear Algebra
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Lecture - 15
Diognalizable Matrices

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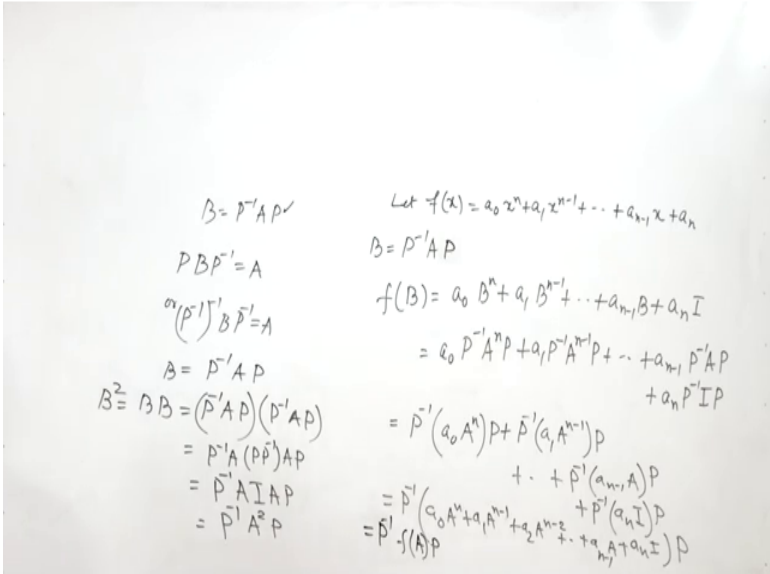

Hello friends, I welcome you to my lecture on diagonalizable matrices. So, let us begin with first be begin with similar matrices. So, let us suppose A and B are square matrices such that we can find a invertible matrix P which can so that we can write B equal to P inverse A P. Suppose A and B are two square matrices of order n, and there is a non singular matrix P such that we can write B equal to P inverse A P. Then we say that we say that B similar to the matrix A.

Now, since we can also write it as P B P inverse equal to A which is same as P inverse inverse B P inverse equal to A. By the definition that B similar to A from here it follows that A similar to B. So, when B similar to A also similar to B and therefore, we say that A and B are similar matrices provided we can find a non similar matrix P such that B is equal to P inverse A P. Now, the similar matrices have very interesting properties.

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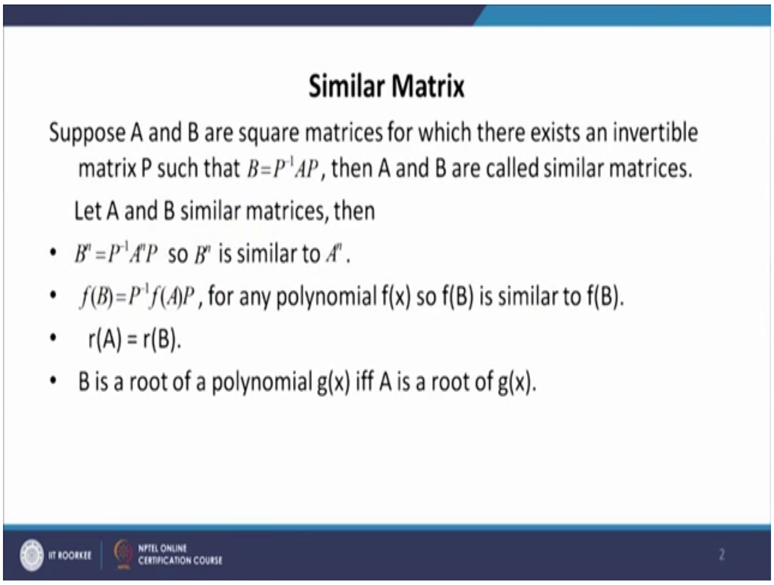

Suppose A and B are similar matrices then we can see that B to the power n is equal to P inverse a to the power n into P. So, A to the power n and B to the power n are similar matrices, where n is a positive integer. So, let us see we can easily see this suppose A and B are similar then B equal to P inverse A P. So, B into B which is B square this will be equal to P inverse A P to P inverse A P or we can say P inverse A P P inverse A P, P P inverse is identity matrix identity matrix into A will give you a, so we get P inverse A or we can say P inverse A square P.

Similarly, we can prove that P B cube equal to B square into B that is P inverse A square P into P inverse A P and then we will get B cube equal to P inverse A cube P. So, we can by mathematical induction we can say that B to the power n is equal to P inverse A to the power n into P n, so B to the power very similar to A to the power n.

Now, let us show that f B is equal to P inverse f A P for any polynomial f x. So, we can say that f B is similar to f A. Let us say f x is equal to a naught x to the power n a 1 x to the power n minus 1 and so on n minus 1 x plus n. Let us take a polynomial in x of degree n and let us assume that A and B are similar matrices, so that B is equal to P inverse A P. Now, we want to prove that f B is equal to P inverse f A P. So, f B is equal to a naught B to the power n plus a 1 B to the power n minus 1 and so on a n minus 1 B plus a n identity matrix of order n. And this is equal to now we have seen that B to the power n is equal to P inverse a to the power n into B, so a naught P inverse A to the power n into P plus a 1 P inverse A to the power n minus 1 into P and so on a n minus 1 P inverse A P. And then we can write a n P inverse I P identity matrix is can be written as P inverse I P.

And then we can write it as P inverse a naught is a scalar, I can take it inside. So, a naught A to the power n in to P then a 1 sorry not a 1 P inverse a 1 A to the power n minus 1 into P and so on P inverse a n minus 1 A into P plus P inverse a n I into P. Or we can write it as P inverse of a naught A to the power n plus a 1 A to the power n minus 1 a 2 A to the power n minus 2 and so on a n A plus no a n minus 1 A and then a n I P. So, this is nothing but P inverse f A P. So, f B is equal to P inverse of A P and therefore, f B and f A are similar.

Now, let us go to the last one B is a root of the polynomial r A is equal to r A, r A is denotes the rank of A, r B is the rank of B. So, when A and B are similar matrices the ranks of A and B are also same, so that we can easily see when A and B are similar matrices, we can write B as B inverse A P. When you post multiply a matrix by an invertible matrix its rank does not change. So, A is pre multiplied by P inverse post multiplied by P which are non similar matrices. So, the rank of A does not change and therefore, rank of P inverse A P is same as rank of A, and therefore, rank of B is equal to rank of A.

Now, then we have B is the root of the polynomial g x; if A is a root of the polynomial g x. So, we can see here if A is a root of f A is a root of f that is f A equal to 0 then A is then B is also root of f. If f A equal to 0, f B equal to 0. So, instead of f we can write here g. So, then B is a root of g x, then f is a root of g x and so on in conversely.

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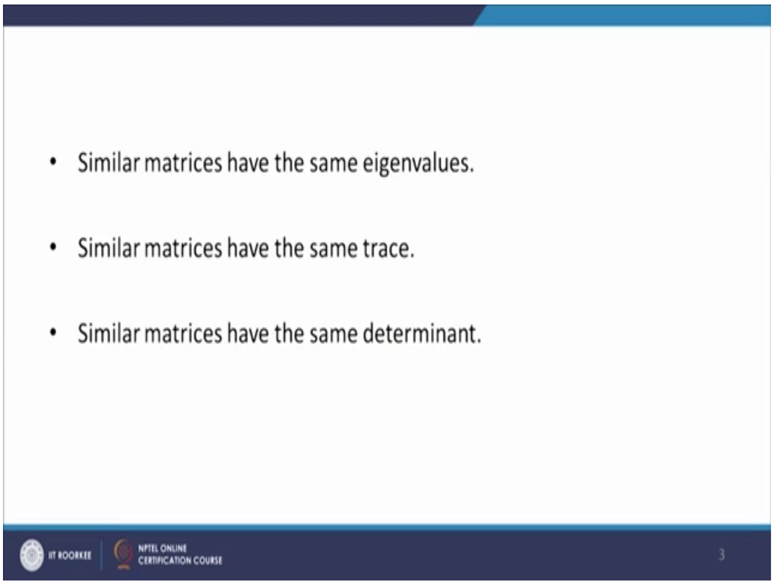

Now, we go to a similar matrices have the same eigenvalues.

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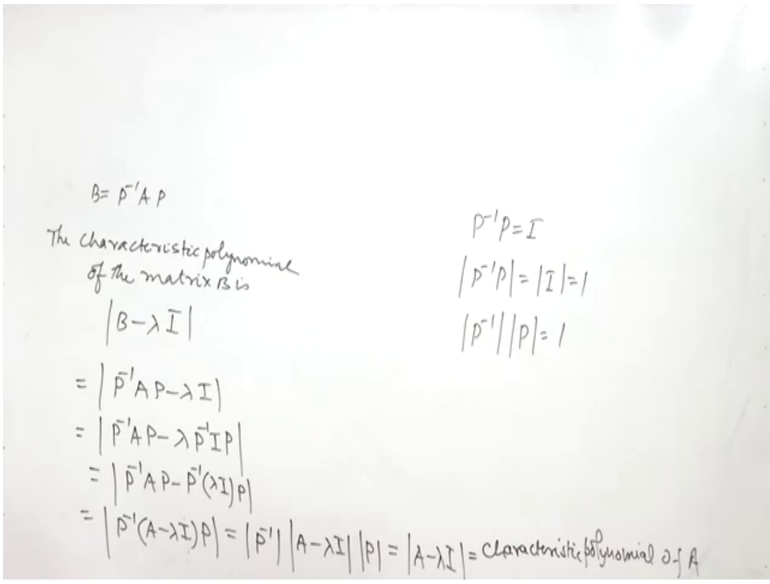

So, let A and B be similar matrices then B are given that B is equal to P inverse A P. We shall prove that the two matrices A and B have the same constructive polynomial. And from there it will follow that they have same eigenvalues. And the characteristic polynomial of a matrix A is given by determinant of A minus lambda I. So, let us write the characteristic polynomial for the matrix B. The characteristic polynomial of the matrix B is determinant of B minus lambda I, which we can write as P inverse A P minus lambda I. And this I can also write as determinant of P inverse A P minus lambda times P inverse I P, I can be written as P inverse I P. And this I can also write as determinant of the P inverse A P minus P inverse lambda is a scalar I can write here lambda I P. Now, which is same as determinant of P inverse A minus lambda I P.

Now, if A and B are two square matrices of the same order, then determinant of A B is equal to determinant of A into determinant of B. So, we can write this as determinant of P inverse into determinant of A minus lambda I into determinant of P. This is equal to determinant of P inverse into determinant of A minus lambda I into determinant of P. Now, P into P inverse are P inverse into P is an identity matrix ok. So, determinant of P inverse P is equal to determinant of I which is equal to 1. Now this is determinant of P inverse into determinant of P which is equal to 1.

So, determinant of P inverse into determinant of P that is equal to 1; we can interchange the position here where real quantities. So, we have determinant of A minus lambda I into determinant of P into determinant of P inverse is equal 1. So, we have determinant of A minus lambdas I, which is the characteristic polynomial of A. So, the two matrices if they are similar have some characteristic polynomial, and therefore, they have same roots. So, we have they have same Eigen values.

Now, similar matrices have the same trace. This follows from the previous result, similar matrices have the same Eigen values, and therefore, they have same trace because trace of a matrix is the sum of the Eigen values. So, they have same trace and then we have similar matrices have the same determinant.

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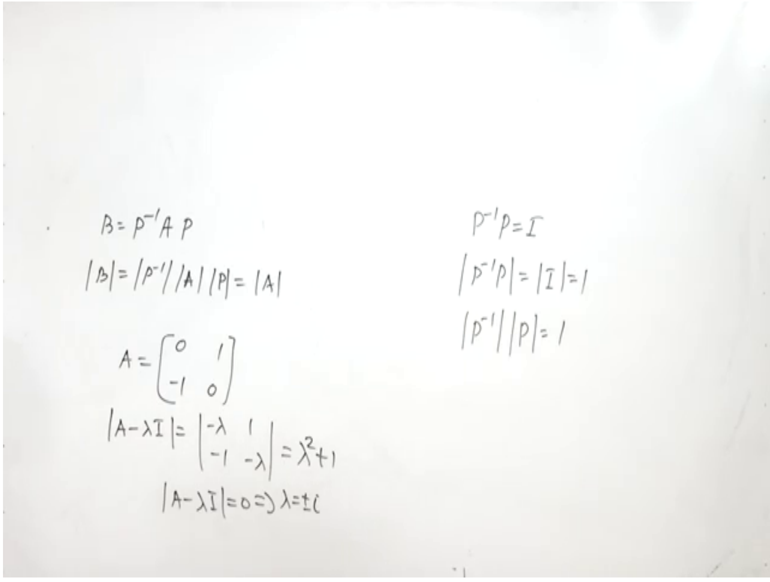

This also follows easily B is equal to P inverse A P. So, you take the determinant of B that is equal to determinant of P inverse A P which is determinant of P inverse into determinant of A into determinant of P as we have seen determinant of P into determinant of P inverse is equal to 1. So, this is determinant of A. So, similar matrices have the same determinate. Ah now comes the question of diagonalization, the diagonalization of A square matrix is very important concept; it is very useful in the study of bilinear forms.

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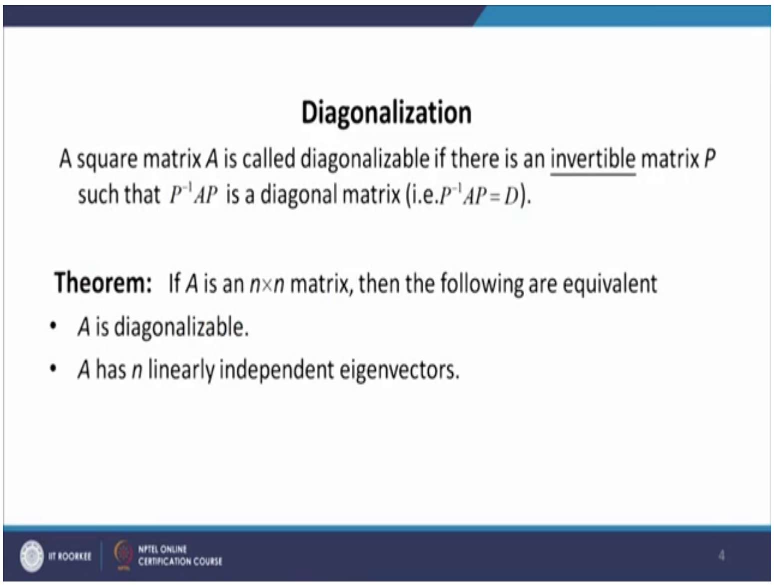

So, let us see how we can diagonalize a given matrix. A square matrix is called diagonalizable if we can find an invertible matrix P such that P inverse A P is a diagonal matrix. Now, not all real square matrices are diagonalizable. There are many examples of real square matrices, which are not diagonalize diagonalizable in fact you can find a real square matrix which does not have any Eigen real Eigen value. Say for example, you can take the matrix A equal to 0 minus 1 minus 1 0. Let us consider this matrix.

Then determinant of A minus lambda I is equal to minus lambda minus 1 minus 1 minus lambda. So, this is this will be I think I should take one here, this is this is one, then that we are like let me write (Refer Time: 13:40) take the matrix like this A equal to 0 1 minus 1 0, so minus lambda 1 minus lambda like this. So, this equal to lambda square plus 1. So, determinant of A minus lambda I 0 gives you lambda equal to plus minus i.

So, there exist matrices which do not have real Eigen values, but we have but there are type of matrices for which the diagonalization is always there says d l c symmetric matrix. If you take a real symmetric matrix then it is always diagonalizable; in fact, we can find a an orthogonal matrix P such that P inverse A P is a diagonal matrix. And one P is orthogonal matrix P inverse is equal to P transpose. So, when a real symmetric matrix we have, then it is always diagonalizable; and there we find an orthogonal matrix P such that P transport A P is a diagonal matrix, so that come that point will come later.

Let us first see foreign for an arbitrary matrix when it will be diagonalizable. So, it will diagonalizable if we can find a in veritable matrix P said that P inverse A P is a diagonal matrix. And the necessary and sufficient condition for the diagonalizability of a n by n matrices that it must have n linearly independent Eigen vectors.

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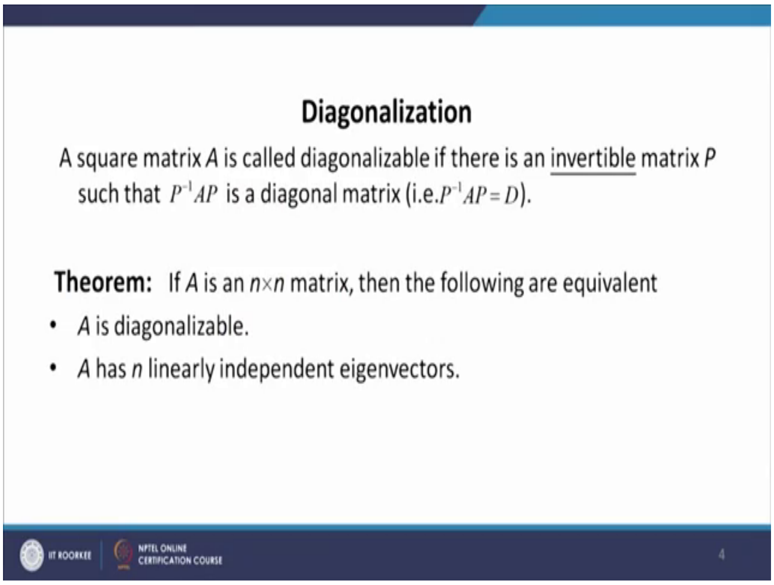

So, A is diagonalizable if and only if A has n linearly independent eigenvectors. Now let us prove this results.

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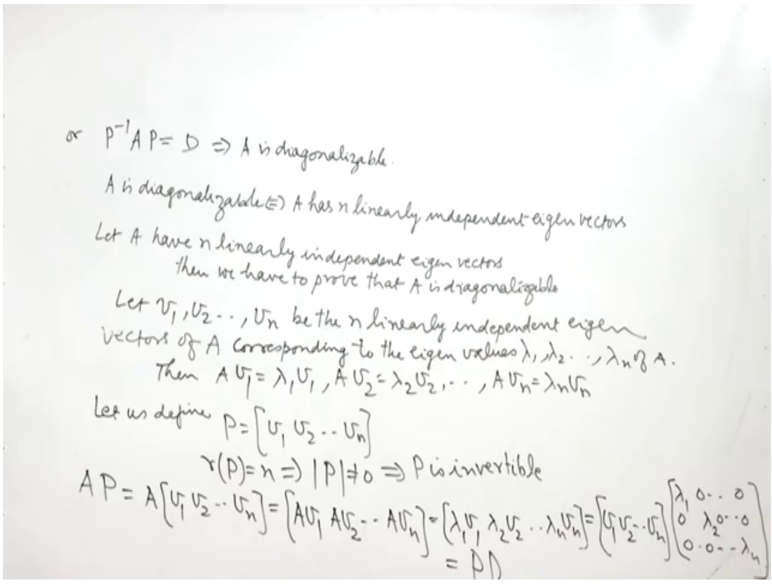

So, we are going to prove that A is diagonalizable if and only if A has n linearly independent eigenvectors, where this n is the order of the matrix. We are considering A to be a square matrix of order n. Now, so let us first assume that let A have n linearly independent eigenvectors then we should we have to prove that A is diagonalizable . So, let us say let v 1, v 2, v n be v eigenvectors of A be the n linearly independent eigenvectors of A corresponding to the eigenvalues lambda 1, lambda 2 and so on lambda n of A. Then we have the matrix equation then A x 1 equal to sorry A v 1 equal to lambda v 1 lambda 1 v 1, A v 2 equal to lambda 2 v 2 and so on, A v n equal to lambda n v n lambda n v n ok.

Now, what do you do is that we have n linearly independent eigenvectors from these n eigenvectors. Let us form the matrix P whose columns are these n eigenvectors. So, let us define P to be equal to let us form the matrix P with these n eigenvectors. Since, v 1, v 2, v n all are linearly independent the matrix P has n linearly independent columns. So, its column rank is equal to n and so rank of the matrix P is equal to n. So, rank of P is equal to n which means that P is a non similar matrix ok. So, determinant of P is determinant of P is not equal to 0, and which implies that P is invertible.

Now, what we do is let us consider A into P matrix. So, A into P matrix is what A matrix multiplied by v 1 v 2 and so on v n. When you multiply the matrix A by P, what you do, you multiply the rows of A by the first column to get the first column of A P. So, when you multiply the rows of A by v 1, v 1 is the first column of the matrix P, but you get the first column as A v 1. Similarly, you multiply by the second column v 2 of the matrix P to all the rows of A, you get A v 2 column second column. And then similarly we get last column A v n which is an n by n matrix.

Now, a v 1 is equal to lambda 1 v 1 a v 2 equal to lambda 2 v 2. So, we get lambda 1 v 1 lambda 2 v 2 and so on lambda n v n. The right hand side this is nothing but v 1, v 2, v n that is the matrix P multiplied by the diagonal matrix lambda 1 0 0 0 lambda 2 0 0 which is diagonal matrix. First column of this diagonal matrix is lambda 1 0 0 0 1. You multiply by this column to v 1 vector, you get lambda 1 v 1. And when you multiply second column to B vector you get lambda 2 v 2 and so on. So, we can write like that.

And the this is nothing but then this is P and that has defined it by D. So, we get P D. So, A P equal to P D or we can say P inverse because P inverse is this P inverse A P is equal to D, where D is the diagonal matrix. So, we have found a matrix invertible matrix P such that P inverse A P equal to D. So, A is similar to the diagonal matrix and eigenvalues of the similar matrices are same which we have already seen. So, eigenvalues of A are lambda 1, lambda 2, lambda n. So, eigenvalues of the diagonal matrix are also lambda 1, lambda 2, lambda n and since the eigenvalues of a diagonal matrix are its diagonal entries, so lambda 1, lambda 2, lambda n where occur the diagonal of this diagonal matrix.

And they occur in the same order in which you have written the matrix P. First column here is the eigenvector v 1. So, in the first column there you have the eigen values lambda 1 second column is the eigenvector v 2 you have the second eigen value lambda 2 there. Last column is the eigenvector v n, so you have the eigen value lambda n there in the last column. So, now this implies that A is diagonalizable. Now, let us prove the converse. Let us assume that A is diagonalizable and then we show that A has n linearly independent eigenvectors.

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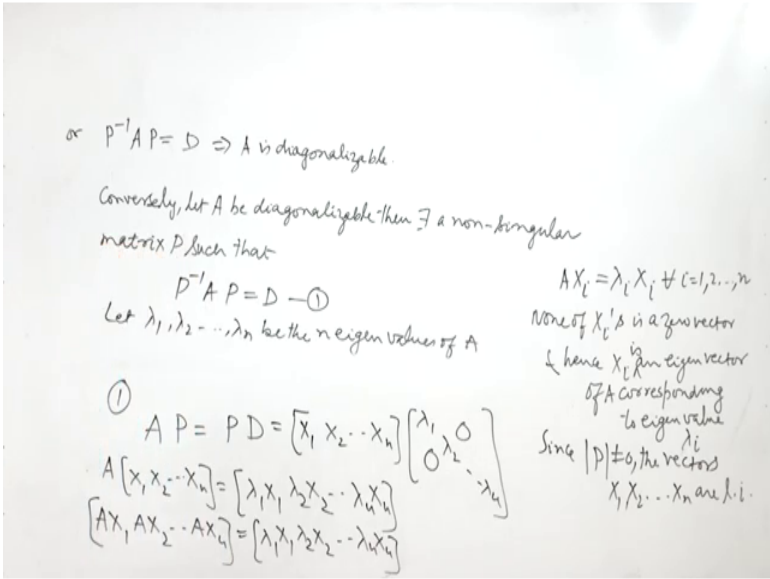

So, conversely let A B diagonalizable, A B diagonalizable. Then there exist a non-singular matrix P such that P inverse A P equal to D. Now, let us say that lambda 1, lambda 2, lambda n be the eigen values of A; and v 1, v 2, v n be the eigen corresponding the eigenvectors ok. So, let lambda 1, lambda 2, lambda n be the an eigen values of a and v 1, v 2, v n be the corresponding eigenvectors. So, what will happen P inverse A P or I can say this equation 1. One is nothing but A P equal to P D, A P equal to PD.

So, what we will have here this is nothing but P matrix let us say the columns of P are wait, wait let us say not like this. Let us assume that lambda 1, lambda 2, lambda n be the an eigen values of A, we shall find their eigenvectors. So, A P equal to P D, then P suppose has columns x 1, x 2, x n; and D has lambda 1, lambda 2, lambda n they are the eigenvalues of A. And here what will have left side will be A x 1, x 2 and so on x n. So, when you multiply this diagonal matrix to this P matrix, what we will get lambda 1 x 1, lambda 2 x 2 and so on lambda n x n. Left hand side will give you A x 1, A x 2, A x n here first column is A x 1 second column is A x 2 and so on. So, what we have by the equality of two matrices A x 1 equal to lambda 1 x 1, A x 2 equal to lambda 2 x 2, A x n equal to lambda n x n. Or we can say a x i equal to lambda I x i for all i equal to 1, 2, 3 and so on up to n. And therefore, for the Eigen value lambda i, X i is the eigenvector ok.

So, now one one one thing more none of the X i is can be 0, because X i if it is 0 then P matrix will be have determinant 0, and we have assume that P is non-singular. So, none of the X i is a 0 vector. And this equation and hence X i is an eigenvector of A corresponding to Eigen value lambda i, corresponding to Eigen value lambda i.

So, what we have corresponding to the Eigen values lambda 1, lambda 2 we have found n Eigen vectors which are x 1, x 2, x 3 and so on x n. And the matrix P whose columns are x 1, x 2, x n is non-singular. So, therefore, the A n eigenvectors are linearly independent. So, since determinant of P is nonzero, the vectors x 1, x 2 and so on x n are linearly independent. So, A has n linearly independent eigenvectors corresponding to the eigenvalues lambda 1, lambda 2, lambda n.

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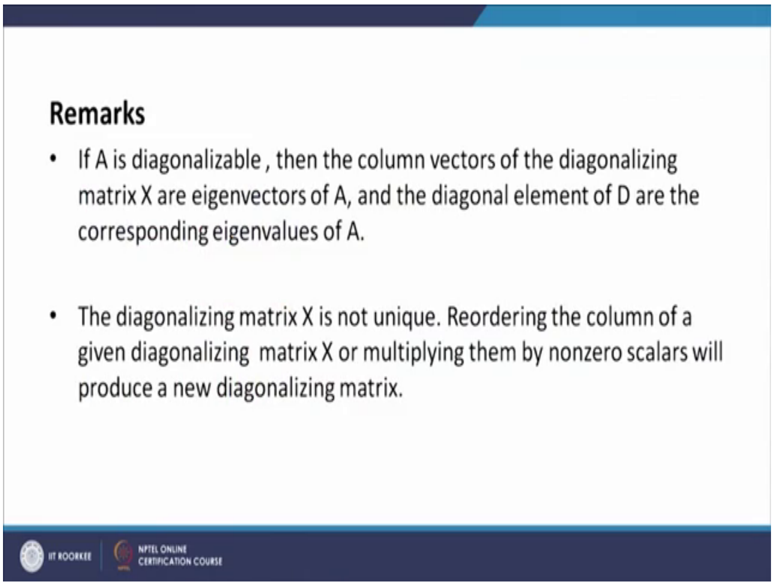

Now, let us make some remarks if A is diagonalizable then the column vectors of the diagonalizable diagonalizing matrix X, here I have written X, but it it it is any matrix you can write P they are the eigenvectors of A. So, the diagonalizing matrix, the columns of the diagonalizing matrix are the eigenvectors of A, and the diagonal elements of D are the eigen values of the corresponding Eigen values of the matrix as I said. The diagonalizing matrix is not unique, because you can write the an eigen vectors as its columns in any order. In whatever order you write them in the same order the eigenvalues will be written in the diagonal matrix. So, the diagonalizing matrix x is not unique. Reordering the columns of a given diagonalizing matrix or multiplying them by nonzero scalars will produce a new diagonalizing matrix ok.

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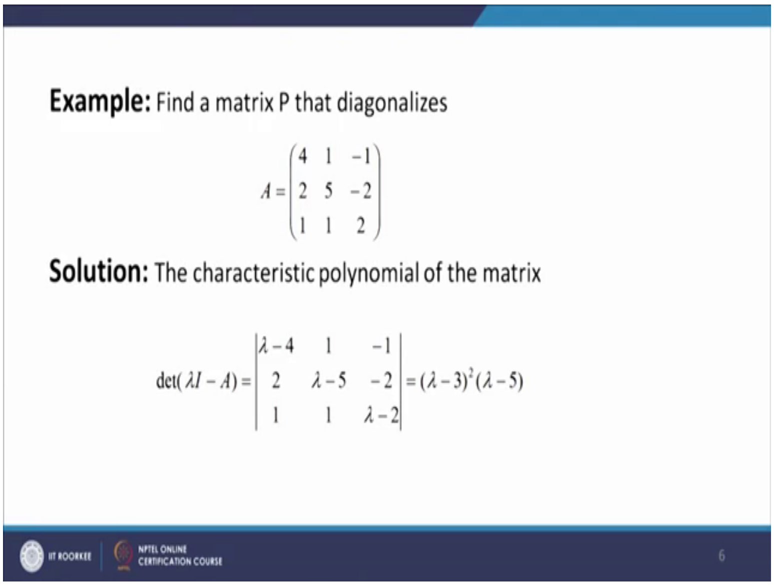

Now, let us find a matrix P that diagonalizes this matrix A. So, is 4 1 minus 1 2 5 minus 2 1 1 2. When we find the characteristic polynomial of this matrix, it is a cubic equation in lambda the determinant of lambda I minus A will come out to be a cubic equation whose vectors are lambda minus 3 whole square into lambda minus 5. So, the eigenvalues of the matrix A are lambda equal to 3, which occurs twice that is its algebraic multiplicity is 3, algebraic multiplicity is 2, and lambda is equal to 5 which occurs once. Now, for the eigenvalue lambda equal to 3, let us find the Eigen vectors.

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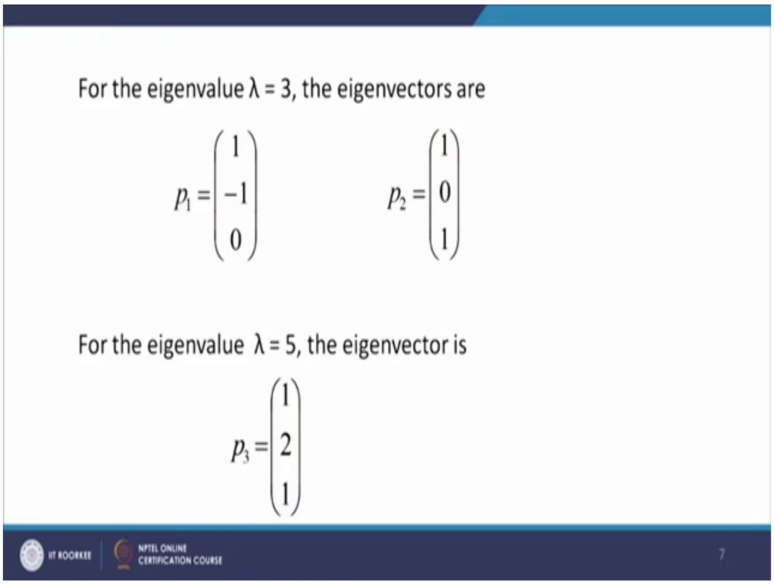

So, we have here the eigen the matrix as 4 1 minus 1.

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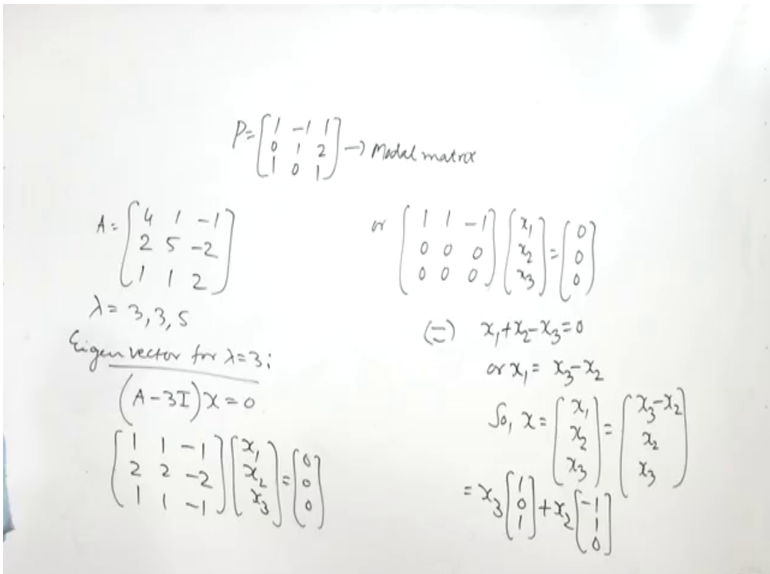

A is equal to 4 1 minus 1, and then we have 2 5 minus 2, and third row is 1 1 2 lambda eigen values are 3 3 5. So, algebraic multiplicity of lambda equal to 3 is 2. Now, eigenvector for lambda equal to 3 let us find so, A minus 3 I x equal to 0. So, if let us subtract 3 from diagonal elements of A, so we will get 1 1 minus 1 then 2 2 minus 2 then 1 1 minus 1. Let us say the components of x are x 1, x 2 x 3; and zero matrix is 0 0 0 column matrix. So, when you multiply this by 2 and subtract here this will become 0;

when you subtract first row from the third row, this will also become 0. So, this system of equations is same as 1 1 minus 1 0 0 0 0 0 0 x 1, x 2, x 3, 0 0 0. The three equations reduce into a single equation, we get x 1 plus x 2 minus x 3 equal to 0.

Now, I can write it as or x 2 by x 1 is equal to x 3 minus x 2. So, x is equal to x 1, x 2, x 3, I can write it as x 3 minus x 2 then x 2 then x 3. So, I can write it as linear combination of two vectors x 3 times 1 0 1, and then x 2 times minus 1 1 0. So, there are two linearly independent eigenvectors they are 1 0 1 and minus 1 1 0. And for lambda equal to 5 we considering find lambda equal to, so the algebraic multiplicity of lambda equal to 3 is 1 geometric multiplicity is also 2. And I had said in my last lecture geometric multiplicity never achieves the algebraic multiplicity, they are equal here.

Now, so and for the lambda similarly if you find the Eigen vectors for lambda equal to five it will come out to be 1 2 1. Here the vectors are 1 minus 1 0, and 1 0 1, I have written minus 1 1 0. If you take instead of x 2 minus x 2 here then it will be 1 minus 1 0. So, they are same things. So, this is how we can find the eigenvectors for the given matrix and then be formed the matrix P ok. So, P matrix be formed from the eigenvectors of the given matrix. So, suppose I write 1 0 1 as the first column second column as minus 1 1 0, and third column as the eigenvector for lambda equal to 5, which is 1 2 1, then this will mat this matrix will diagonalize A this matrix is also called as model matrix.

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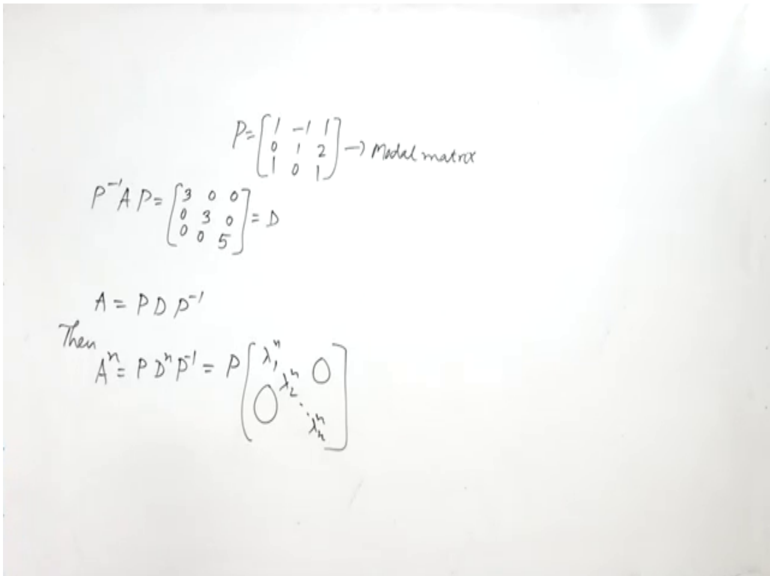

And when we will find P inverse A P we can verify very easily you can find P inverse here. And then if you find P inverse A P, it will come out to be the diagonal matrix where in the first column we will have the Eigen value corresponding to the first eigenvector which is 3. Then in the second column the Eigen value corresponding to the second eigenvector which is again 3 and in the third column the eigenvector corresponding to the 3, third eigenvector which is 5. So, P inverse A P will be equal to this diagonal matrix. So, here diagonalize matrix we have found and also we have found this is D. Now, this diagonalization can be used to find the powers of the matrix A.

So, suppose you are given the matrix A you want to find A to the power k, I can write here A equal to P D P inverse, P inverse P equal to D. So, P I pre multiply P P inverse will become identity and here we will get P D. So, A P will be equal to P D then I post multiply by P inverse we will get A equal to P D P inverse. So, this then A to the power n equal to P D to the power n P inverse we have earlier seen when A and B are similar matrices A to the power n and B to power n are also similar. So, A to the power n is equal to P D to the power n P inverse and D. D has eigenvalues same as eigenvalues of A. So, P lambda 1 to the power n, lambda 2 to the power n, lambda n to the power n and we have 0 here ok.

So, in order to find A to the power n, we need to find the matrix for the matrix A, we need to find simply P matrix. And then you can find any power of A, say A to the power 10 or a to the power 100, even we can find A to the power 100. We have taken an example in the previous lecture, we are we have taken a 2 by 1 matrix and we found A to the power 100 that A to the power 100 can also be found here can also be found here very easily by following this diagonalization process.

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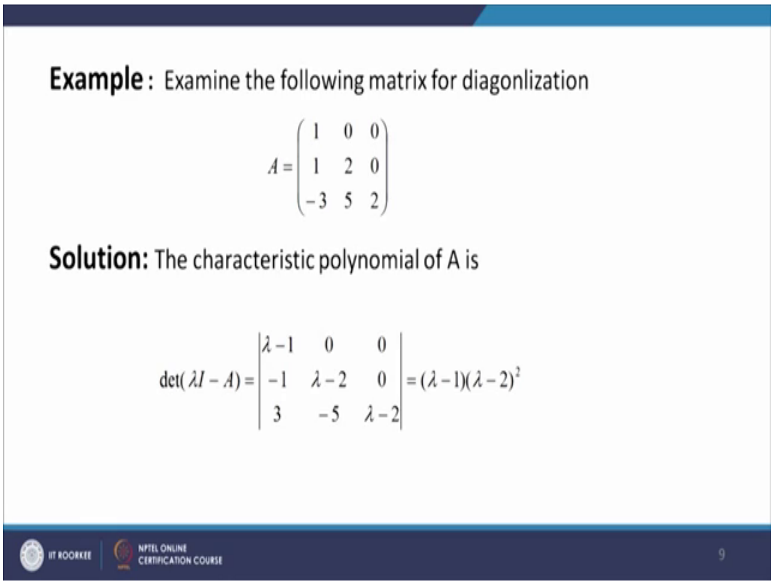

Now, let us taken a example of a matrix where we shall see that the matrix is not diagonalizable. So, here we have an example of a real matrix A equal to 1 0 0 1 2 0 minus 3 5 2. If we write the characteristic polynomial of this n factor factorize it we will have the factors lambda minus 1, lambda minus 2 whole square. So, lambda equal to 1 is an Eigen value lambda equal to 2 Eigen value occurs twice.

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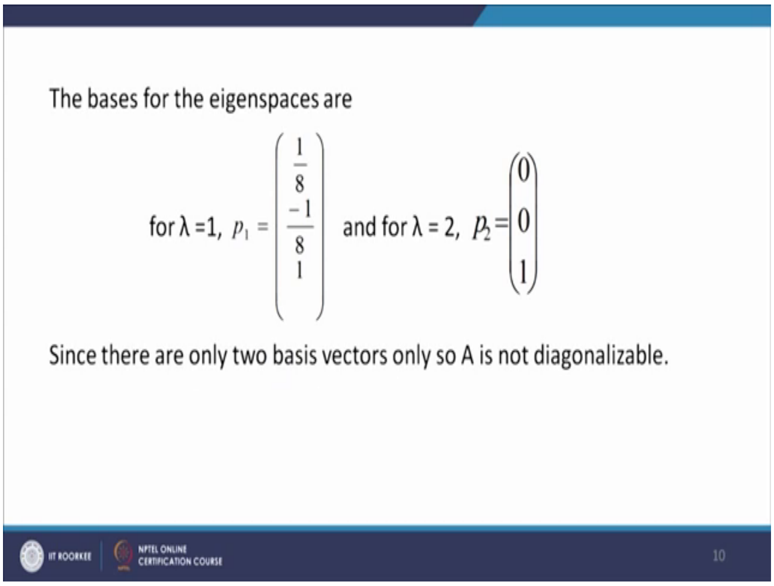

And then if you find the eigenvectors for lambda equal to 1, it turns out that the eigenvector is P one equal to 1 by eight minus 1 by 8 1. You can also write it has one minus 1 and 8 because n is scalar multiple nonzero scalar multiple of an eigenvector is also eigenvector. But in the case of lambda is equal to 2 only one eigenvector we get 0 0 1. So, the total number of eigenvectors here are only two while the matrix is of order three. So, we do not have three linearly independent eigenvectors. And therefore, the theorem says A cannot be diagonalizable, because the theorem says that A is diagonalizable if and only if we have three linearly independent eigenvectors.

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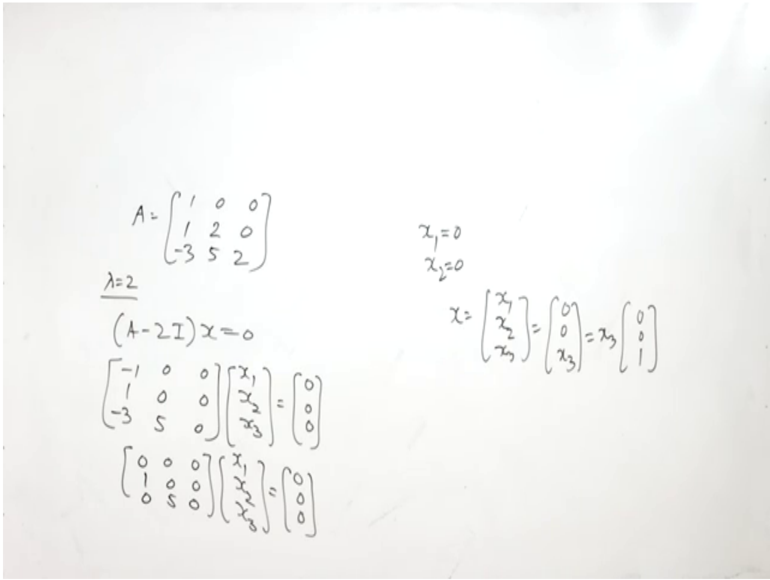

So, here let me see show you how we get a corresponding to lambda equal to 2, only one eigenvector. I would like to show that. So, A is 1 0 0 and then we have A B 1 2 0, and then we have minus 3 5 2. For the eigenvalue lambda equal to 2 let us see find the eigenvector. So, lambda is equal to 2. So, A minus 2 I x equal to 0 will give you A minus 1 0 0 1, we are subtracting 2, so 0 0 and then we are subtracting 2, so minus 3 and we have 5 here and we have 0 here.

So, what do we get x 1, x 2, x 3 equal to. If I add this second row to the first row first row becomes zero row, adding first second row to the first row. And adding three times the second row to the third row makes it 1 0 0, and this we are adding three times the second row to the third row, so 0 5 0. So, the given system of equations is by linear elementary row operations we have reduced to this system.

So, what we have first row equation is a 0 equation. Second equation says that x 1 into 0, x 2 into 0, x 3 into 0, 0. And third equation is give us x 2 5 time x 2 equal to 0, wait this is x 1 ok. Second equation gives x 1 equal to 0, x 1 equal to 0. Third equation gives you x 2 equal to 0; you see 5 x 2 equal to 0. So, x is equal to x 1, x 2, x 3. So, this is equal to 0, 0, x 3, and therefore, it is scalar multiple of 0 0 1 vector. So, here corresponding to the repeated eigenvalue lambda equal to 2, we have only one eigenvector. So, geometric multiplicity is 1, by algebraic multiplicity is 2. So, we do not have three eigenvectors and therefore, this matrix is not diagonalizable. So, so this is what we have.

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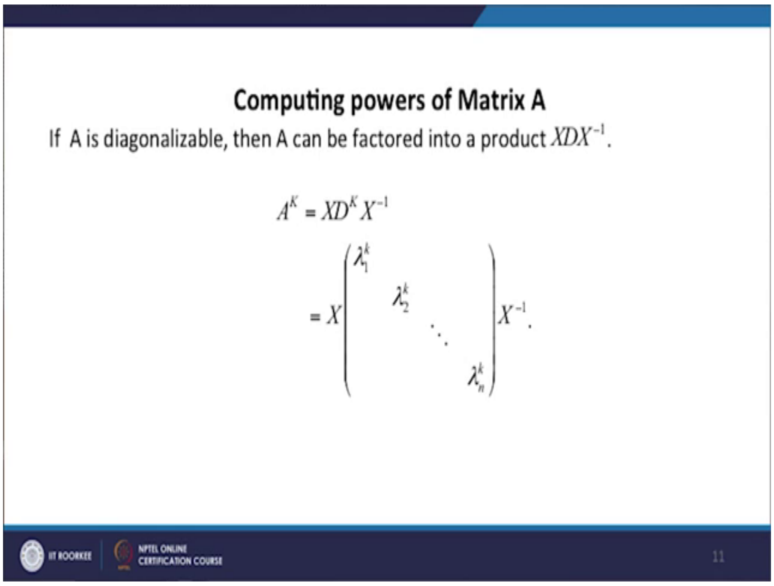

And as I said when be A is diagonalizable, we can find any integer power of A, A to the power k is equal to X D to the power k X inverse, so that is what I have to say in this lecture.

Thank you very much.