Numerical Methods
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Lecture 38
Numerical Methods - 2

Hello friends. So welcome to the third lecture of this module and in this lecture we will continue from the last lecture in which we have introduce Euler's method and in the Euler’s method we have seen that we need to reduce the step size for getting a better accuracy and hence reducing the step size means you need to do more calculations. So in this lecture our aim is to develop a whole family of numerical methods that can attain any order of accuracy. Unlike the Euler’s method where we are having a accuracy of order each.

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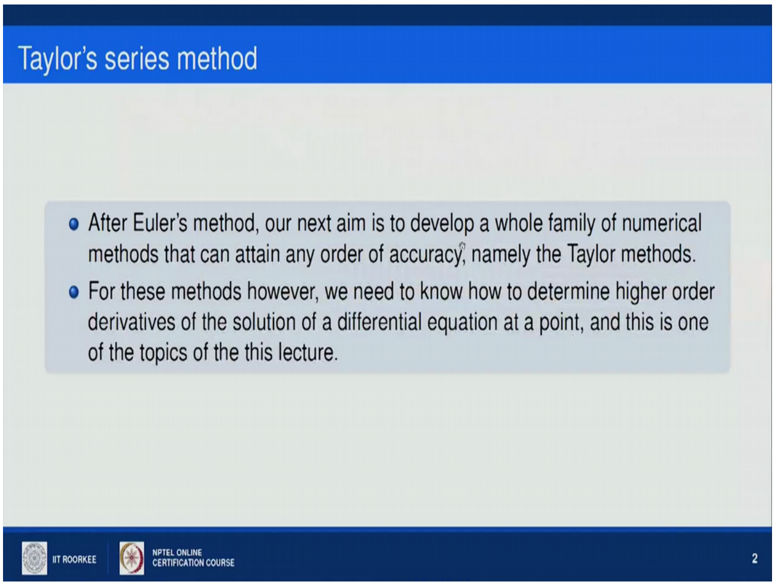

So here we will do that Euler’s method in this lecture and specifically I will talk about quadratic Tailor method and then I will tell you how we can generalize Tailor method up to any order of any order. But in Tailor’s method we need to know how to determine higher order derivatives of the solution of a differential equation at a point and this is also we will explore in this lecture that how to calculate higher order derivative for a given function. So let us start with quadratic Tailor method.

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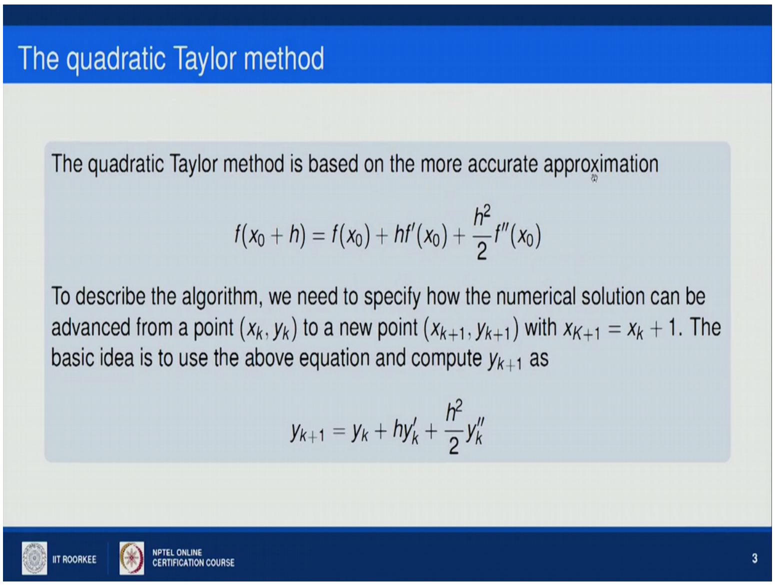

So the quadratic Tailor method approximation is based on the more accurate approximation that is the approximation of second order derivative like in Euler's method we have taken only upto first order derivative but here we are taking up to second order deri derivative so a function can be approximated about a point X nought by this expression that is F of X nought plus H times F prime X nought plus H square upon 2 F double prime X nought.

To describe the algorithm we need to specify how the numerical solution can be advance from a point XK YK to a new point SK plus 1, YK plus 1. Where XK plus 1 is XK plus H. Now basic idea is to use the above equation and compute YK plus 1 so by the Tailor series expansion we can write YK plus 1 HYK plus H time Y prime K plus H square by 2 Y double prime K.

So in this expression you can see we are having Y prime K and realize Y double prime K. So Y prime K can be given by the differential equation from our initial value problem. However we need to calculate Y double prime K here. So for calculating Y double prime K we will use the lemma.

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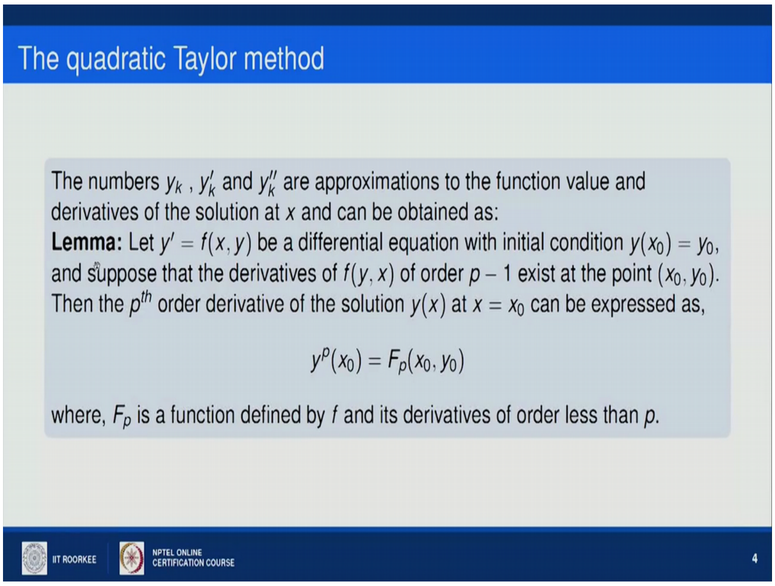

And in this lemma we are having a its function Y prime that is equal to F of XY that is the given differential equation with initial condition Y X nought equals to Y nought. In the same time suppose that the derivative of F of order P minus 1 exists at the point X nought, Y nought then the Pth order derivative of the solution Y X at X equal to X nought can be expressed in terms of FP capital FP X nought Y nought.

Where capital FP is a function defined by F and its derivative of order less than P. So let us take an example to get a better understanding of this lemma.

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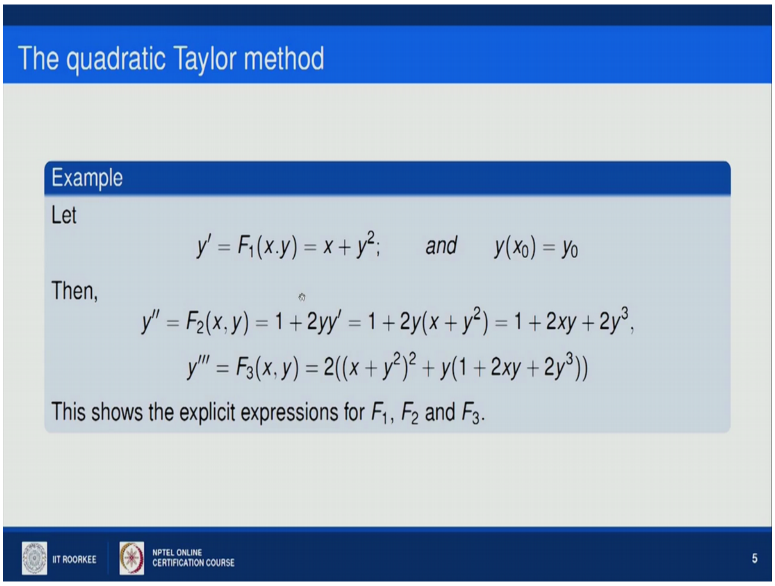

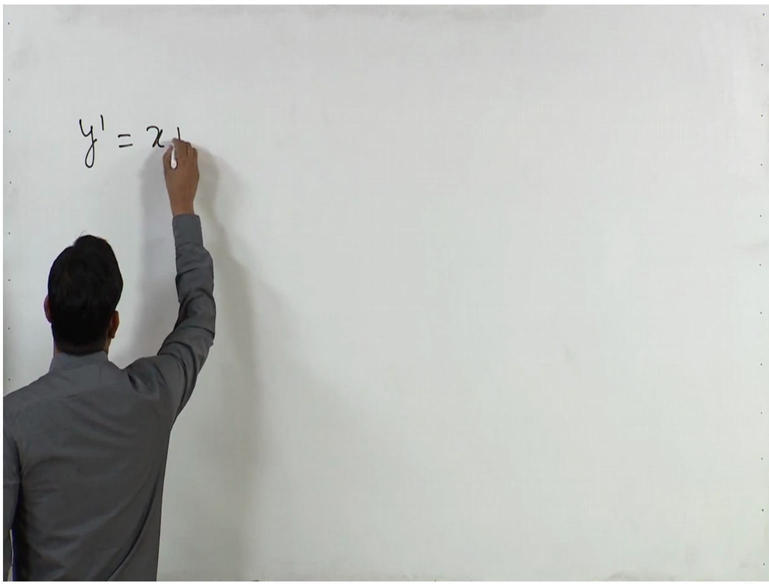

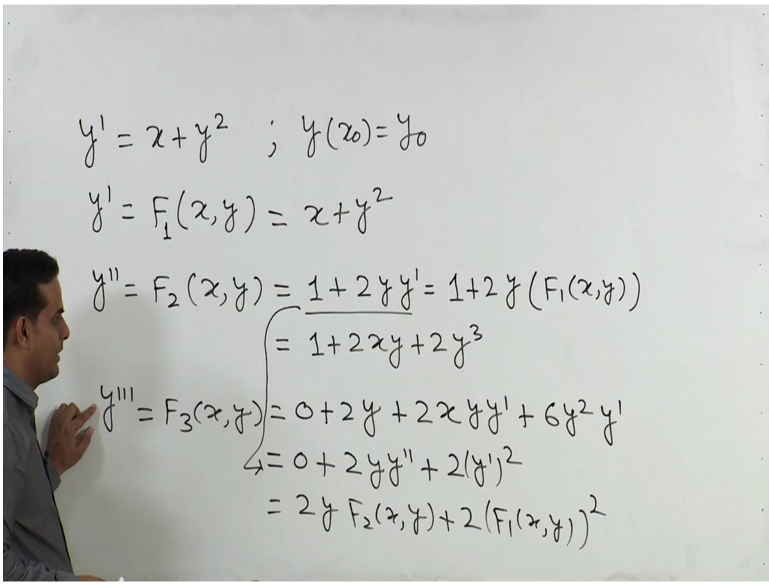

So in this example we are having a differential equation Y prime equals to X plus Y square together with initial condition Y of X nought equals to Y nought. Now let us assume that Y prime equals to F of XY or I will write F1 of XY. Now if I calculate Y double prime that is some according to previous lemma I am having F2 of XY. And this is the differentiation of this particular term with respect to X.

So it will be 1 for this particular X and then 2 Y into Y prime so that is 1 plus 2 times Y and Y prime is F1 of XY. So here you can note down for calculating the second order derivative of Y that is the Y double prime we need the value that is value of Y as well as value of capital F1 XY. That is the derivative of Y which is less than order 2. And this is coming out now if I substitute the value of F1 XY from here that is your X plus Y square.

So it will be 1 plus 2 XY plus 2 times Y cube. Similarly we can calculate higher order derivative for example if I want to calculate Y triple prime that will be F3 XY so it will become 0 plus 2 times Y plus twice XY into Y prime plus 6 Y square into Y prime. Or somewhere I can write if I want to calculate from here directly 0 plus 2 times Y into Y double prime.

So if I am coming from here so it will be 0 plus 2 times Y into Y double prime plus 2 Y prime whole square. So basically each two times Y into F2 XY plus 2 times F1 XY whole square. So here again you cannot (())(6:57). I am using the value of Y, I am using the capital F1, I am using capital F2 for calculating capital F3 according to previous lemma and by substituting all these values I can get Y triple prime. Now how to use quadratic Taylor method?

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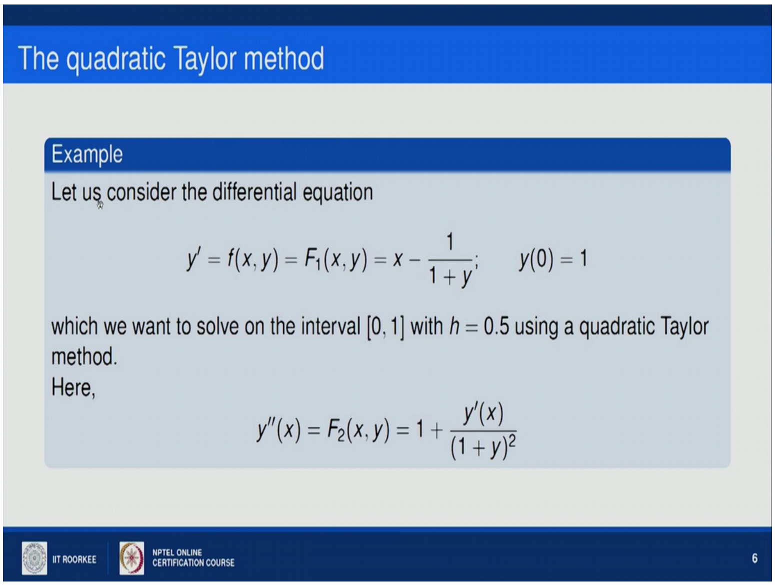

For that again we will consider an example of an initial value problem that is given as Y prime equal to a small f of XY that is also my capital F1 of XY and it is given as X minus 1 over 1 plus Y and initial condition is Y at X equals to 0 equals to 1. If we want to solve these particular initial value problem on the interval 0 to 1 with step size h equals to 0 point 5 using quadratic Tailor method.

So first of all we need to calculate the second order derivative of Y for applying the quadratic Taylor method and for doing that Y double prime X can be given as F2 of XY that is basically 1 plus Y prime X upon 1 plus Y whole square. So now after that what I will do?

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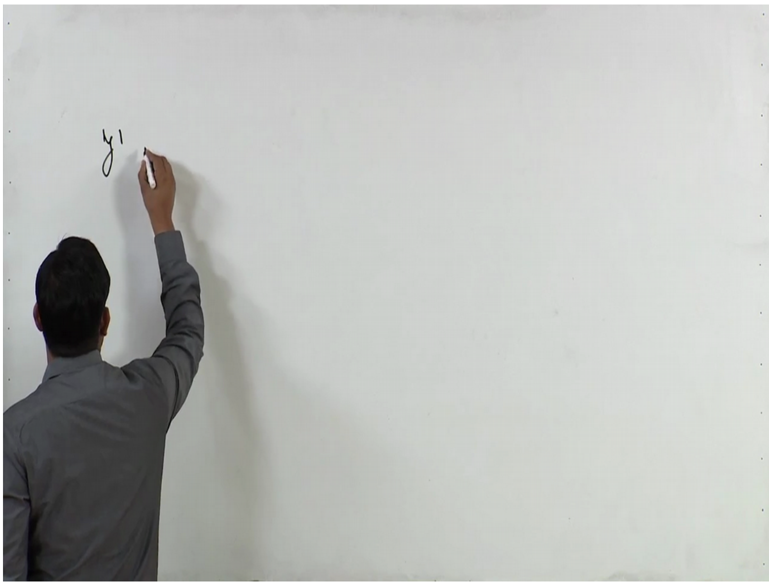

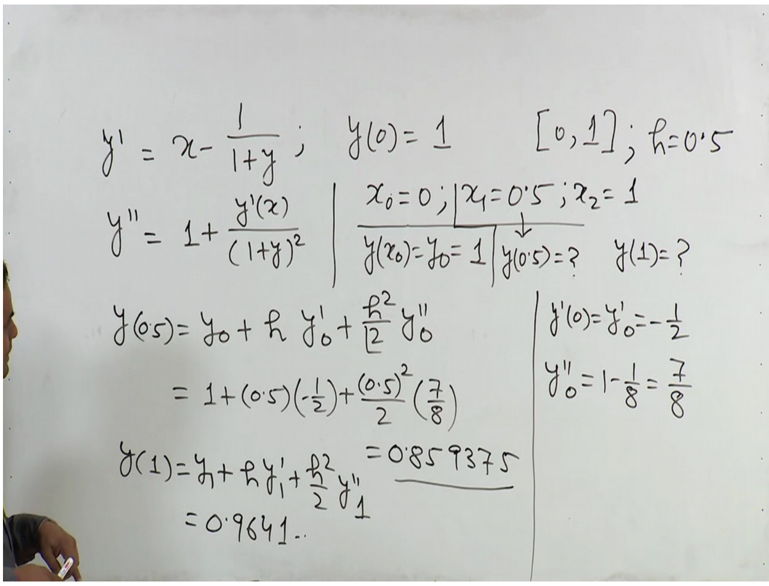

I need to solve a differential equation Y prime equals to X minus 1 over 1 plus Y Y at X equals to 0 is given as 1. So I want to solve this on the interval 0 to 1 with h equals to 0 point 5. So now Y double prime is 1 plus Y prime X over 1 plus Y whole square. So here X nought is 0, X1 is 0.5 that is 0 plus H and X2 is1. At initial Y at X0 that is my Y0 it is 1. Now I need to calculate Y at 0 point 5 and Y at 1 using the quadratic Tailor method.

So for doing this what I want I will write that YX equals to Y0 plus H times Y prime 0 plus H square by 2 Y double prime 0. Now Y0 is given as 1. If I want to calculate Y prime 0 that is Y prime 0. so from here I will calculate Y prime 0 X is 0, 0 minus 1 upon 1 plus 1 so it is minus Half. Y double prime 0 will I can calculate from here. So 1 plus Y prime 0 that is minus Half upon 1 plus 1.

So 1 plus 1 will become two. A two square will become four. So 1 minus 1 upon 8. It is basically 7 upon 8. So after putting these values here Y0 is 1 plus H is 0.5 Y prime 0 is minus 1 by 2 plus 0 point 5 whole square upon 2 and then Y double prime 0 is 7 by 8 and after simplifying this I will get a value 0 point 85937. So this is the approximation of Y at X equals to 0 point 5.

Again if I want to calculate Y at 1 what I will use? I will use Y1 plus H time Y prime 1 plus H square upon 2 Y double prime 1. So I am having value of Y1 which I will kept H 0 point 859375. I will calculate the value of Y prime 1 from this formula. After putting the value X 0 point 5 and Y is this value and then similarly I can calculate Y double prime with the help of this expression.

And then finally I will get this value which is coming out something 0 point 9641 and so on. So this is the overall procedure for implementing quadratic Tailor method For solving initial value problem. And here we are getting more accuracy compare to the Euler's method without reducing the step size means taking the large steps.

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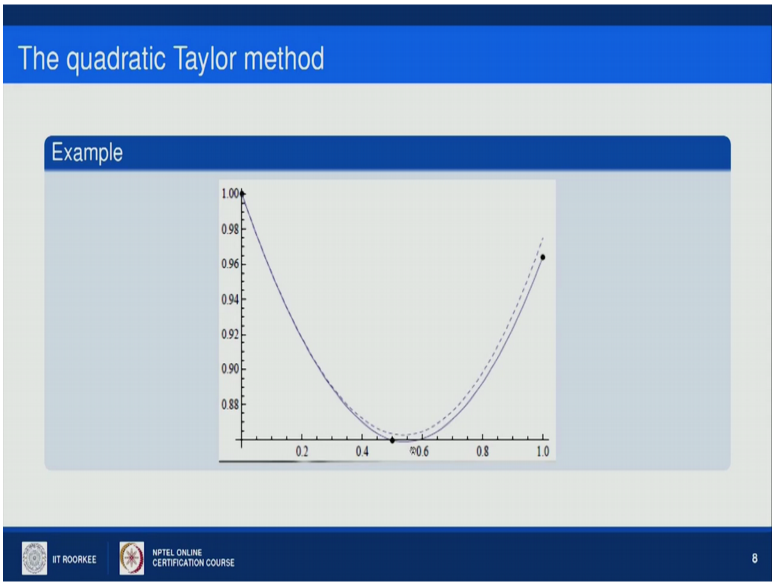

So this is the curve of the approximate solution and the exact solution. Here you can see between 0 to 1 the curve the two curves are quite similar and hence we are getting a good approximation with larger step size using the quadratic Tailor method. If we talk about Tailor methods of higher order we can do it.

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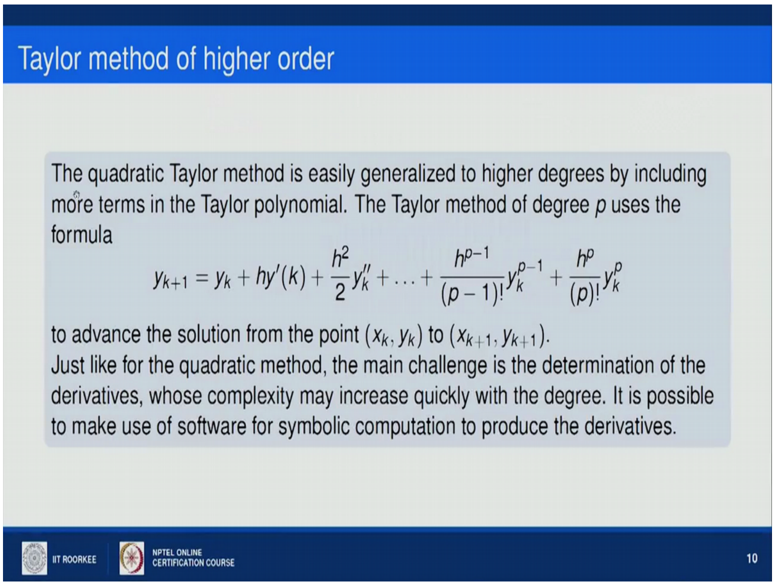

The quadratic Tailor method is easily generalized to higher degrees by including more terms in the Tailor polynomial. For example that Tailor method of degree p uses the formula Y at X K plus 1 equals to YK plus H times Y prime K plus H square upon 2 Y double prime K and upto Pth order derivative. That is the last term will be a H rest to power P upon factorial P into YK Pth derivative.

To advance the solution from the point XK YK to point XK plus XK plus 1 YK plus 1 we will use this formula. So just like for the quadratic method the main challenge is to determination of the derivatives. How to determine higher order derivatives? And for that we can use the same lemma however we have to do a lot of computations and hence the complexity increase quickly with the degree.

It is possible to make use of software for symbolic computation to produce the derivatives of higher degree. Now if we talked about error in quadratic Tailor method so we can drive it in this way. So Y at XN plus H can be given by Y of XN plus H times Y prime XN plus H square upon 2 Y double prime XN plus third and higher order derivatives.

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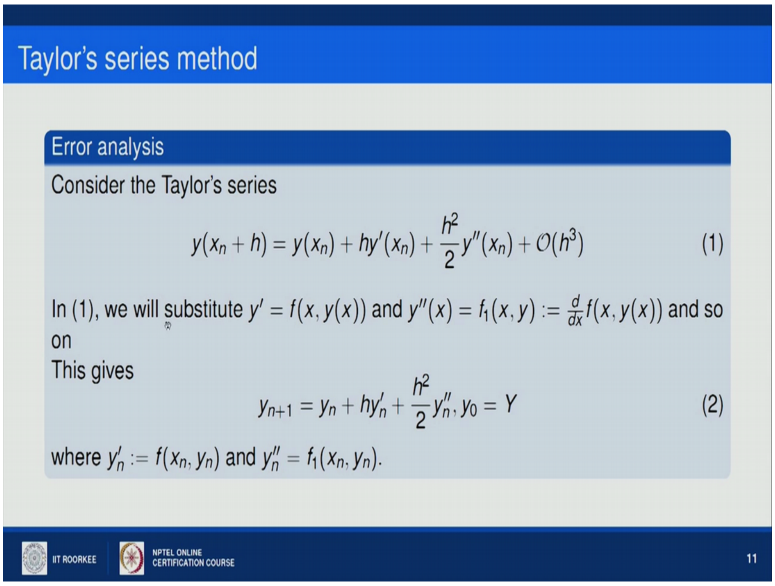

So in this equation we will substitute Y prime is F of X YX and Y double prime is F1 of X Y that is the derivative of F with respect to X and so on. So this gives YN plus 1 equals to YN plus H times Y prime n plus H square upon 2 Y double prime n with initial condition Y0 equals to capital Y. Here Y double prime n as you know is F at XN YN and Y double prime n is F1 XN YN.

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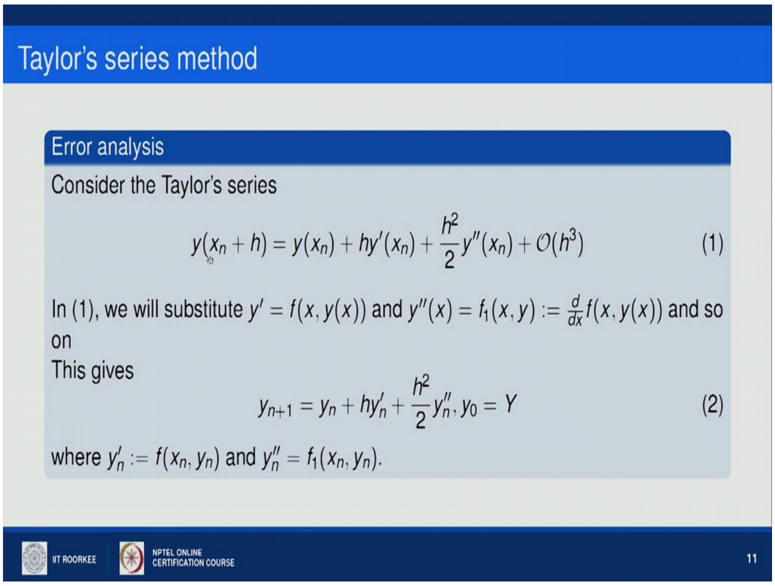

So taking the difference between 1 and 2 we get the in the left hand side it will be YN plus 1 minus Y at XN plus H. So that will be the error in n plus 1 iteration.

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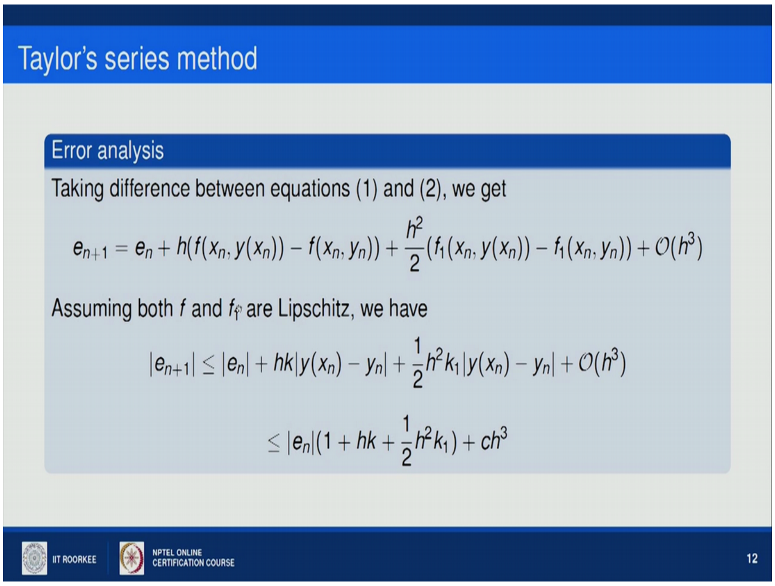

And let us denote it E of N plus 1 equals to if you take the

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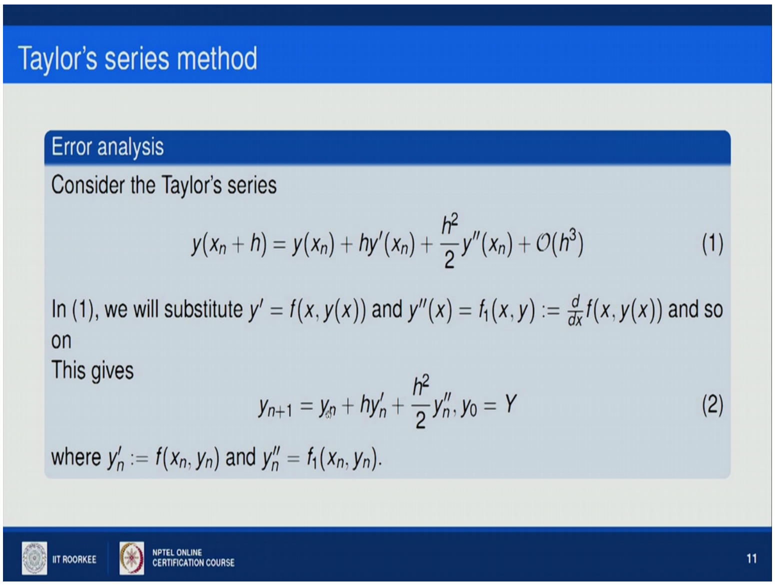

See the first time in right hand side it will be YN minus YXN.

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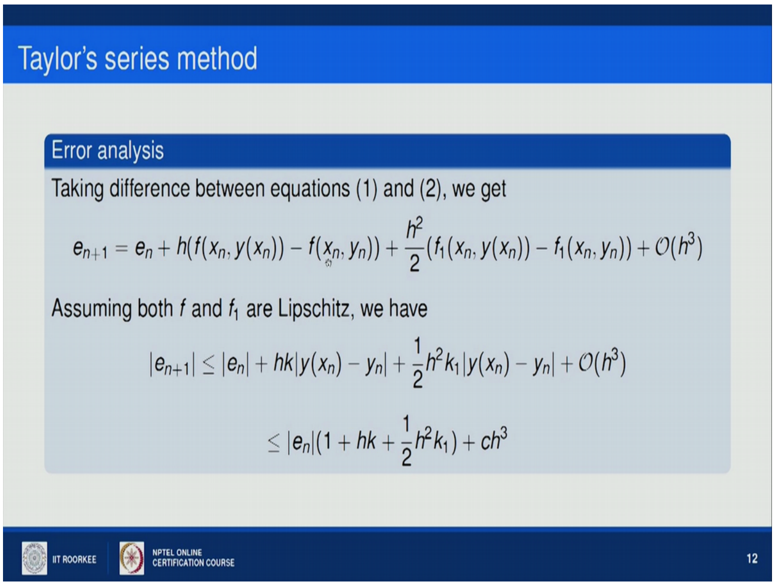

So this I am writing EN plus H times F at XN YXN minus F XN YN plus second order term and then third and high order terms. If we assume that both F as well as F1 are Lipschitz continuous then we can replace these two expressions in round brackets like this one and this one by the definition of lipschitz continuity in this way that EN plus 1 will be less than equal to en plus H times K Y XN minus YN.

So K is lipschitz constant for this particular expression plus Half time H square K1 and again this term. That is basically EN. It is again EN. So this after simplification I can write EN if I take out 1 from here plus HK from here plus half times H square K1 plus third and higher order terms. This expression can be written in this form.

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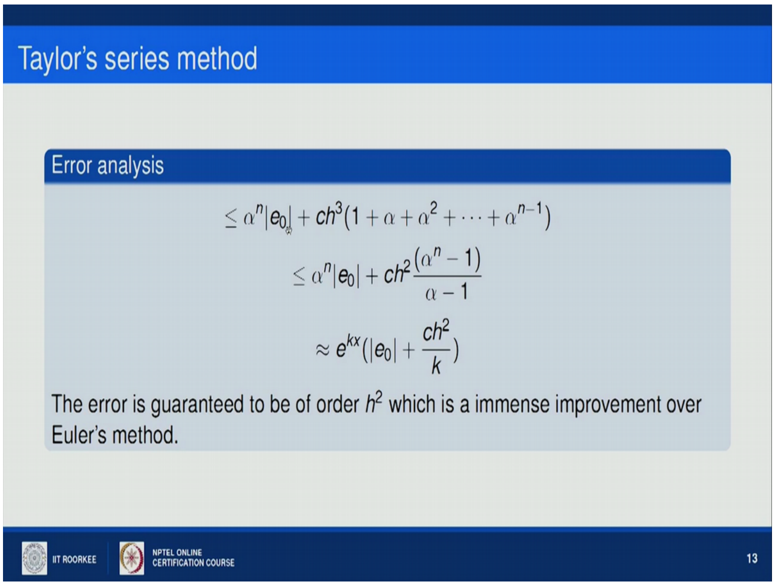

If I used the expression for the error from the first iteration that is E nought that is the initial error and finally this sum can be written in this way that is 1 plus alpha plus alpha square plus upto alpha N minus 1. In this particular form and finally it is coming out in this way. So this particular expression gives a guarantee to be that the error will be of order H square which is an immense improvement over Euler's method where we were having the accuracy of order H.

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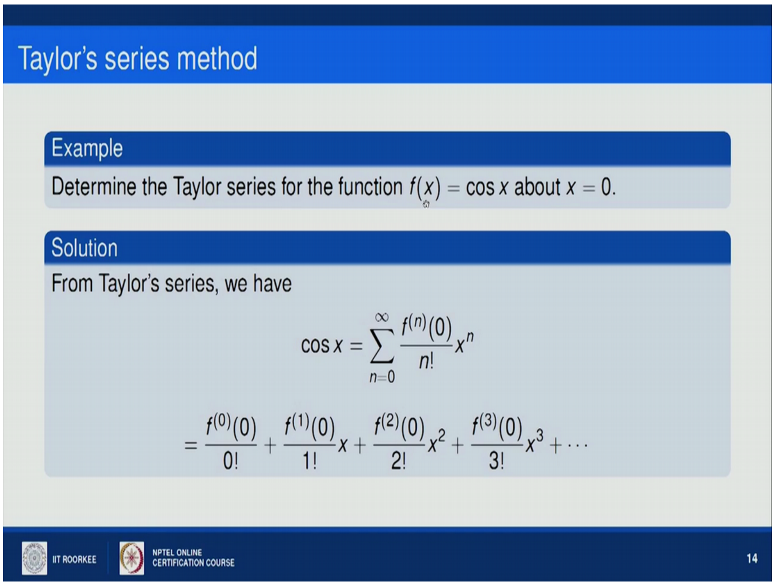

We can also use this Taylor's series for finding the approximation of a function about a given point for example if I want to find out the expression of f X equals to 2 cos X about X equals to 0. So Taylor's series is given by this 1 and hence we I am having different order derivatives in different terms after calculating all these at x equals to 0, so I will get 1, 0, minus 1, 0, 1.

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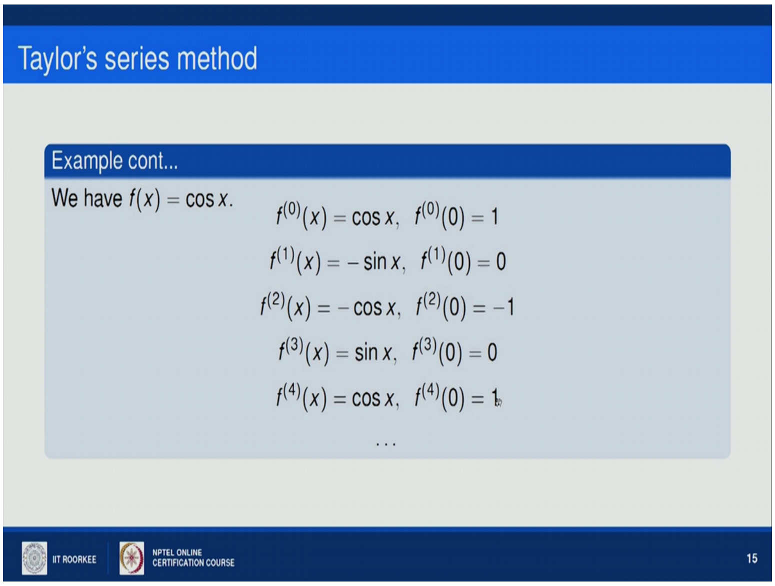

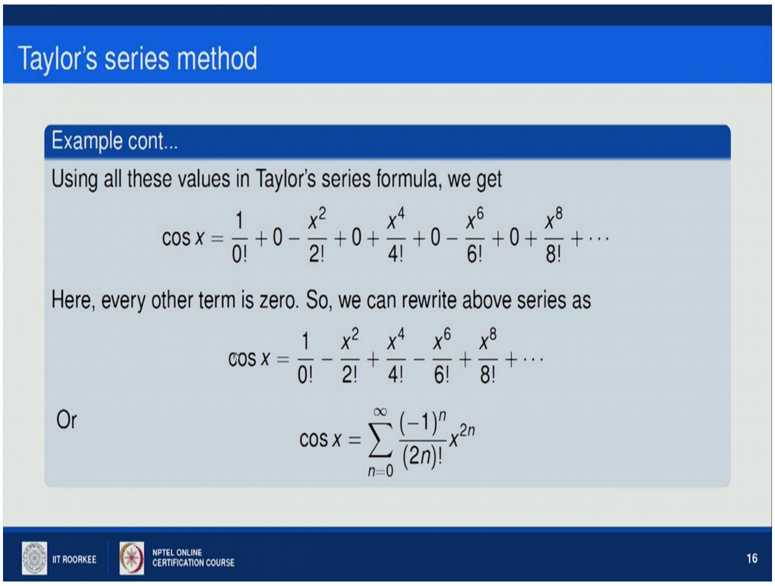

And then substituting in expression I will get this particular 1 minus X square by factorial 2 plus X rest to power 4 upon factorial four and so on. So this was about the Taylor's method and here I told you that we can use quadratic Tailor method which is having the error of order of H square however we can use higher order Taylor method for getting better accuracy.

Now I will explain one more method that is Euler's modified method and that is just an improvement of the Euler's method which we have discussed in earlier lecture and what is this method?

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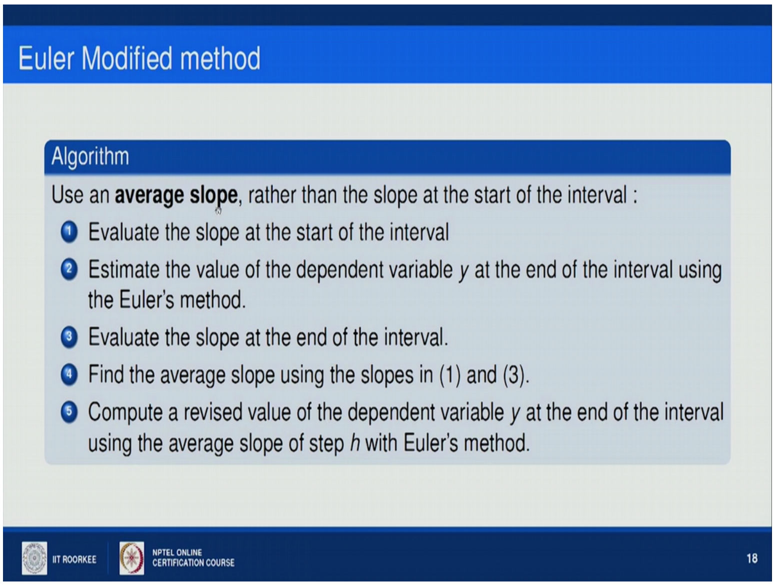

Basically in this method we will use an average slope rather than the slope at the start of the interval like we have taken the slope at the starting point in of the interval in Euler's method. So what I will do? I will evaluate the slope at the start of the interval, I will estimate the value of the dependent variable Y at the end of the interval using the Euler's method evaluate the slope at the end of the interval.

Find the average slope using slopes in step 1 and step 3 and compute a revised value of dependent variable Y at the end of the interval using the average slope of step H with Euler's method.

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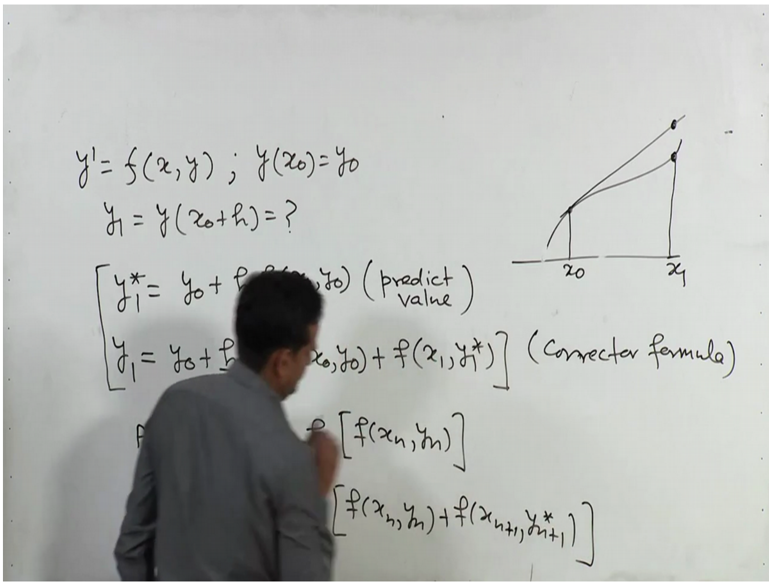

So if I want to explain this method so basically problem is I need to solve this problem by prime equals to f of XY, YX not equals to Y nought. And I want to calculate Y1 which is the value of Y at X nought plus H. So what I have done in the Euler's method I have used Y1 equals to Y0 plus H times FX0, Y0. Where I am approximating the value of Y at X equals to X1 by the slope which I have taken at the initial point of the interval.

For example if this is the function, this is the point X nought, this is the point X1. So here I know the value and I am taking the slope here and I am approximating this value by this one in Euler's method. Here I am calculated this value of Y1 and I will use this as the predict value. Then what I will do? I will correct this predict value by a new formula that will be Y nought plus H times F of X nought Y nought plus F of X1 Y1 star.

So what I am doing using the Euler's method I am finding the slope at the end point of the interval means at this particular point and then what I am doing I am taking the average of the slope over the whole interval by 1 by 2 of these two. And here what I am doing? This formula I can use again and again because this is the corrector formula and this formula I can use again and again in an implicit manner.

How? You can see here I am having Y1 as well as Y1 star, so I will calculate Y1 from this again I will substitute that Y1 here. I will get a new Y1 that is the more better approximation and again and again I will repeat this process and I will get a better approximation. So this these two formulas jointly is called Euler's modified method. So in first you need to find out a predict value of Y1 and then you can correct the predict value by this formula.

So in general setting I can write this as YN plus 1 is equals to YN plus H times f of XN, YN. If you know the value of YN at X equals to XN and you want to calculate value at x equals to XN plus H that is XN plus 1. So this is predict formula and a corrector formula is YN plus 1 equals to YN plus H upon 2 F at XN, YN plus F at XN plus 1, YN plus 1 star.

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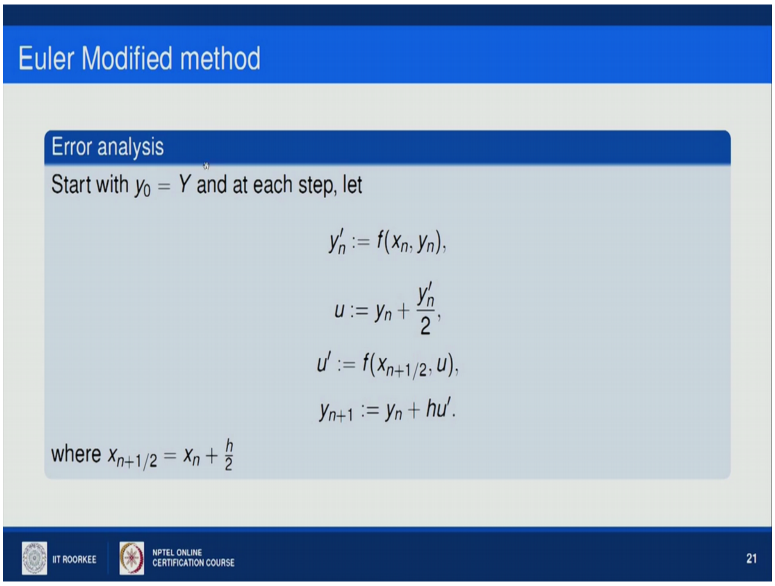

So if we talk about error analysis in this modified method so suppose at starting y at F0 is given by capital Y and at each step I am having y prime n equals to f of XN, YN. That is a given differential equation and u equals to YN plus YN prime upon 2. U prime is F XN plus Half. So XN plus half is basically XN plus H by 2. I am taking the half length of the interval comma U and finally YN plus 1 is given as YN plus H times u prime.

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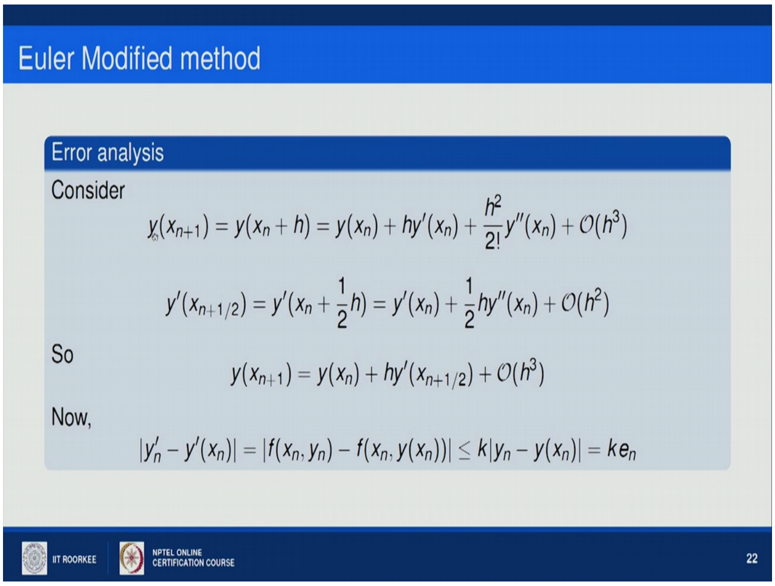

Then if I use the Taylor series expansion of Y about X equals to XN then I can write in this way. If I calculate y prime XN plus half is given by Y prime XN plus half h so that will come in this form. And here you can note down just look at this two terms that are similar so I can substitute this particular thing here. So Y XN plus 1 is given as Y XN plus H time Y primes XN plus half plus order of H cube.

Now the difference between Y prime N and Y prime at XN is given by these two functions and we are assuming that F is Lipchitz continuous with Lipchitz constant K. So I can write in this way and that is equals to K times EN. That is error in (())(25:26).

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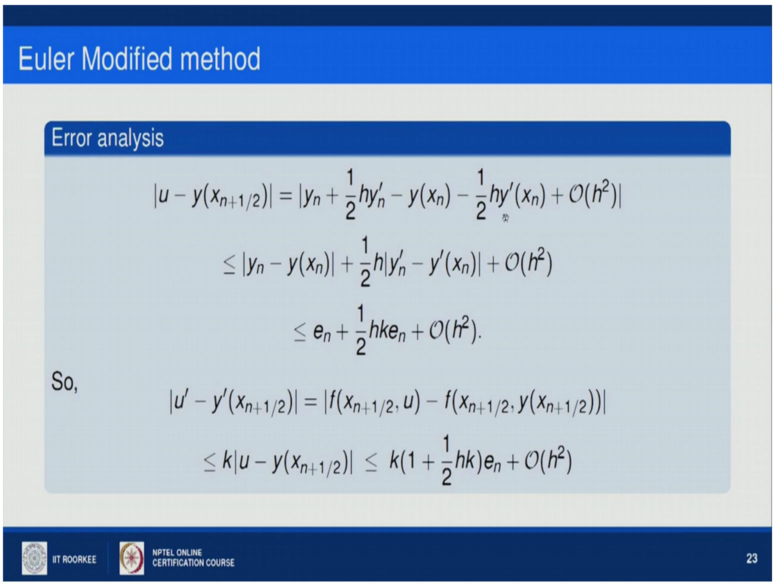

So U minus Y XN plus half can be given now by this particular equation which is less than equal to YN plus YXN. I have taken this term and this term together plus half H I have taken common. So Y prime N minus half H I have taken out. So Y prime XN plus order of H square which is EN plus half HK EN plus order of H square. Similarly I can get U prime minus Y prime XN plus half and that is given by K times 1 plus half HK into EN plus order of H square.

So finally YN plus 1 minus Y XN plus 1 that is the error in N plus 1 step is can be calculated YN plus H times U dash minus YXN minus HY prime XN plus Half plus order of H cube that is less than equal to YN minus YX at X equals to XN plus H time U prime minus Y prime XN plus half plus order of H cube. So EN plus 1 will be less than EN plus HK I have taken common.

So 1 plus half K H into EN and these values I have substitute in from the previous slide which I have calculated earlier. So if I take alpha equals to 1 plus HK plus half H square K square and so on. So after substituting this values and writing the error in (())(26:57) in terms of error in initial error that is error in the initial iteration.

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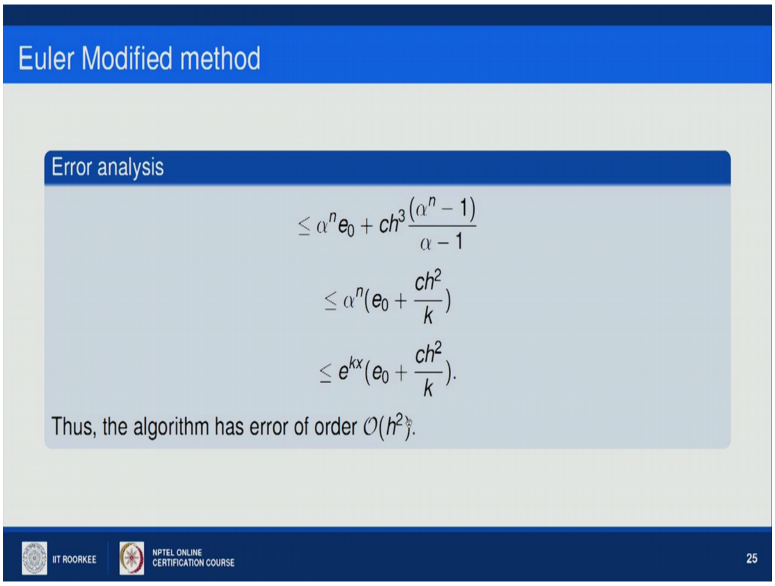

Then I will get this particular approximation and again like the quadratic Taylor method this particular approximation and again like the quadratic Taylor method here the algorithm has error of order H square which is again an improvement over the simple Alex method that is given error of order H.

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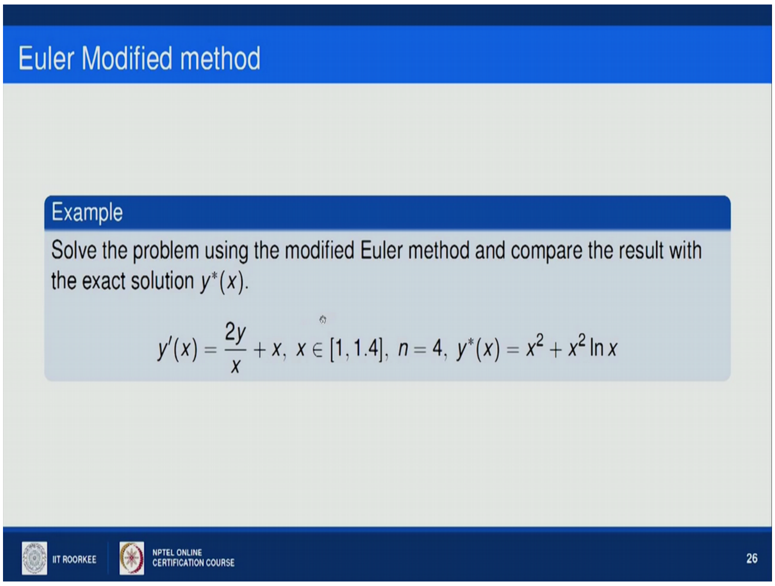

So after this we will take one example of Euler's modified method so example is given by this particular initial value problem. So Y prime X is 2Y upon X plus X and I need to find out the value of X in interval 1 to 1 point 4 by taking N equals to 4. The same time a X solution is also given for this particular differential equation which is X square plus X square log X with natural base and now so here if I calculate the initial value of Y at X equals to 1, it will be 1.

So Y1 is 1. I need to calculate y at 1 point 1, y at 1 point 2, y at 1 point 3 and y at 1 point 4 and at a same time we will compare the approximate value with the exact value and we will see how much error we are getting in our solution. So how to apply this method?

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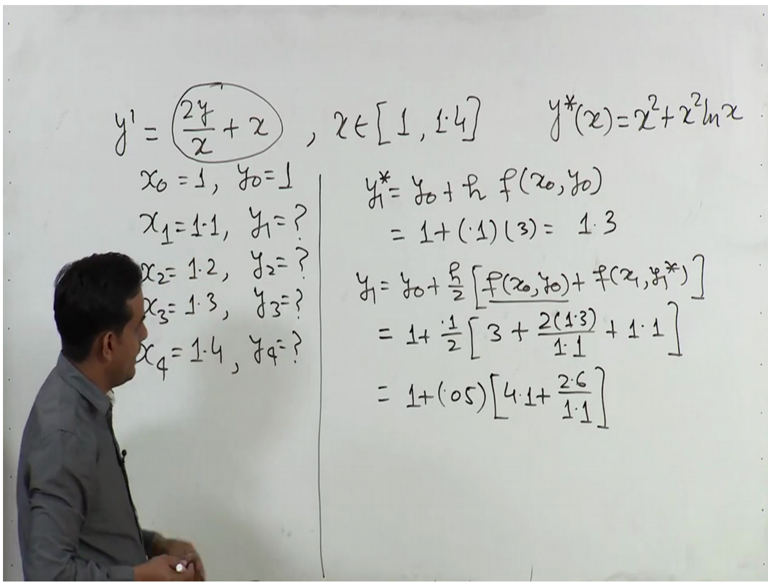

So by initial value problem is given as Y dash equals to 2Y upon X plus X. X belongs to interval 1 to 1 point 4. So the two solution analytics solution also given that is Y star X is X square plus X square log X. Now I need to solve this problem. So here X nought is 1. So Y nought is come at this equation I can calculate when X is 1 Y will become 1. X1 is 1 point 1. Y1 I need to calculate.

X2 is 1 point 2, Y2 I need to calculate. X3 is 1 point 3, Y3 I need to calculate and so for Y4, X4 that is 1 point 4. So let us first calculate Y1 using the Euler's modified method. So Y1 is given as Y0 plus H time F X0 Y0. So Y0 is 1 plus H is point 1 here F of X0 Y0 can be calculated from this because it is my F of XY. So 2 upon 1 2 plus 1 3 so it is coming at 1 point 3 and this is the predict value.

Now I will correct this value. So Y1 will be Y0 plus H by 2 F of X0 Y0 plus F of X1 Y1 star. So Y0 is 1 plus H is point 5 upon 2. F of X0 Y0 is 3 plus F XN YN star will become 2 into 1.3 upon X1 is 1.1 plus 1.1. So 1 plus sorry it is 4 point 1 point 05 into 4 point 1 plus 2 point 6 upon 1 point 1. So after simplifying this particular expression I will calculate the value of Y1 that is the value of Y at X equals to 1 point 1.

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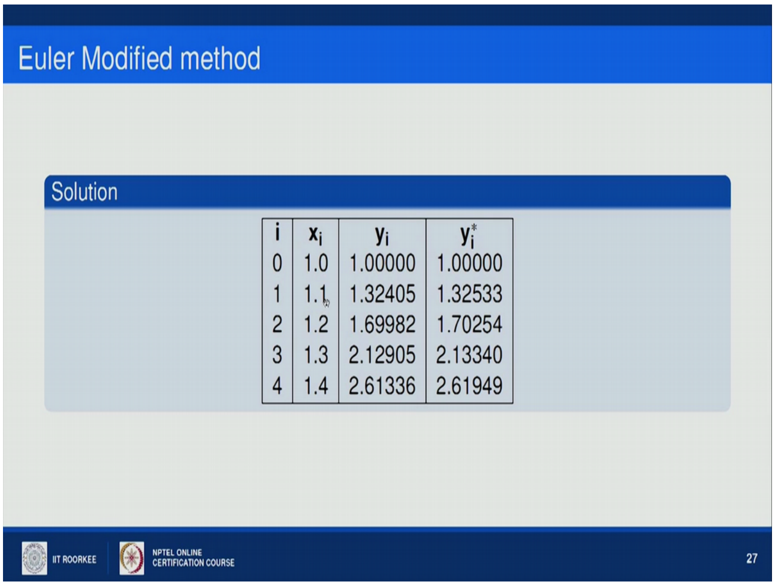

i	x_i	y_i	y_i^*
0	1.0	1.00000	1.00000
1	1.1	1.32405	1.32533
2	1.2	1.69982	1.70254
3	1.3	2.12905	2.13340
4	1.4	2.61336	2.61949

This value is coming Y at X equals to 1 point one is 1 point 32405 where the exact value was 1 point 32533. So a very small difference we are having here that is after third place of decimal. Then Y at X equals to 1 point 2 is 1 point 69982. The exact one is 1 point 70254. Y at X equals to 1 point 3 is 2 point 12905 exact one is 2 point 13340 and and finally these are the approximate and exact values of Y at X equals to 1 point 4.

So here you can note down from these two columns that the approximate solution is quite close to the exact one. And this is the implementation of the Euler's modified method for solving initial value problem. So in this lecture we have seen 2 methods those are having error of order H square. That is the quadratic Taylor method and then Euler's modified method.

In the next lecture we will learn another class of numerical methods for solving ordinary differential equation and those methods are called Runga Kutta method. So thank you very much for listening this lecture.