Complex Analysis

Prof. Pranav Haridas

Kerala School of Mathematics

Lecture No – 39

Argument Principle

We have not defined what is meant by the logarithm of a complex number however if we have a logarithm of complex number say $\log z$, then it is desirable that the derivative of $\log z$ is equal to $\frac{1}{z}$ and by the chain rule we would like to have the derivative of $\log\big(f(z)\big)$ to be $\frac{f'(z)}{f(z)}$.

DEFINITION 1 (Logarithmic Derivative). Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. We will define the log-derivative of f to be the meromorphic function $\dfrac{f'(z)}{f(z)}$.

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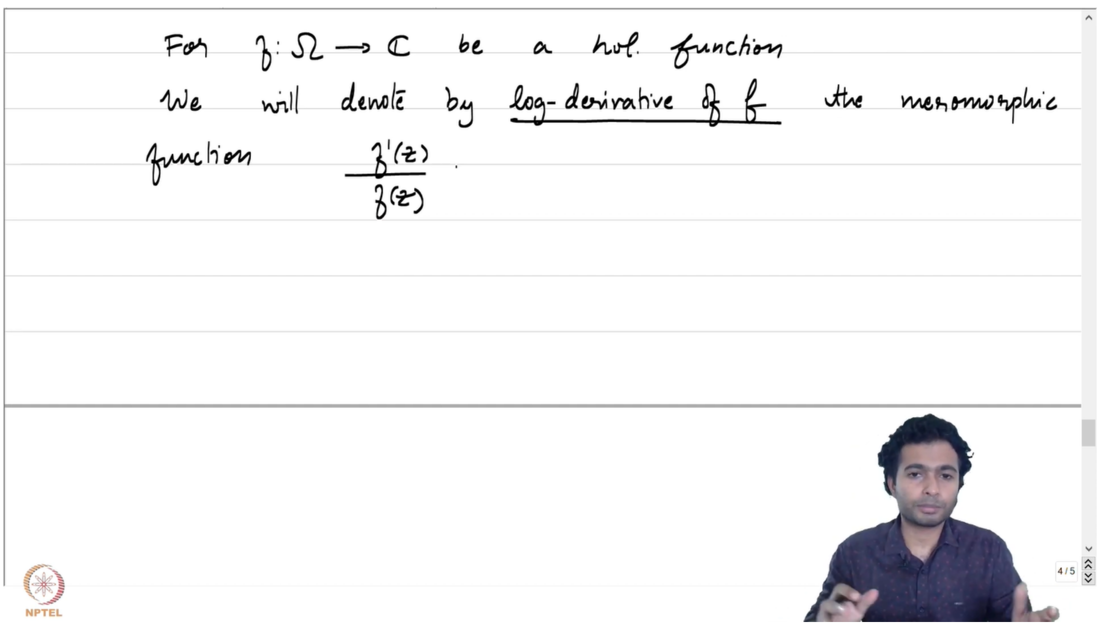

EXAMPLE 1.

(1) Let $f:\Omega\longrightarrow\mathbb{C}$ and $g:\Omega\longrightarrow\mathbb{C}$ be any two holomorphic functions on Ω. Then the log-derivative of the product fg will be,

$$\frac{(fg)'(z)}{(fg)(z)}=\frac{f'(z)g(z)}{f(z)g(z)}+\frac{f(z)g'(z)}{f(z)g(z)}=\frac{f'(z)}{f(z)}+\frac{g'(z)}{g(z)}.$$

If g is non-zero on Ω, then we can talk about the log-derivative of the quotient $\frac{f}{g}$ and which will be,

$$\frac{\left(\frac{f}{g}\right)'(z)}{\left(\frac{f}{g}\right)(z)}=\frac{f'(z)g(z)-f(z)g'(z)}{f(z)g(z)}=\frac{f'(z)}{f(z)}-\frac{g'(z)}{g(z)}.$$

(2) Consider $p(z)=a(z-z_1)^{d_1}\dots(z-z_n)^{d_n}$ and $q(z)=b(z-w_1)^{e_1}\dots(z-w_m)^{e_m}$, where z_i's are distinct and w_j's are distinct. Let $R(z)=\frac{p(z)}{q(z)}$. Then

$$\frac{R'(z)}{R(z)}=\frac{d_1}{(z-z_1)}+\frac{d_2}{(z-z_2)}+\dots+\frac{d_n}{(z-z_n)}-\frac{e_1}{(z-w_1)}-\frac{e_2}{(z-w_2)}-\dots-\frac{e_m}{(z-w_m)}.$$

(3) Let $f:\Omega\setminus S\longrightarrow\mathbb{C}$ be a holomorphic function.

- If $z_0\in\Omega\setminus S$ is such that $f(z_0)\neq 0$, then $\frac{f'}{f}$ is holomorphic in a neighborhood of z_0.
- If $z_0\in\Omega\setminus S$ and $f(z_0)=0$ with order m, then $f(z)=(z-z_0)^m g(z)$ where $g(z_0)\neq 0$ and the log-derivative of f will be

$$\begin{aligned}\frac{f'(z)}{f(z)}&=\frac{m(z-z_0)^{m-1}g(z)}{(z-z_0)^m g(z)}+\frac{(z-z_0)^m g'(z)}{(z-z_0)^m g(z)}\\&=\frac{m}{(z-z_0)}+\frac{g'(z)}{g(z)}.\end{aligned}$$

  Note that $\frac{g'}{g}$ is holomorphic on z_0.
- If $z_0\in S$ and z_0 is a removable singularity of f, then $\frac{f'}{f}$ behaves as above.
- If $z_0\in S$ and z_0 is a pole of order m, then

$$f(z)=\frac{g(z)}{(z-z_0)^m}$$

where $g(z) \neq 0$ in a neighborhood of z_0 and we have

$$\frac{f'(z)}{f(z)} = \frac{\frac{g'(z)}{(z-z_0)^m} - \frac{mg(z)}{(z-z_0)^{m+1}}}{\frac{g(z)}{(z-z_0)^m}} = \frac{g'(z)}{g(z)} - \frac{m}{(z-z_0)}.$$

THEOREM 2 (Argument Principle). *Let Ω be an open set in $\mathbb{C}$ and f be a meromorphic function defined on Ω such that f has zeroes of order $d_1, \ldots, d_n$ at $z_1, \ldots, z_n$ respectively, after removing the removable singularities, and f has poles of order $e_1, \ldots, e_m$ at $w_1, \ldots, w_m$ respectively. Let γ be a closed curve which is null-homotopic in Ω such that zeroes and poles don't lie on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{n} d_i W_\gamma(z_i) - \sum_{j=1}^{m} e_i W_\gamma(w_j)$$

where W_γ is the winding number.

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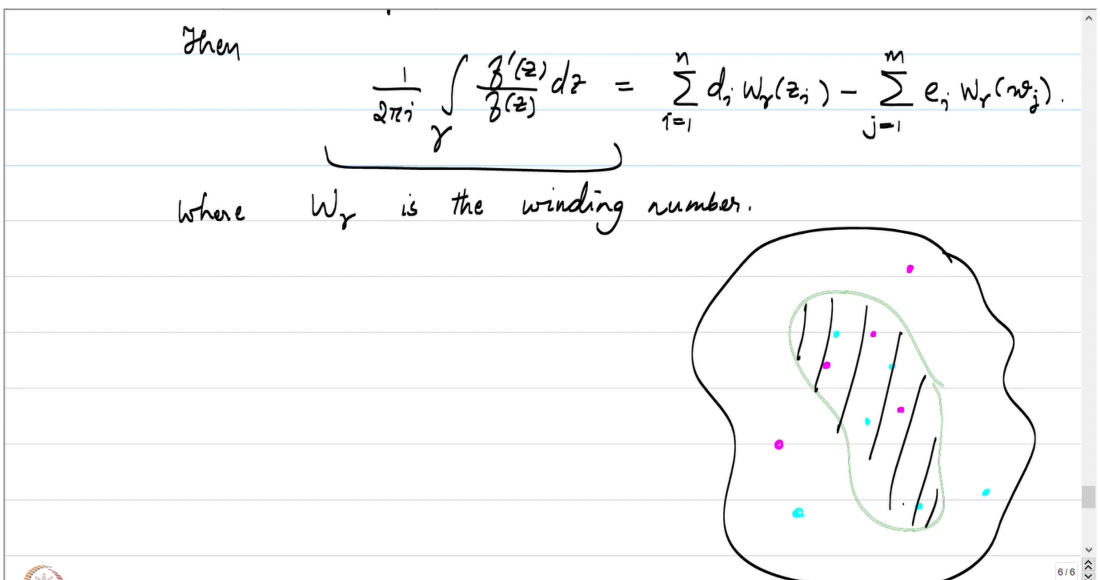

PROOF. We have

$$f(z) = \frac{(z-z_1)^{d_1} \ldots (z-z_n)^{d_n}}{(z-w_1)^{e_1} \ldots (z-w_m)^{e_m}} g(z)$$

where g has neither zeros nor poles in Ω.

Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\gamma \left(\frac{d_1}{(z-z_1)} + \cdots + \frac{d_n}{(z-z_n)} - \frac{e_1}{(z-w_1)} - - \cdots - \frac{e_m}{(z-w_m)}\right)dz +$$

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz$$

$$= \sum_{i=1}^{n} d_i W_\gamma(z_i) - \sum_{j=1}^{m} e_j W_\gamma(w_j).$$

□

EXAMPLE 3. Let γ be a contour in Ω and f be a holomorphic function on Ω. Define $\sigma : [a,b] \longrightarrow \mathbb{C}$ given by $\sigma(t) = f \circ \gamma(t)$. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_a^b \frac{f'\left(\gamma(t)\right)\gamma'(t)dt}{f\left(\gamma(t)\right)} \\ &= \frac{1}{2\pi i}\int_a^b \frac{\left(f\circ\gamma\right)'(t)dt}{\left(f\circ\gamma\right)(t)} \\ &= \frac{1}{2\pi i}\int_\sigma \frac{dz}{z} \\ &= W_\sigma(0).\end{aligned}$$

Suppose $z_1, z_2, \ldots, z_n$ be the zeroes of f on Ω. If γ is a null-homotopic simple closed curve which does not pass through z_i's, then $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$ gives the number of zeroes of f counting multiplicities in $H([0,1]\times[a,b])$, where H is the homotopy between γ and the constant curve.

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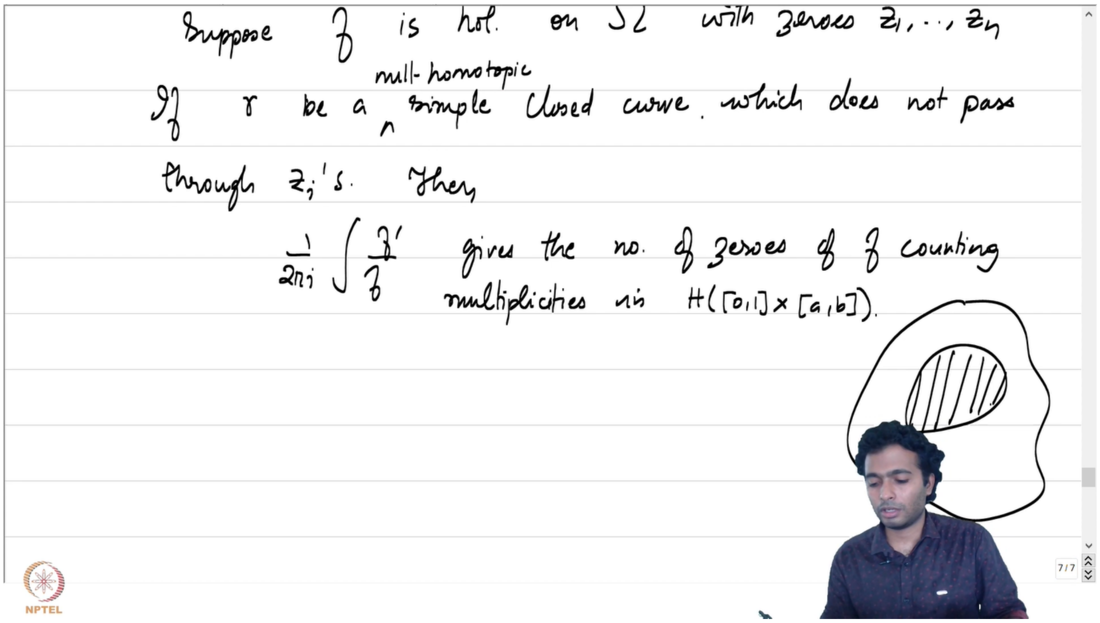

THEOREM 4 (Stability of zeroes). *Let* Ω *be an open set and* $\gamma_0 : [a,b] \longrightarrow \mathbb{C}$ *be a null-homotopic closed curve in* Ω. *Suppose* $H : [0,1] \times [a,b] \longrightarrow \Omega$ *be a homotopy of closed curves in* Ω *from* γ_0 *to* γ_1. *Suppose* f_0 *and* f_1 *are holomorphic functions on* Ω *such that there exists a continuous function* $F : [0,1] \times \Omega \longrightarrow \mathbb{C}$ *such that* $F(0,z) = f_0(z)$ *and* $F_1(1,z) = f_1(z)$. *Further, for each* $s \in [0,1]$, $F(s, H(s,t)) \neq 0$ *(i.e.,* $F(s,.)$ *does not vanish on* $H(s,t) = \gamma_s(t)$*). Then the number of zeroes of* f_0 *in the interior of* γ_0 *counting multiplicities is the same as the number of zeroes of* f_1 *in the interior of* γ_1.

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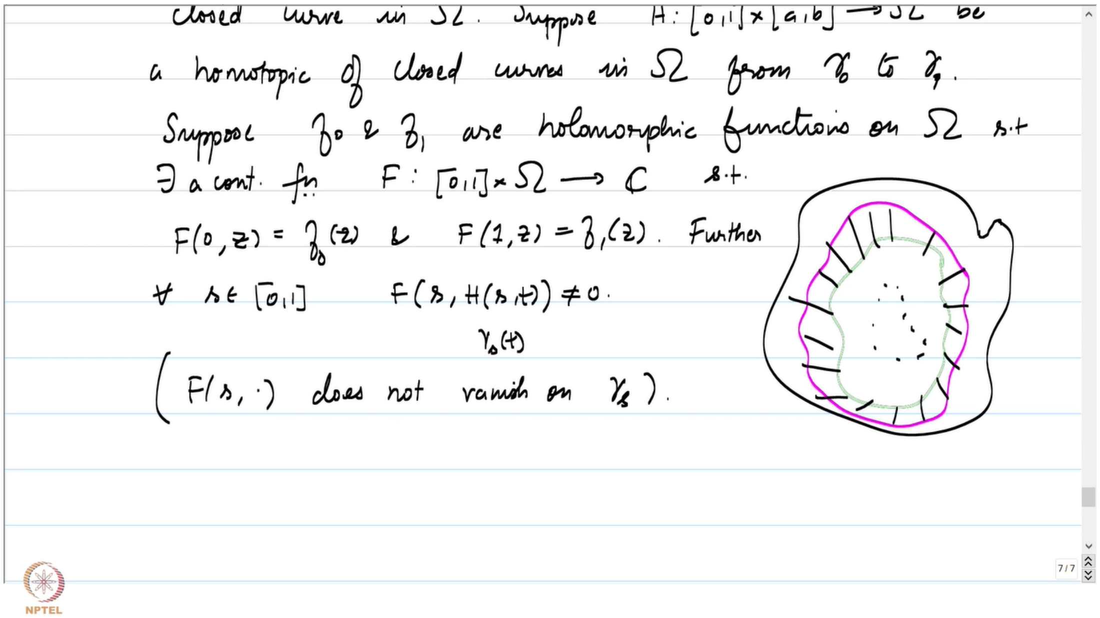

PROOF. The number of zeroes with counting multiplicities of f_0 in the interior of γ_0 is given by

$$\frac{1}{2\pi i}\int_{\gamma_0}\frac{f_0'(z)}{f_0(z)}dz = W_{\sigma_0}(0) \tag{1}$$

where $\sigma_0 = f_0 \circ \gamma_0$.

Similarly, the number of zeroes with multiplicities of f_1 in the interior of γ_1 is given by

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f_1'(z)}{f_1(z)}dz = W_{\sigma_1}(0) \tag{2}$$

where $\sigma_1 = f_1 \circ \gamma_1$.

Define $G : [0,1] \times [a,b] \longrightarrow \mathbb{C}\setminus\{0\}$ given by $G(s,t) := F(s, H(s,t))$. Then G is a homotopy of closed curves in $\mathbb{C}\setminus\{0\}$ from σ_0 to σ_1 which gives us $W_{\sigma_0}(0) = W_{\sigma_1}(0)$ and hence the result. □

EXAMPLE 5. Let $f(z) = z^2$ and $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then $z = 0$ is the only zero of f and the number of zeroes of f, counting multiplicities, is 2. Now consider $f_\epsilon(z) = z^2 + \epsilon$.

Then the zeroes of f are $z = i\sqrt{\epsilon}$ and $z = -i\sqrt{\epsilon}$. Note that the homotopy here will be the constant homotopy, $H(s,t) = \gamma(t)$.

THEOREM 6 (Rouche's Theorem). *Let γ be a closed curve which is null-homotopic in Ω. Suppose f and g are holomorphic in Ω and $|g(z)| < |f(z)|$ on γ. Then f and $f+g$ have the same number of zeroes counting multiplicities on the interior of $\Gamma([0,1]\times[a,b])$ where Γ is the null-homotopy from γ to a constant path.*

PROOF. Define $F(s,t) = f(t) + sg(t)$. Also define $H(s,t) = \gamma(t)$. By the stability of zeroes, since $F(s, H(s,t)) \neq 0$, we have the number of zeroes of f and $f+g$ in the interior of γ are equal upto multiplicity. □