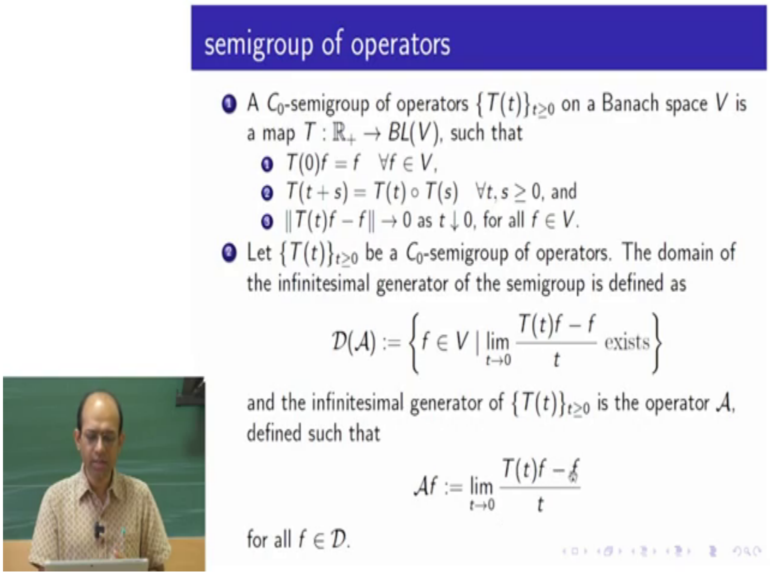
Probabilistic Methods in PDE
Professor Dr. Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture 51
Semi-group of bounded linear operators on Branch space Part 1

(Refer Slide Time: 00:21)



Today we are going to see the definition of Semi-group of operators, linear operators and given a C 0 semi-group that means semi group of linear operators which has some continuity property, we would define what do we mean by generator of the semi-group. So, here before talking about these we must understand that a semi-group of operators is essentially a collection of operators.

And that collection is parameterized by a parameter, we call that say t. So we can think t is like a time parameter, and there in that collection of operators. The members of the collections are related, related in the sense that they have an algebraic structure that is semi-group structured they have.

So that means that collection has one identity operator of course I operator, and with composition operations they form a semi-group that means that given 2 operators, their composition operator also lies in that same collection, and that composition operator is here, associative.

So here we start the definition here, so C 0 semi-group of operators T t, t greater or equals to 0. So, these operators, which are defined on a Banach space V is a map T from R plus that is closed 0 to open infinity to B L V that is set of all bounded linear operator on the Banach space V. So here, that small t is coming from R plus and capital T of small t is an operator which belongs here that means capital T of t is a bounded linear operator on V.

So that means, capital T of t takes a member from V to V and it is a linear operator and this operator also is continuous. So, what we say in the language of functional analysis that we call that bounded linear operator, fine basically it takes bounded set to bounded, so this operator so we have a collection of such operators satisfying 3 conditions.

One is that T 0 of f is equal to f for all f in V. So V is the Banach space, f is a member of that, why am I writing f because actually the kind of Banach space what we would consider space of some functions so, f would be a some kind of functions on some set. So that is the reason to actually resemble with those notations, I am keeping this notation f.

Then this semi-group property that capital T of small t plus s is nothing but capital T of t composition with capital T of s, for any t, s non-negative. And the third property here that continuity property, so that is saying that from the, say the right continuity that T t of f minus f, this is a vector now in V but you take the norm, so, you get a positive real number and as small t goes to 0 that should go to 0.

Anyway, we know that this f can be written as T 0 of f. So you are saying that T t of f minus T 0 of f is going to 0 as small t goes to 0. So this is basically continuity property of capital T at times 0 from the right side. So, if this happens for all f in V we call this C 0 semi-group, okay, fine.

So, this type of continuity is like the for every f we ask this correct, so this is like point wise continuity. However, one can define some stronger notion also, so we would to come to that latter. So, let T t this collection of operators what we call as semi-group satisfying these properties. So, let T t be a C 0 semi-group of operators, the domain of the infinitesimal generator of the semi-group is defined in the following way.

So here when I say generator, so that would be also an operator, so that would be an operator, but what should be the domain of the operator? So, that is also one interesting question one should ask, so necessarily, it is not necessary that the domain of the operator which is generated is not necessary that the way we are going to define the generator, its domain would also include the whole space V, it is not necessary.

So, here we actually define both together, the operator and its domain, the domain of the infinitesimal generator of the semi-group is defined as. So, here first is the domain we are defining the set of all possible f in V such that this limit exists, this is basically right derivative at 0, correct, T t of f minus T 0 f divided by limit t tends to 0, so basically this is right derivative of this, if this exists, okay.

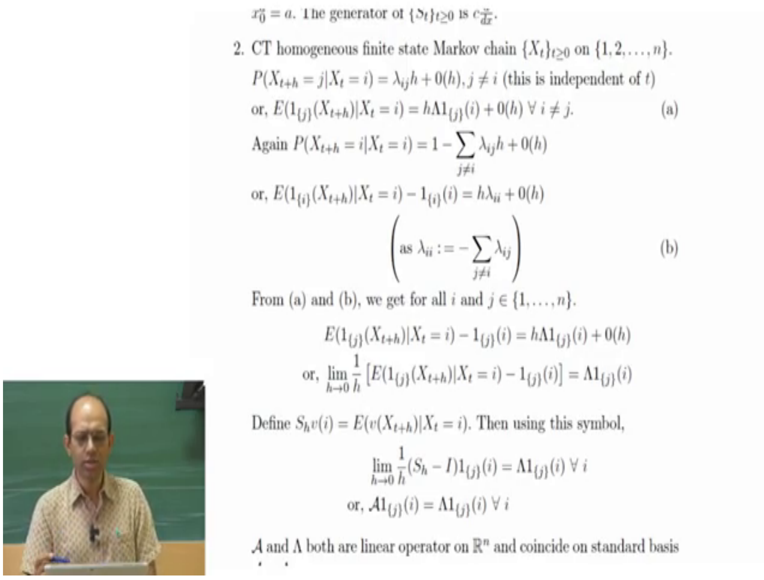
So, it may not exist for all f in V then we do not consider those f to be in the domain of operator generator, we are not going to define the operation of the generator for those f, but for only those f where for this exists those we would retain and this collection we call as D of A, or we say it is a domain of A.

And then the formula of A is given below that A f is defined as limit t tends to 0 T t f minus f divided by t. So this is a definition, so now, one can ask that why should these be well defined for f in D A, because that D is the definition that is the way we have defined the domain of A. We have defined domain of A so that this limit exists so, we have no problem.

So, when this limit exists, those fs are here, and whatever fs are here for that when you compute this limit, what we get would be now we would say that okay, A f. So A is the operator, if it acts on f that would give me right derivative at 0. So right derivative of T t f at t is equal to 0. So that is the definition of generator of a C 0 semi-group, we would see some examples how to compute this, then it would be a little clear.

So here, so let us first consider the case where we have a dynamics, dynamics say for example a Markov chain on finite state space. From that Markov chain we would articulate a particular semi group, and for that particular semi group after we define the semi-group it would be trivial to check that is indeed a semi-group because it is very simple and then after that we would compute what should be the generator of that semi-group.

(Refer Slide Time: 10:12)



So, let us do that now so, we take C T stands for continuous time, homogeneous finite state, so, homogeneous means time homogeneous, does not, the law does not change over time. So, C T homogeneous finite state Markov chain X T on a finite state space on 1, 2, 3 up to n.

Now, we consider since it is only a finite state Markov chain, so we consider the transition time so the transition probabilities. So, at time t which is at i, given this information the probability that X t plus h is equal to j when h is a small positive number so, that probability can be written as lambda i j h plus small o of h, when j is not equal to i.

So, this is the description of continuous time Markov chain given a rate matrix lambda. So assume that you have that rate matrix. So, assume that the rate matrix is already given and that rate which is lambda matrix which has n square entries, n cross n square matrix and the rate matrix we recall quickly that is a square matrix which has row sum 0 and of diagonal elements are all non-negative.

So, that means it's all elements apart from diagonals must be 0 or positive and row sum should be 0. That means all the negatives entries should appear on diagonals and this is a square matrix. So, any such matrix is eligible to be treated as a rate matrix. Imagine that rate matrices and then rate matrices actually considered as parameter for the Markov chain, whenever we talk about some random variable, we talk about what is the parameter.

So, or stochastic process we talk about what is the defining parameters coefficients. So, here for Markov chain this rate matrix is the parameter. So, given a rate matrix there is only one single law of Markov chain. So, the law of a Markov chain is determined from the rate matrix uniquely. So, that is a description you assume that we have that Markov chain, so we have this lambda so we have this thing for all j not equals to i. So, this left hand side we rewrite in terms of expectation.

So, probability of a random X belongs to A is same as expectation of integral function of A at X. So, that thing we are using now here, probability that this belongs this is equal to j is same as expectation of this function X t plus h if it is in j then it is 1 otherwise it is 0, so we are going to get these to left hand side.

So, left hand side is rewritten as expectation of indicator 1 j of X t plus h given X t is equal to i, right hand side is as it is, but this lambda i j we are rewriting in a little different way that lambda matrix so this is the rate matrix capital lambda and then 1 j, what is 1 j? 1 j is unit vector, column vector whose jth element is 1 all are 0s. This is just standard normal basis, jth element in the standard normal basis, so that would give me jth column.

So Jth column is and then the ith row of that, ith element of the jth column so that is nothing but lambda i j so, this lambda i j is rewritten in this fashion. So this is whatever I have written earlier I am just rewriting this fashion that really helps for computing the generator of the semi group, but the semi-group I have not defined yet, so I would soon define, okay and then small o of h here.

What does it mean that, so if I take divide both sides by small h, and then take limit h tends to 0, so then this divided by h limit and limit h tends to 0 would go to lambda i j. So that thing only rewritten in this fashion using small o notation. So again, now here for both these lines I have taken j is not equal to i, what happens when j is equal to i?

So, when j is equal to I, I know that from the law of probability that from the rule that sum of all event is 1. So, this is the only thing which was kept aside, so the probability X t plus h is equal to i given X t is equal to i is 1 minus of this quantity. So, 1 minus this lambda i j summation j not equals to i all this quantity so, this is pretty much clear what is expected so this should be this thing.

Now, again this line I am rewriting in terms of expectation, in terms of conditional expectation, so that I write down exactly this way that expectation of indicator function at the ith set of X t plus h, given X t is equal to i, so this is written as this one. And then this the right hand side thing this one I am rewriting as indicator function i at i so, of course it is 1, it is always you know whatever i you chose, so this 1 subscript this i stands for the column vector which has only 1 at ith proposition.

Basically this is the ith element of the unit basis and then the ith element of that vector is 1 actually. So, this one is rewritten this way and then minus lambda i j can be rewritten as lambda i i because i ith element of lambda matrix is nothing but minus of summation of all other elements in that same row.

Why is it so? Because capital lambda is rate matrix which has row sum 0, so its diagonal elements are actually negative of the sum of all diagonal elements in the same row, so that I am rewriting this way. So this is nothing but rewriting of this, okay. So, I have one this thing here another this thing here, so I call this b. So I have a and b the two equations here, what we have obtained just from directly this description.

So from a and b, now we get that for all i and all j in 1 to n, so expectation 1 j so, see that here in a, I have expectation 1 j X t plus h, here I have expectation 1 i X t plus h here, and here if you see carefully, I have only one term on the left hand side, here I have two terms on the left hand side, but here in this term instead of i if I put j, where j is not equal to I, this term would become 0 so, as if this is also there, but we have 1 i of j .

So, I can actually see that both are actually same equations, this is the evaluated at j when j is not equals to i, here it is evaluated at i because lambda i i is also can be written as lambda 1 i i so, this way. So, I combine these two things to write down expectation of 1 j X t plus h given X t is equal to i minus 1 j i is equal to h times capital lambda 1 j i plus small o of h.

So, this factor you know, if i divide it by h and limit h tends to 0 that would be 0 basically. So now, we divide small h from both sides so, here I get expectation of 1 j X t plus h given X t is equal to i minus 1 j i, divided 1 over h but this here I have small h here so h by h would be just 1 here so I would be left with lambda 1 j i this thing.

Now, this calculation we retain and then we define the semi group operator. Although there is a description, we have denoted semi-group operator as capital T of t, but here for some reason I have written capital S, but it is the same thing. So, assume that we have a family of operators, S h which is defined as S h acting on v because v is the vector.

So, what is the Banach space here? The Banach space is set of all vectors on the set 1, 2 up to n, so n tuples, set of all n tuples or you can also view that set of all functions on the set 1, 2 up to n. So that is the space and what is the norm we are going to take? We are going to take Euclidean norm so because all norms are equivalent so it does not matter what norm we choose, for now we choose Euclidean norm there.

And then S h v is defined as expectation of v of t plus h given X t is equal to i, since it is homogeneous Markov chain, it does not matter what my t is, t could be 0 or any other t, I will get the same value on the right hand side so, the left hand side does not depend on t. So it is S h v i, this is defined by this, is it a semi-group?

That you can take as an exercise that to see that this is indeed a semi-group, just by doing tower property and conditioning you can see that, it has satisfies of all 3 properties that S 0 v is equal to v, I mean that is actually I mean just immediate, but then only when put h is equal to 0 that means expectation v of X t given X t is equal to i, there is nothing like v of i.

So S 0 v of i v of i so that S 0 is equal to identity and then constructing S of say S subscript h 1 plus h 2 of v i, you need to show that is S h 1 composition S h 2 of v i that you can do with conditioning using tower property. So, this is a semi-group of operators and it is also C 0 semi-group, why? Because as h tends to 0 this part as you know the expressions what we have obtained earlier from here we see that as h tends to 0 this part goes to 0 here.

So and then for all unit vectors, so this you know instead of v, I have this thing written for all unit vectors, but any vector can be written as finite sum of the unit vectors with the basis and from there one can establish that it is also C 0 semi-group. So this is the semi-group here, and for this semi group now we are planning to find out the generator as we have defined in the last slide.

So, that definition of the generator was that you just S h minus identity matrix and divided by h and operate on a particular function v and that would give you the generator but instead of operating on any arbitrary function v, we are calculating it on the basis vectors, in that Banach space. So what are the basis vectors?

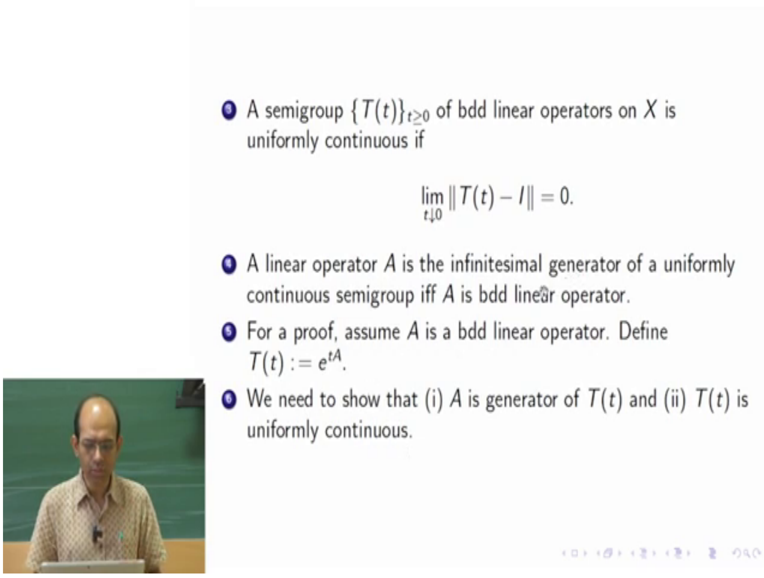
They are all unite basis we are taking, 1 j. So, this operator 1 j and after operating this whatever the vector we are going to get evaluate that vector at point i this i, so this thing we want to find but that we have already found here, we have already found here. We have found that this is S h is nothing but this thing and this is identity operator on this thing and we have already obtain that is lambda 1 j i, we have already obtained that.

So, from the definition that this thing is A operator. So, A operator acting on 1 j at i is same as this lambda matrix operating on the 1 j vector and evaluted at i, so that is true for all i and all possible j vectors. So, what does it mean? That these two operators are coinciding on the basis vectors and both are linear vector linear operators. So, what is the conclusion?

The conclusion is that this A the generator is nothing but lambda. So, this is a proof that the rate matrix of finite state Markov chain is essentially the generator of the semi-group generated by that Markov chain. What is the semi-group? The semi-group is generated by this, okay.

So here we write down this conclusion that A and lambda both are linear operator coincidence yes this is the comment. So now you get back so this after looking at this example we get back to again the further properties etc.

(Refer Slide Time 25:36)



So, so this is a stronger notion of continuity as I was talking about that okay, we would come back to this. A semi-group T t of bounded linear operators on Banach space X is called uniformly continuous if following holds, what is this that the operator T t minus operators identity operator. So then the operator norm so, this is the subordinate norm.

So, as we know that for now when I have a linear operator on Banach space X then from the norm of the Banach space I can induce one norm in the BLX the set of a bounded linear operators that is supremum over norm of A v. So, where v is running on the unit's sphere in that Banach space. So, that is going to give me the norm of the BLX.

So, in that norm, if we take limit t tends to 0 if that goes to 0, then we say that this is uniformly continuous semi-group. A linear operator A is the infinitesimal generator of a uniformly continuous semi-group if and only if A is a bounded linear operator. So, this is an important very important result.

So, when I start talking about say semi-group of operators always you know semi-group of bounded linear operators that T t but that does not necessarily imply that the generator operator what we are going to derive that would also be a bounded linear operator, that would be linear but that is okay, but that would also be a bounded linear operator need not be true.

The generator what we are going to get need not be continuous operator. So here this theorem is saying that a linear operator A is infinitesimal generator of a uniformly continuous semi-group if and only if A is bounded linear operator. So, that means if we have a semi group C 0 semi-group, which is not uniformly continuous then the generator what we are going to get would not be bounded linear operator.

So, we know there are some examples of linear operators which is not bounded linear operator correct I mean say differential operators are like that. However, the example what we have computed recently so, there the generator was just a matrix, a matrices of course a bounded linear operator.

So, this result would then imply that semi-group what we have obtained there is actually not only C 0 semi-group that is uniformly continuous semi-group, so we are going to prove this theorem actually. So, for a proof what we are going to do, we are going to assume A is a bounded linear operator. So and then from there we are going to show that the semi-group T t so we are going to do that semi-group T t would be uniformly continuous.

So we define, so given A so here I mean, be careful about this statement it does not give any particular semi-group it talks only about generator, talks only about A it is just saying that A is the infinitesimal generator of S uniformly continuous semi-group, okay some semi-group it is, you can identify A as you know generator of some continuous semi-group then A is bounded linear operator it is saying that.

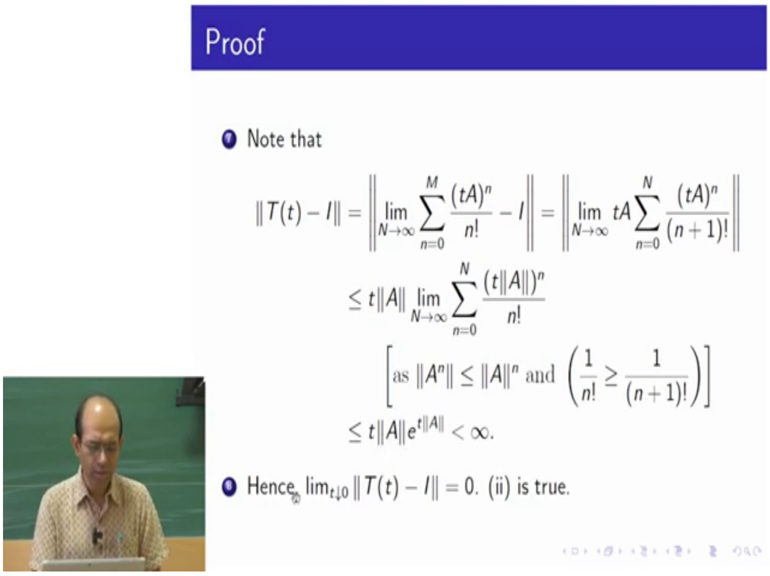
So, that means to prove this type of statement we have to cook up the semi-group also. So, the most natural way to cook up a semi-group from a generator is taking this e to power t A, why is it so? Because remember that semi-group has this property that T of t composition T of s is T of t plus s. So here, product composition that thing is giving me the T t plus s. So, that thing is naturally true for exponential maps because here if we have T t composition T s, we can define using the series expansion of exponential map.

So that would be e to the power of A, it could be defined as limit of I plus A plus half A square plus 1 by 3 factorial A to the power 3 plus etc, etc, etc. So, these series convergence and that converges because if you take the norm, that norm would be convergent because this is a bonded linear operator. So that converges so that limit I can always define so, this is well defined thing.

So, using that we can define this semi-group and for this kind of definition of course, this satisfy the first property T 0 is e to the power of 0 that is identity operator and second property that semi-group property satisfies that and then one should also check that this semi-group what we have defined whether indeed A is the generator of the semi-group that also needs a proof. So, these are the things what we are going to see step by step, so for the time being we just define this.

So, this is a candidate, candidate semi-group for which A is actually generator, but we need to show each and every of these things. So, we need to show that first A is generator of T t and second is the T t is uniformly continuous, if you show these two things, then we are done with this reverse side.

(Refer Slide Time: 32:23)



So, we start with this that T t minus I so, first we prove the second property T t is uniformly continuous. So, T t minus I norm so, I need to show that this norm goes to 0 as small t goes to 0. So, this norm of this difference is equal to limit n tends to infinity. So, T t now I am writing this expression, e to the power t A, but e to the power t A we understand as limit of that series expansion, so I am writing down the series itself, n is equal to 0 to capital M t A to the power n by n factorial.

And here there is a typo, it should be N not M, capital N is equal to goes to infinity, small n is equal to 0 to capital N this thing minus I. So, this thing is just it is just rewriting these things. So t A, I am just taking common here because when n is equal to 0 appears in this series, then I am going to get t A to the power 0 that is identity by 0 factor that identity and this identity would cancel each other.

And then then what I would be left with on small n is equal to 1 to capital N. And then there we would always have t A, and t A whole square, etc those terms so I can take t A common. So, I take t A common from there and then what remains is t A to the power of small n divide by n plus 1 factorial, I mean I could have written n here, n minus 1 here, but then I should have written n is equal to 1 to capital N.

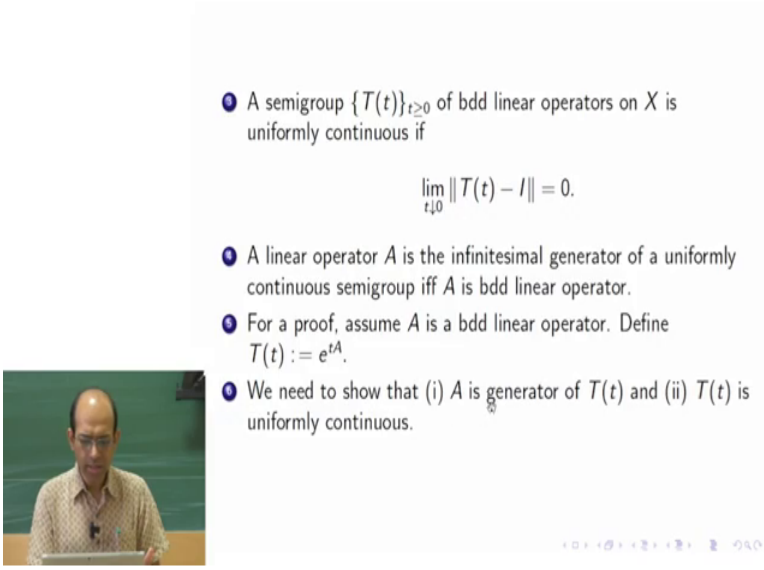
But I would like to substitute this thing so, that I get n plus 1 factorial to the power n, and small n is equal to 0 to capital N. So, this is a nice expression and then from this norm we get this upper bound. So, this norm of this thing is less than or equal to t times norm A, and then the norm is going inside the summation so, we would get less than or equals to sign and small n is equal to 0 to capital N, t norm A to the power of n divided by n factorial.

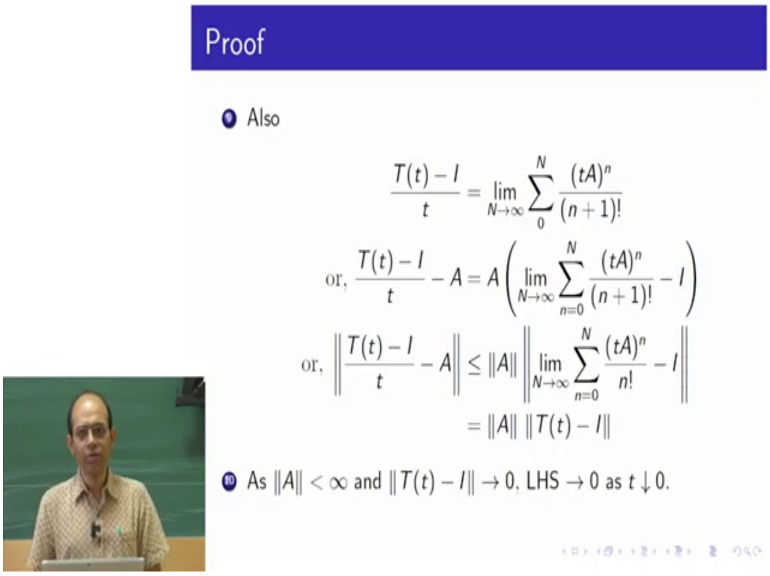
So here, instead of n plus 1 factorial I have written n factorial, why because after all less than equal to sign is there. So, if I replace n plus 1 factorial by n factorial, so I am replacing this by smaller number, so but this is the denominator so I am actually getting a larger number. So we write down this thing.

Okay, fine, so here I am just using this property that norm of A to the power n is less than or equals to I mean nth power of norm of A and also this property the 1 over n factorial is greater or equal to 1 over n plus 1 factorial. So now, this thing is nothing but exponential series as n tends to infinity. And this we can write down as e to the power of t norm A.

So, we write e to the t norm A and then t norm A is here. But this is you know, if some finite quantity here and now this thing is a finite quantity and as t tends to 0 this e to the power 0 so this is bounded quantity and this part is going to 0. So, we are going to get that this is 0 so, hence limit t tends to 0, T t minus I is norm is equal to 0, so 2 is true.

(Refer Slide Time: 36:26)





Now, we look at the first property, the first condition that A is generator of T t so, that we need to check here. So, we take T t minus identity divided by small t and we take limit we are having this quantity and then here we take the full expression of T t that is e to the power of these things and then that we write down in the full form limit n tends to infinity this thing. So that we have done in the earlier slide exactly the same manner so, you get this expression.

So, now T t minus I divided by t and minus A. So, I subtract A from both sides so, when I do that, so, here we see that n is equal to 0 case, I would have t A to the power of 0 that is identity and here just 1 factorial, so I get just identity here. I had this thing there is some typo here, T t minus I by t so, that is this quantity without small t because small t would be there and this A times n is equal to 0 to capital N t A to power n by n plus 1 factorial, t A to the power n, n plus 1 factorial, but something is missing, A is missing.

So, there is a typo here that I should get A here so, T t minus I divided by t is equal to limit n tends to infinity A times this summation. So, now I subtract both sides by A, so left hand side becomes Tt minus I divided by t minus A here and right hand side I will get minus A another and here I have A, here I have minus A, I take A common so, A times limit N tends to infinity, n is equal to 0 to capital N t A to the n by n plus 1 factorial minus I.

Now, we take norm both sides and here norm of product is less than or equals to product of the norms. So, here we get norm of A here and norm of this thing and this thing is what this thing is already you know this thing is actually T t as n tends to infinity so T t minus I and norm of A is here.

However, we have already seen that T t is uniformly continuous, so as small t tends to 0, this norm goes to 0, so this right hand side goes to 0, so that means the left hand side also goes to 0 as t tends to 0. So, what does it mean? That means that this limit converges this to A. So, or in other words we have proved that this A is indeed the generator of the semi-group.

So, T t minus I, so left hand side goes to 0 as t tends to 0, so this A is indeed the generator of the semi-group T t. So this part is done and the reverse side as I told that showing that A is having norm A is finite, thank you very much.