Introduction to Probabilistic Methods in PDE
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Lecture 39
A System of stochastic differential equations in application

So, now, there are more improvisation as I told that okay there was beginning and then geometric Brownian motion is more or less pretty much understood, but that has some limitations.

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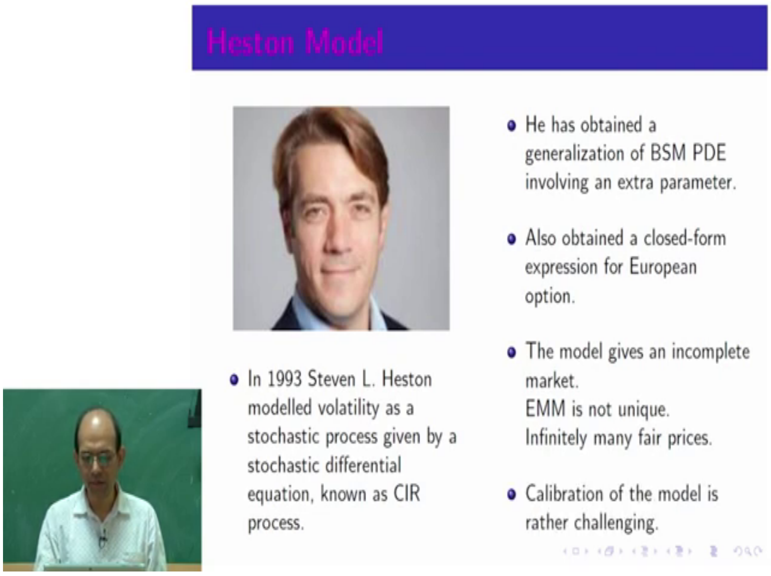

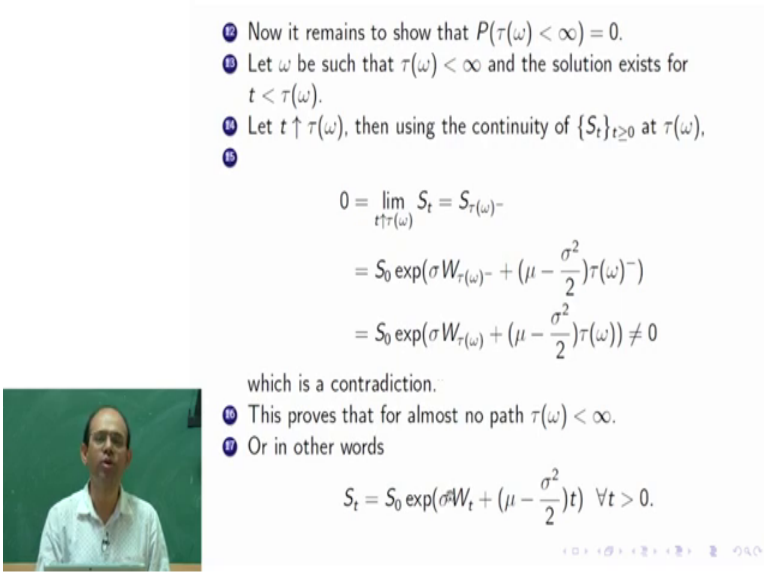

So, in 1993, Steven L. Heston, okay this is a photo of Heston I obtain from open domain. So, L. Heston model volatility okay, so volatility is pretty much like you know, just here for this sigma appears correct, this sigma, so this sigma is called volatility okay, this is the coefficient of Brownian motion.

So, here what he did, he allowed volatility coefficient to also vary over time, as also a stochastic process okay given by a stochastic differential equation known as CIR process, Cox Ingersoll Ross process okay. So, he has obtained a generalized BSM PDE involving an extra parameter. So, that was his work.

So, we are interested about you know, looking at the SDE, what he came up with okay or what he proposed. So, also obtained a closed form expression for European options, these are success stories, okay the model gives an incomplete market okay etc and how that the calibration of the model is challenging.

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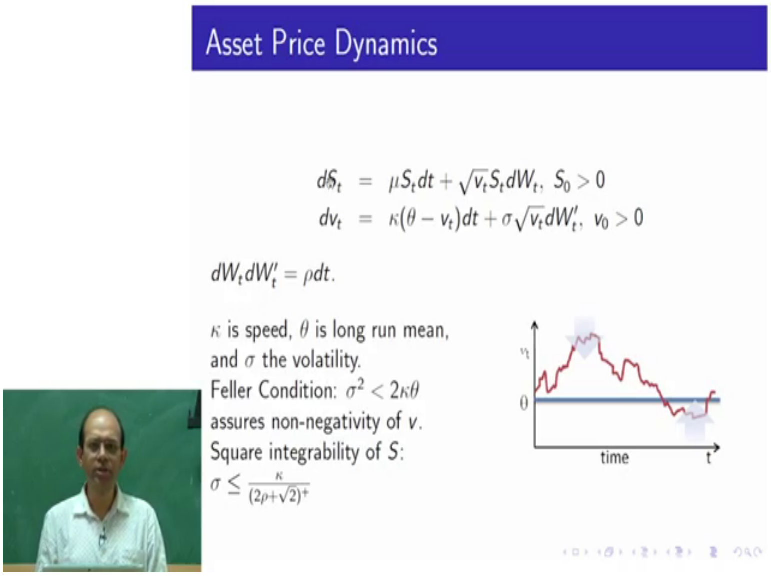

So, we come to this thing okay, the small change of the process, this is a stochastic differential equation as earlier. Here, we had mu also, we now have mu, earlier we had sigma here. Instead of sigma, now we have square root of v t, this is time dependent okay, square root of v t, where v t is a stochastic process which solves another stochastic differential equation.

dv t is equal to, so some drift parameter which also involves v t, okay affine linear function dt plus, so here some constant sigma times square root of v t dW prime t okay. So, these unknown appears here also. So, here is also unknown appears here the coefficient, but here unknown also appears, but here unknown appears as a linear function okay.

The coefficient is a linear function of S. Here this coefficient is not linear function of V correct, it is a square root of this and the coefficient here is of course constant okay. So, these we call volatility of volatility, so this appears like volatility of this, correct. So, volatility of volatility, you call this, okay. And then this W and these W prime need not be the same and they could be too different Brownian motion.

They could be independent. They could be correlated. So, if they are correlated, so dW t, dW prime t is rho dt, rho is the correlation coefficient. So, that is the market traders, they believe that this rho should be negative term. okay That is a different story that why, I mean or how they believe and why do they believe that? So, that is the model, okay. So, now, imagine that we have such type of stochastic differential equation okay.

Forget about this, just look at this thing. So, how can we assure that this has a solution? How can you assure this has a solution? Because it turns out that for this equation, it is not possible to get any closed forms solution. One can not write down the solution of this equation in terms of elementary functions.

However, this equation was largely studied, we call this CIR process. This is studied because of its very interesting feature okay. So, let us look at the feature of this process without solving it just looking at this equation, we can actually understand how does it behave little bit, okay. So, that way, I am going to discuss here, once one get is v t, then using this v t one can solve S t.

So, here this is actually system of stochastic differential equation. So, one should perceive this as system differential equation, it has two unknowns okay S t and v t and two unknowns, two equations. So, now, let us see what does it do? So, here are all these term has some particular meaning. So the kappa is called speed parameter. Theta is called long run mean. And this sigma is called the volatility of volatility, okay.

So, now we try to understand that how v t moves, okay. Imagine that theta is a positive constant and kappa is some positive constant. And your v t is very large so at some time t, v t become very large, very large, much larger than theta, then what would happen? So, since v t is very large and larger than theta, so theta minus v t will be negative.

Since theta minus v t is negative, so that means the drift is negative okay, so the small change of the dv t would be negative term times dt. So, it would force to the lower side, so minus d, negative dt. So, that means the next change would be smaller than v t. So, v t plus delta could be less than v t. However, it is not as simple as that because it is not ODE, it is SDE some other term is also involved.

This is a Brownian motion okay, that this dW t could be positive or could be negative, although this coefficient is positive of course, there is no doubt about it, but this could be positive or negative okay. Now although this term is negative, however this term could be very high positive because Brownian motion you know the increments are like almost like square root of dt type.

Because dW t whole square is the dt, okay that product table says that. So, when number is very small, then their square root is actually larger than the number, correct. If the number is smaller than 1, the square root is actually larger than that number. So, here this term would be larger than this dt term okay, will be dominant.

So, there is some kind of, you know fight between these 2 terms. However, if v t v is significantly large then perhaps it would be able to win over it okay and then it would give a push towards the lower side because then change of v t is negative that means the next v t, I mean v would proceed to the lower side.

So, on an average it would go to come below. So, when it goes up to up then there is kind of force to push it below, that we understand from this equation, okay. It would not directly come just like, you know exponential function but it would fluctuate but on an average it would come down because this part is negative and this part is positive negative both, okay.

Also another thing when v t is very large, the square root of v t is not that large, okay. So, this coefficient would dominate over this coefficient also. So, that is also one thing one should take account of. Okay now the other side, imagine that v t is very small very close to 0, what would happen? If v t is very close to 0, so that means it is smaller than theta.

So, this term would be positive, okay this term would be positive, this kappa is anyway positive. So, dv t is positive that means, you know change would be upper side. However, story is not as simple as that because there is some noise term also. So, the noise term, what is it going to do? Sometimes it could be positive, sometime it could be negative, so although it is going to push up this can push down also okay.

However, it is multiplied with square root of v t. So, if v t is very-very small this is also a small quantity, this is also small quantity multiplied with the Brownian motion here the increment Brownian motion. However, if v t is very small, square root or v t is not that small also.

So, it is not an intuitively we think that okay it would be able to push up, but it is not very clear why it should win over, why it would not allow the process to go to negative side, okay I mean we understand this intuitively very vaguely that we do this, but there is a very clear sufficient condition this we call feller condition.

If the sigma is sufficiently small that means sigma square is less than 2 times kappa theta that assures non-negativity of v, okay. Because here, we understand if sigma is large than although this is small, these you know Brownian motion you know, the negative term could push to touch to 0 okay.

But this feller condition says that okay if sigma is sufficiently small, we can actually, can assure that it would remain positive, okay, good. So, theta is something so which is actually allowing us to do this analysis because things is lesser than theta then things are, I mean that the process goes up, if processes more much higher than theta then it comes down. So, that also gives the justification of the name the theta is called long run mean okay, long run mean.

And kappa is speed, why is it so? Because imagine the kappa is very-very large quantity, then even if v t is little more than theta, this top is kicked up, it is very high. So, it would sharply, you know the process v t would sharply come down. So, speed of reverting around this mean is proportional to kappa correct, so it is coming from kappa. So, we that is the reason we call kappa as speed.

If kappa is very small then actually v t may allowed to go very large and may not and then after that it would come back and then the time it would take to come back and etc would be longer, okay slower. So, that also justifies the name that kappa is called speed and why this is called volatility what that is very clear because this is coming from here.

If sigma is 0 then there is no volatility, there is no fluctuation, it is just O t, how would that behave? It would be just exponential function correct, the solution and it would just call if it starts from here, it would converge to this theta exponentially okay, asymptotically. And if it starts from below, it will just converge like this, okay.

So, this is the understanding of SDE. So, now we have some sort of feeling about what stochastic differential equation mean, and for one particular SDE which is for Geometric Brownian motion, we also solved that equation, we found that the solution in the closed form, but for this type of equation for which we cannot apply, Ito's lemma here or something you cannot do that. Okay for that, we have discussed some properties using mostly you know, handwaving manner because it is little far from our goal.

However, we have quoted some theorem which is called fellers condition that studied this equation actually without solving it that this sufficient condition for non-negativity. There are some other theorems also like you know under another condition on sigma, if sigma is smaller than kappa by 2 rho plus square root of 2 positive part because rho could be negative, right.

If rho is negative, you know then you do not take this you know, if it is positive then only you take this. So, that thing assures that S t would be square integrable. Okay so, these are the results what people study. So, what we have realized that for application purpose, stochastic differential equation do arise and sometimes we can solve explicitly, sometimes we cannot.

However, it is important to justify existence of solution. And sometimes, you know after that we also studied few more other properties desirable properties stochastic process or the solution of the stochastic differential equation. So, next week onward, we are going to discuss the topic of stochastic differential equation and the notion of its solution, strong solution, weak solution etc.

Also, that would allow us to enlarge the scope of the differential equation, the PDE we are discussing now, because we have discussed only those PDEs where the differential operator is just Laplacian the second order differential operator is just Laplacian, okay.

However, when we are going to discuss this type of stochastic process, I mean beyond Brownian motion this thing, so for that would allow us to discuss, you know some other Cauchy problems or the Dirichlet problem where the operator is not Laplacian, but some variable coefficients second order partial differential operators okay. So, I stopped here. Thank you very much.