Introduction to Probabilistic Methods in PDE
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Lecture 15
Brownian motion as the building block

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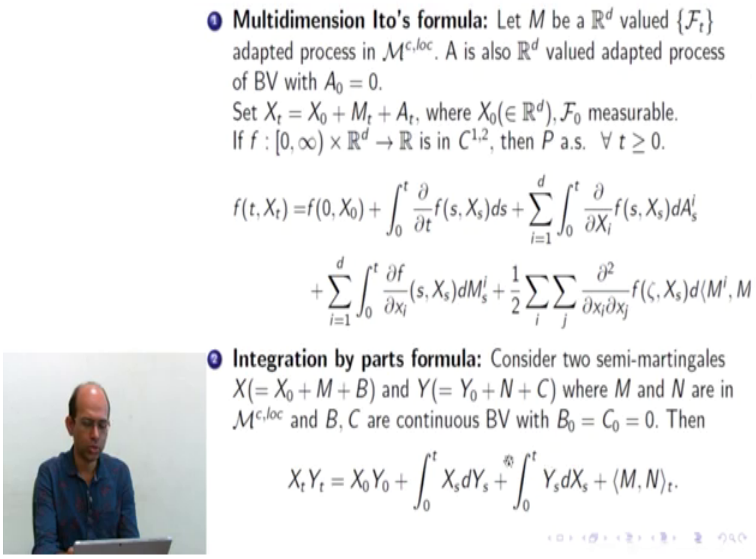

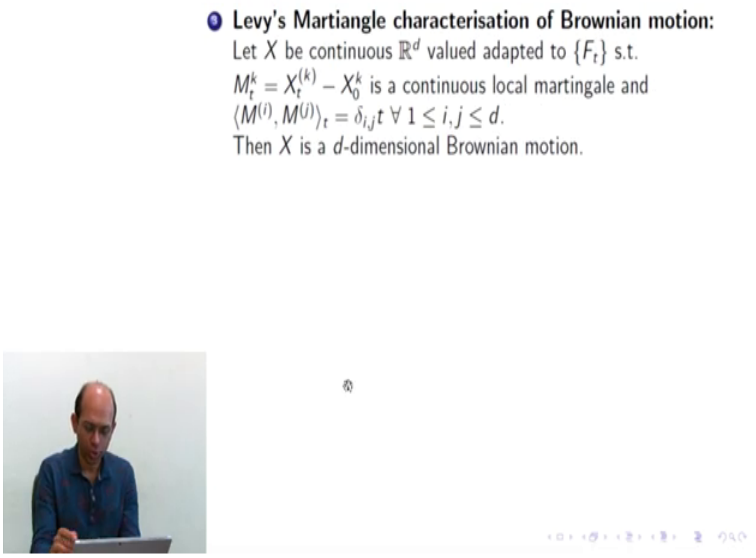

So, now we would see some generalization of Ito’s formula. We are not going to prove these results. We are going to state these results, which is immediate, immediate in the sense that exactly the similar techniques is required. This is just multidimensional version of that.

So, imagine that your X semi martingale is not a real valued but Euclidean space valued. So imagine, so let us start reading from here. So let, M be a Rd valued, Euclidean space valued Ft adapted process and that is continuous local martingale, and A is also Rd valued, adapted process of bounded variation with A0 is equal to 0, and if we add, you know Mt, At and some X0 which is F0 measurable then what you are going to get is a semi martingale.

So, if we consider this semi martingale Xt and then you know, function f is just a function of time, earlier our function f was not explicitly dependent on time t, only dependent on the space X but here we allow f to be a function of time and also function of Xt, Xt is coming from Rd so it is, the domain is close 0 to open infinity and cross Rd, Cartesian product of these two sets. And we need to assume some kind of, you know differentiability of f, otherwise we cannot write down the Ito's formula. Here we consider f to be C12, 1 means it is once differentiable with respect to time, and twice differentiable with respect to space.

Then, the statement of the theorem says that with probability 1 for all t we are going to get this value. What is this equal to, left hand side you have f of t, Xt and then right hand side you have, this is the value of f evaluated at time 0 and the process also at 0, so starting point, so value of function at 0.

And then we have some other integral terms, first term is the first order derivative of f with respect to the first variable. So, del del t of f, denotes the first order derivative of f with respect to the first variable, where that function is evaluated at point s, Xs and then this becomes a function of S, and you integrate that with respect to s from 0 to t. So, then you get a process, by varying t you are going to get a various different random variables, you are going to get a stochastic process with respect to time t.

And then this term is, the first order derivative of f with respect to second variable; second variable is multidimensional, it has d number of components. So, one needs to take the gradient from here, like you know one has to take all the derivatives. So, we take sum of that i is equal to 1 to d del delXi of f of s, Xs, Xi is, stands for, actually I wanted to write the small x, this is typo, so del del small xi stands for, derivative of f with respect to the ith component of space variable, and then integrate with respect to the boundary variation processes, I mean that is also part of the Ito's formula which we see that , with respect to the bound variation process and with respect to the local martingale process.

So, this is the only new term we are seeing here because we have taken f to be a function which depends on time explicitly, so that is why you are getting this extra term. And here, so this is integration with respect to As, and this part is the stochastic integration of del f del xi with respect to Mis, and then there is a remaining term that is, there is a typo, this will be small s here, so this is summation half times, summation over all possible i and j del2 delxi delxj, f of s, Xs, then quadratic covariation of Mi and Mk, d Mi Mj s. And then, I think integration sign is also missing, there should be integration sign here 0 to t. So, this is the multidimensional Ito's formula.

So, there is another important result. So, this is integration by parts formula. Consider two semi martingales, X and Y. X is having the decomposition X0 plus M plus B, where Y is having the decomposition Y0 plus N plus C, where M and N are continuous local martingales, and B and C are adapted boundary variation processes, and then B0 is equal to C0 is equal to 0.So, these are the two semi martingales.

Then we know that, product of X and Y, XtYt is equal to X0Y0 plus integration 0 to t Xs dYs plus integration 0 to t Ys dXs plus the quadratic covariation of M and N, at time t, so that appears. So, this is the integration by parts formula. Actually, if you compare this formula with the classical calculus by parts formula, you would get all these terms except this one.
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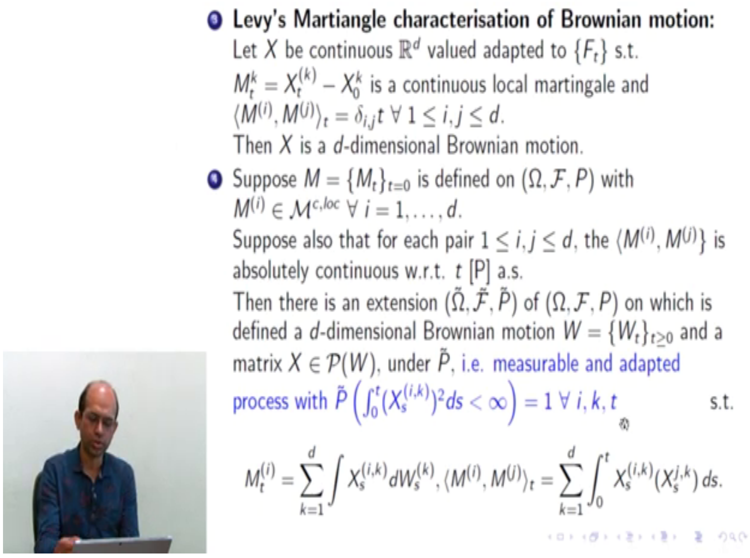

So, this is also one very important result but this result is related to one property of Brownian motion that, Brownian motion has quadratic variation as identity function. What does it mean, that quadratic variation process of B, by B is a Brownian motion, quadratic variation process of B is nothing but the time itself. So, we are going to see the detailed proof of that in another class.

So, here for the time being we assume that result. So, for explanation, you can see that this result is actually saying that the reverse is also true. What is the reverse statement of that, that let X be continuous Rd valued adapted Ft process, such that difference of the process with the initial value so that you get another process which starts from 0. I mean just for the sake of starting from 0, we just subtract the process by its, I mean subtracted the initial point from the process, so the initial value becomes 0.

Now this Mkt, if that is a continuous local martingale and its components, it has, it is vector correct, it has d number of components so, and for every components, i, and pair of components i and j, the quadratic covariation is delta i j t. If that is the case then what can we say, that if i is not equals to j, then delta i j would be 0, so the quadratic variation will be 0. If i is equal to j, then Mi and pair, actually, the same thing, so this is actually quadratic variation of the ith component of the vector and that is exactly t.

So, imagine that we have one such local martingale X, such that Mk which is you know centralized, for that we have this particular property. If that is a case, then X is a

d-dimensional Brownian motion. This is called Levy's Martingale characterization of Brownian motion. Brownian motion is the only process which has this property, mindfully continuous is very important..
There could be some other local martingales which is not continuous and not Brownian motion but having the same property. But for continuous local martingale this Brownian motion is the only member which has this property, that quadratic variation of one-dimensional Brownian motion is just the time itself.

Suppose, M is a stochastic process defined on the probability space omega F,P with each and every component Mi, is continuous local martingale. And suppose, we have that quadratic covariation of Mi and Mj, is absolutely continuous with respect to t. So, here we are considering a larger class. We saying that, okay it is not exactly t but it is absolutely continuous with respect to t.

What does it mean,it means that we can actually, I mean if we take one interval where there this quadratic covariation, which is increasing process, that increasing process, the difference would be made smaller and smaller if we make the increment of the time smaller and smaller. And I mean in the sense that, I mean, if I take a finitely, you know finite union of intervals with measure delta, sum of the length is delta, does not matter where are these, you know intervals located.

So, I can always find out one small random variable epsilon such that, we can, that, given epsilon I can always find out small delta such that whenever these, you know small sum of the intervals of the time is, size is less than delta, the difference, corresponding difference of increments would also be less than epsilon.

So, that is the meaning of this, this is basically absolutely, continuous with respect to t is same as saying this is just absolutely continuous, this is absolutely continuous. Then, there is an extension.

This extension in which sense, that, like you know, we can augment omega to make a larger omega tilde s a and then correspondingly we have to increase, you know, we have to change the sigma algebra, and also the probability measure. But it is possible, it is possible to, you know extend the probability space, such that on which we can define a d-dimensional Brownian motion W, and a matrix X which is in the P W class.

What is PW, PM class we have already seen. That you know, it is progressively measurable, I mean, not progressively measurable, that is P star. P class is just the adapted and this probability of the integration with respect to the quadratic variation, here quadratic variation of W is just time, is finite, that probability is 1.

So, what is the conclusion, conclusion is that for such process M, we can find out a larger probability space on which you can construct a Brownian motion W. And one process, one matrix valued process X, such that this M can be written as integration of X with respect to W.

So, M can be written as, I mean here we cannot claim M is exactly W, earlier as we have done but here because you know, here quadratic variation is not, there is a typo, there should be, the right angle notation So, because quadratic variation of this is just absolutely continuous with respect to t, but using that thing one can construct this you know, integrand X, a matrix, and this vector W, you know this Brownian motion, d-dimensional Brownian motion.

And then this integration has a proper meaning. that matrix and the vector, so you are going to get a vector in d-dimensional process only, and that d-dimensional process is going to coincide with Mt. So, this matrix multiplication is here written in this manner, k is equal to 1 to t, Xik dWk. This is nothing but the matrix multiplication. This is matrix, this is a vector, and 0 to t, here I should have written 0 to t.

So, that is, basically saying that, does not matter what is your martingale, as long as your martingale's quadratic variation process is an absolutely continuous process. Then that Mc loc, that you know M is not even martingale, just a local martingale. Then that local martingale Mt can be written as integration of a matrix, adapted and which is in the P W class, matrix with d-dimensional Brownian motion.

That means we can represent that martingale in terms of Brownian motion, basically. So, that actually tells the Brownian motion is actually building blocks of the stochastic processes, as long as you are dealing with the cases where the quadratic variation is absolutely continuous process.

And also the formula we see now, that if we look at the quadratic covariation of the ith and jth component of this local martingale, that can also be written in terms of this matrix X , that Xik Xjk and sum over k is equal to 1 to d, so what you are going to get is that this ds, so this is the matrix, correct for i, j. So you also are going to get, this is matrix multiplication. So, you are going to get this matrix and this matrix you are just taking integration, 0 to t, so you are going to get this matrix, Mij.

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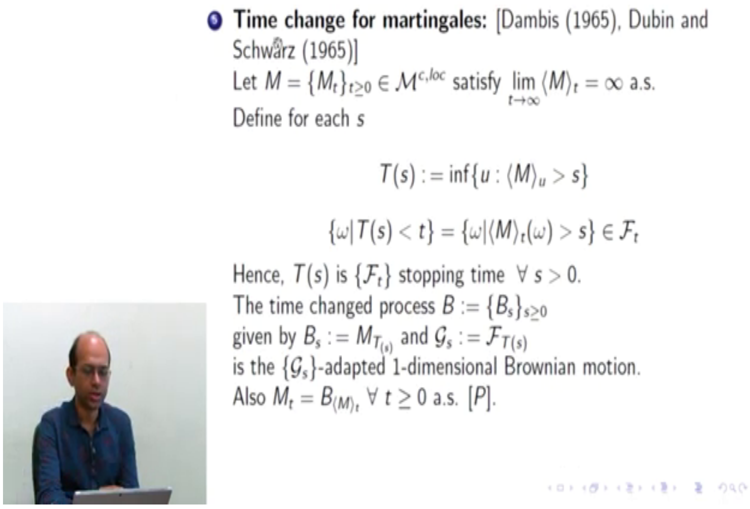

So, this is time change for martingale. This is another way, of viewing Brownian motion as building block, another way. So, what is this? this is the work of Dambis in 1965, Dubin, Schwarz in 1965. This is quite old work.

So, this says that when M is a continuous local martingale and which satisfies this particular condition, so this is just saying that it is not converging to a finite number.

So,as t tends to infinity, quadratic variation of Mt is going to infinity, I mean, Brownian motion has that of course, and which process does not have that. For example, you take an arbitrary local martingale and then you consider a stopped process, localization where using a stopping time, which is the hitting time of a boundary of some kind of big ball.

So, then that would be like, you know, does not matter what t is, this would be bounded, it is fixed, it would not go to infinity. So, only thing is that you should, I mean if we do not consider that type of processes. So, here the quadratic variation process grows to infinity. So, under this condition you can define this random variable Ts for each and every positive number s, such that this is infimum of u, such that quadratic variation of Mu is greater than s.

So, what does it do, this is actually some sort of generalized inverse, inverse of a function. If a function, you know Mu is exactly a bijection, from close 0 to infinity, close 0 to infinity, then this would actually coincide with the M inverse.

However, sometimes you know, M may have a some kind of, may have constants. So, those kinds of things, there you cannot talk about inverse but we can of course talk about this infimum, so when it crosses s for the first time, when it crosses s for the first time. So, that is called Ts. If you are finding this formula little, you know, non intuitive you can think just inverse of M, just generalized inverse of M. Also, we use like, you know percentile function for, you know inverting CDF etc. also in the same manner.

So, here this Ts, we first see that Ts is a, is an Ft stopping time. So, let us see, how to do that,this is actually very trivial. So, we considered this as event, set of the sample points, such that Ts of omega is less than t.

So, when Ts is less than t what does it mean, that before time t, that Mu has crossed this, because Mu, when Mu crosses s for the first time that is Ts and Ts is less than t; that means, I mean before t, Ts has crossed s. So, that means at t, Mt of course, has crossed because Mt is increasing process. The quadratic variation process is increasing process.

So, Mt, so this is same as, I mean, actually first you get that this is a subset of that, because this implies this and the reverse also, is also you can expect, so this is equal, these two events are equal. And here, if you look at this event, what is there, this is that Mt is Ft measurable, and then we are asking that, the set of events under which Mt is more than s, s is some fixed real number. So, that event is of course, in the same sigma algebra Ft. So therefore, this is in Ft.

So, we have therefore shown that this is true, capital Ts is , capital Ts is less than t, that event is in Ft for every t positive. So, that in turns says that capital Ts is in Ft stopping time. Now, we have a family of stopping times. For each and every s we have a stopping time, Ts is family stopping time.

Now we consider, the time changed process B, so B is the process Bs, where Bs is given by this. What is this, Bs is defined as capital M evaluated at time capital T of s. So, now this is important. You can think that, Ts is the inverse of quadratic variation of M, and then if you put small t here, this Ts equals small t, then s would be that quadratic variation at t, because this is the inverse.

So, t inverse would be Ms, so B Ms, I mean quadratic variation of Ms would be Ms. So, that we are going to get here, that Mt is equal to B quadratic variation of t. Now what would be the, what should be the measurability of B, that, for that we have to change the filtration also, depending upon, because now time has changed, right therefore filtration should also change.

So, we are going to take F subscript Ts. So, because this is a stopping time, so for this particular stopping time we have stopped sigma algebra and as s is increasing Ts is also non decreasing process. So, this would give me another filtration, coming from the stop sigma algebra. And if you call this as the filtration Gs, so this Bs is Gs adapted, and this is also Brownian motion.

This is a Brownian motion and this Brownian motion, it has this property, that BM this t is equal to Mt. And now, you can actually take quadratic variation of this thing, so quadratic variation of this process Bt on the right hand side, and left hand side going to Mt , you are going to see that, that thing would match with this value. So basically, from here one can easily find out this is Brownian motion. If you can show this, the quadratic variation of this would be exactly, would match, this would match.

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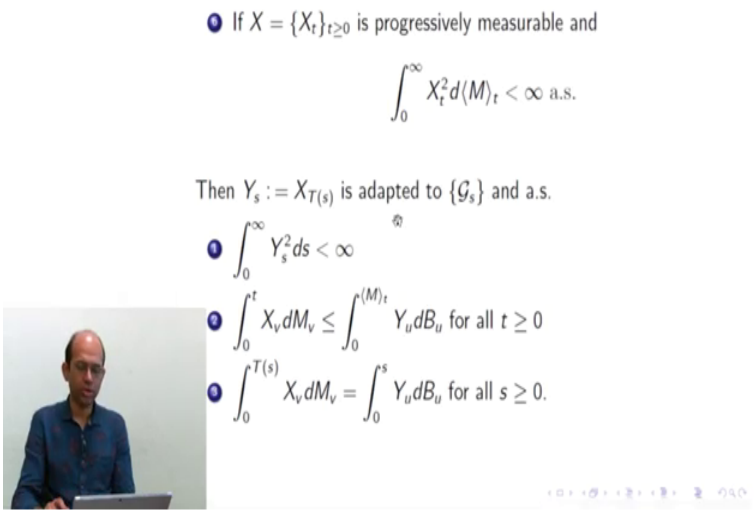

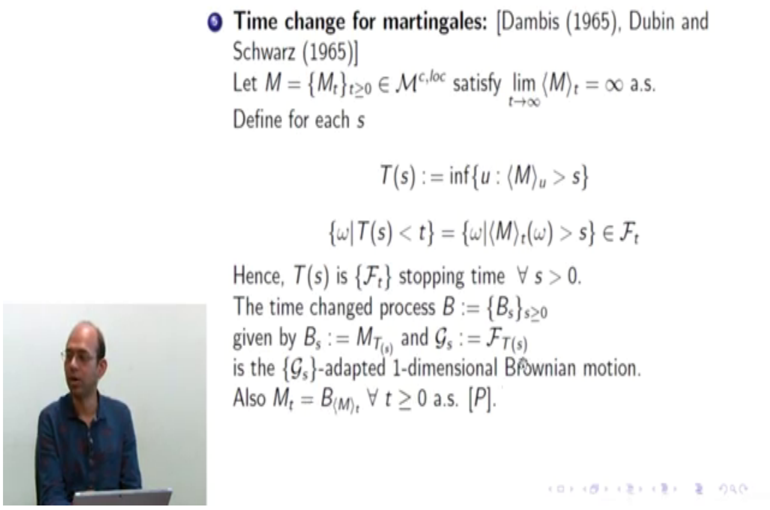

So, this is the last slide, so fifth one, sixth point. If X is progressively measurable, so X is equal to measurable process such that, this is actually, I mean P star of M basically, that probability of integration 0 to infinity Xt square d you know, quadratic variation Mt is finite with probability 1 and is progressively measurable. Then, Y defined as time change process of X, X subscript capital Ts is adapted to Gs because you know the time change, since you have to change the time, so we have to change the filtration also, is adapted to the Gs filtration and almost surely, we have following properties.

So here, this is not very surprising. Let me explain why, because you know, this Xt square integration 0 to t dMt is finite with probability 1. So, if we can, you know take control of this thing then Xt square is not growing much. So here, Xts is Ys, so Ys square is like the time when you know Mt, you know just crosses s. So, that thing is there, so for a fixed s. And then we are considering s running from 0 to infinity, that is finite.

This is like a time change formula, and here we have integration of X dM, for example you consider. And now, we apply the time change formula. So here, if we replace v by Ts you know so here, like Ts is there, so if we replace this by Ts. So, then we are going to get, for X also Ts is Ys, so instead of that I am going to get Ys here, instead of Mts I am going to get the Brownian motion here, and then instead of small t, we are going to get the quadratic variation of M at t.

There is one more identity. So, this identity is saying that, here we had 0 to t, instead of that here if we have integration 0 to capital Ts, but integrand and integrators are same, there the limit has changed. So then, for small t we have got, the quadratic variation of M and this is like inverse of that, so we are going to get this s here, correct, s here. So here, all others are same, for all s greater than or equal to 0. Thank you very much.