Advanced Computational Techniques
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Lecture 15
Linear Boundary Value Problems (BVP)

Next, we will talk about the boundary value problem. Now boundary value problem means how it is different from the initial problem is that here the conditions are given at different points at the 2 boundaries.

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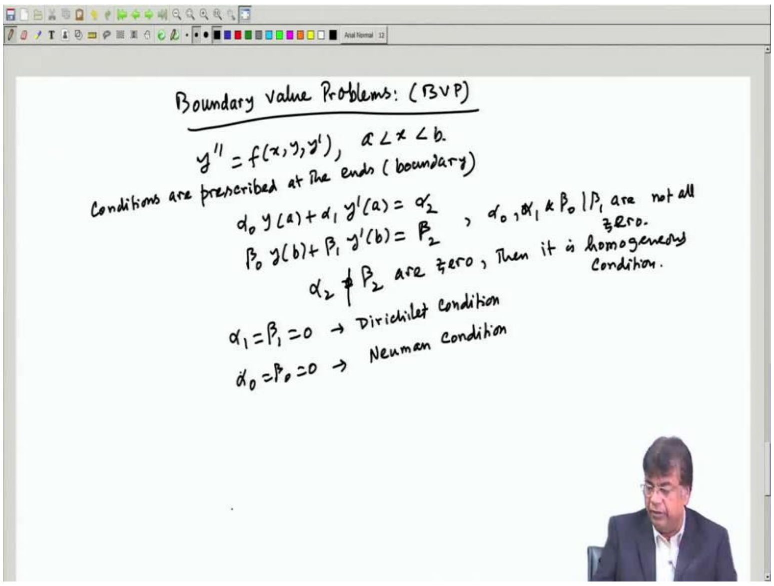

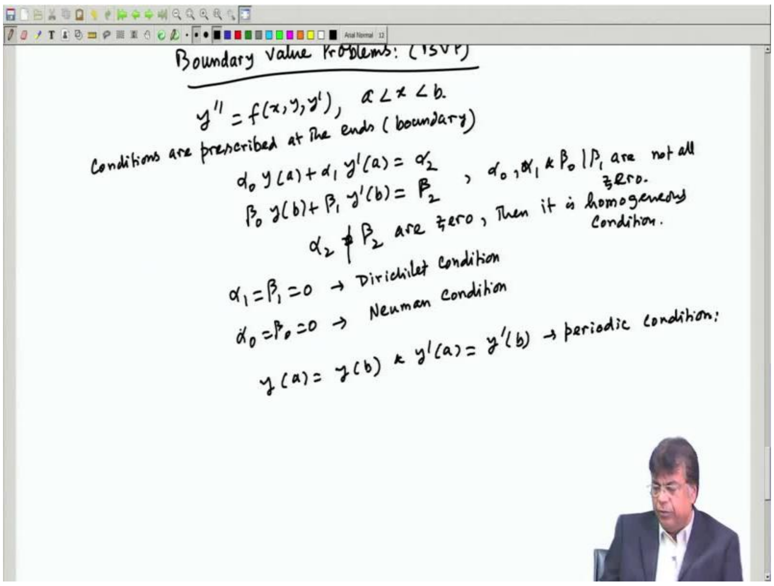

Suppose in a very simple way. Let $y'' = f(x, y, y')$ and $a < x < b$ are the 2 points between which this x is varying. Conditions are prescribed at the end or boundary. So the conditions can be prescribed in different form. So

$\alpha_0 y(a) + \alpha_1 y'(a) = \alpha_2$

$\beta_0 y(b) + \beta_1 y'(b) = \beta_2$

So, α_0 , α_1 & β_0 , β_1 these are not all 0. So, α_0, α_1 I would say > 0.

Conditions can be homogeneous. So, if $\alpha_2 \neq \beta_2$ are 0 then this is a homogeneous condition, then it is homogeneous condition. If $\alpha_1 = \beta_1 = 0$ then it is called the Dirichlet condition so, that means only the function values are prescribed at the boundary there is y(a) and y(b) is given and if $\alpha_0 = \beta_0 = 0$ so, that means only the derivative conditions are prescribed so that is called the Neumann condition. And this kind of condition as we stated above is the mixed condition so that means, the function value as well as its derivative prescribed.

Also there can be a condition like $y(a) = y(b)$ and $y'(a) = y'(b)$. So, this is a periodic condition. So, some cases we may impose the periodic condition. So, these are all forming a boundary value problem. So, that means, here the conditions are described in different points. Now, unlike the IVP we cannot reduce to a first order system for this case. So, we have to solve this problem as it is.

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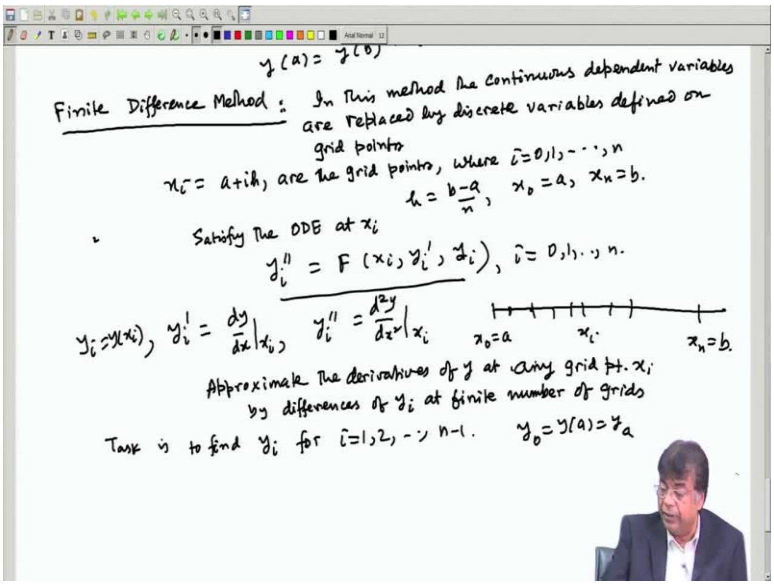

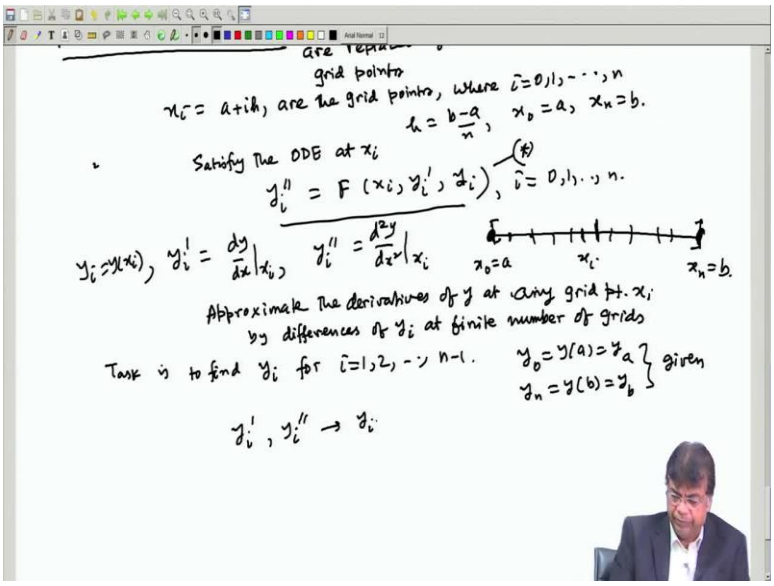

So one of the simple technique is finite difference method. So, again we have assumed this is a well posed problem so, that means the conditions are all solution exist, solution is unique and the solution depends continuously on the auxiliary conditions. So, these are already we have considered to be governed, so in those case if this is the situation then our task is to find out the solution of the boundary value problem by some numerical method.

Now, what we do in this finite difference method basic essence is that we reduce this is a continuous function form that is the function and its derivatives related. So, what we do is to maybe I can write in this way in this method dependent variables are in replaced by discrete variables defined on the grid points. So, first of all we define a set of grid points $x_i = a + i h$ the grid points, where i =0, 1, 2,…, n say or maybe small n. So, $h = \frac{b-a}{n}$ and without notation $x_0 = a$, $x_n = b$ are the two endpoints.

So, first of all if I satisfy this equation the ODE at the grid point x_i because it is valid between a to b. So at any point if I satisfy the ODE so what I get is

$$y_i'' = F(x_i, y_i, y_i') \qquad \text{-(*)}$$

So, this is say i =0, 1, 2,..,n. Now, so, this is no longer a continuous function. According to our notation

$$y_i' = \frac{dy}{dx}\Big|_{x_i}, \quad y_i'' = \frac{d2y}{dx\,2}\Big|_{x_i}$$

which is evaluated at x_i. So, these are the values and define at those discrete points.

So, this is say $x_0 = a$, so like this way we have x_i and this is $x_n = b$. So, this y_i , y_i'' these are all y_i. So $y_i = y(x_i)$. So, these are the discrete values are defined at a finite number of discrete points. So what do you find that this now become a algebraic relationship it is no longer a differential equation. Now, what is our next task is to relate or approximate the derivatives of y at any grid point x_i by differences of y_i at finite number of number of grids.

Because our task is to find y_i for i =1 to n – 1. Suppose we consider only the DD select condition say $y_0 = y(a) = y_a$ is given to be and how we are set to say α_2 or something and $y_n = y(b) = y_b$ is given. So, there is a function values at the two end x = a and x = b is given. So, our task is to find out the values of y at any arbitrary point y_i .

So, what we are intending to do is these equations (*) which is now a algebraic relation between the function value, its derivative, first derivative, second derivative and x_i . So, now, what you want to do is y_i', y_i'' we have to replace by y i’s we want to express in terms of y i’s. So, that is the one that approximation is called finite difference approximation.

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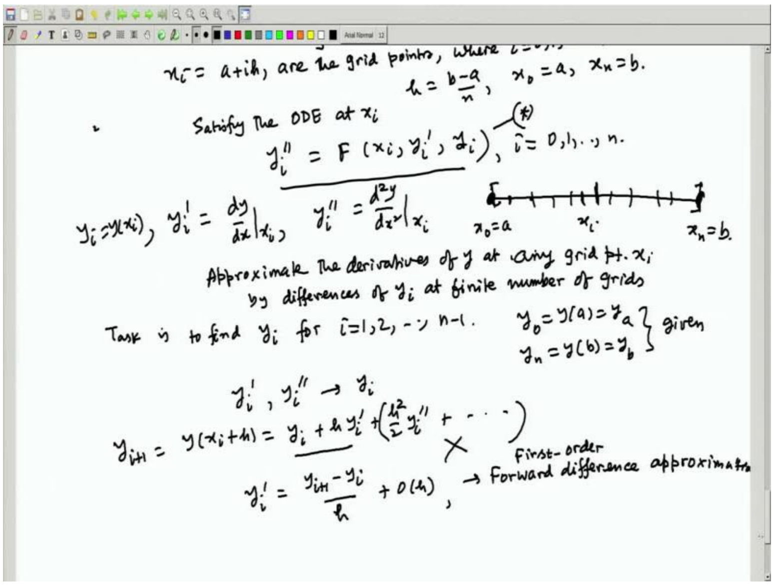

How we do that? So,very simple way we should expand by Taylor series.

$$y_{i+1} = y_i + h\, y_i' + \frac{h2}{2}\, y_i'' + \ldots \qquad \text{-(1)}$$

and so on. Now, as we did for the Euler method that retaining only up to linear terms in h, so if I neglect these terms, so what I get is

$$y_i' = \frac{y_{i+1} - y_i}{h} + O(h)$$

So, if I do that approximation, which is order h because, as you said the truncation error that means, the terms which we are dropping or chopping out or the terms we have dropped, so highest or the least order term is order h, so this is called the first order for a difference or this is called the forward difference approximation. And since it is order h so this also can be said as first order forward difference approximation.

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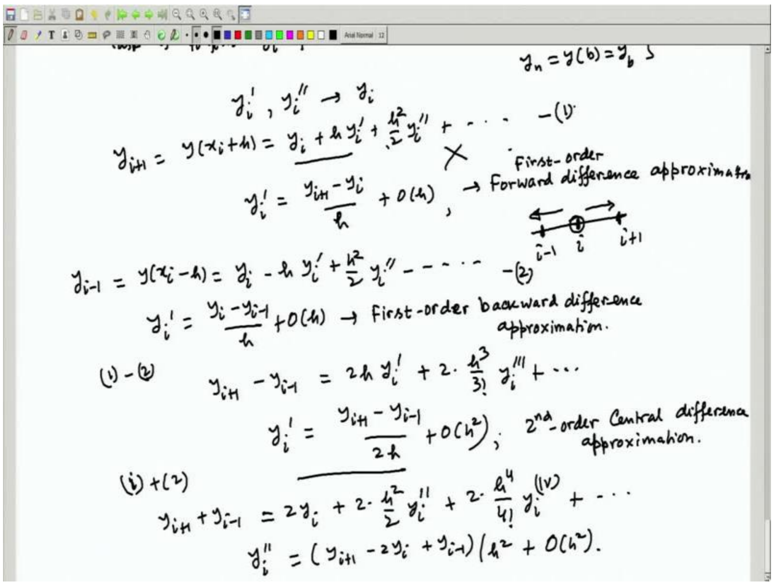

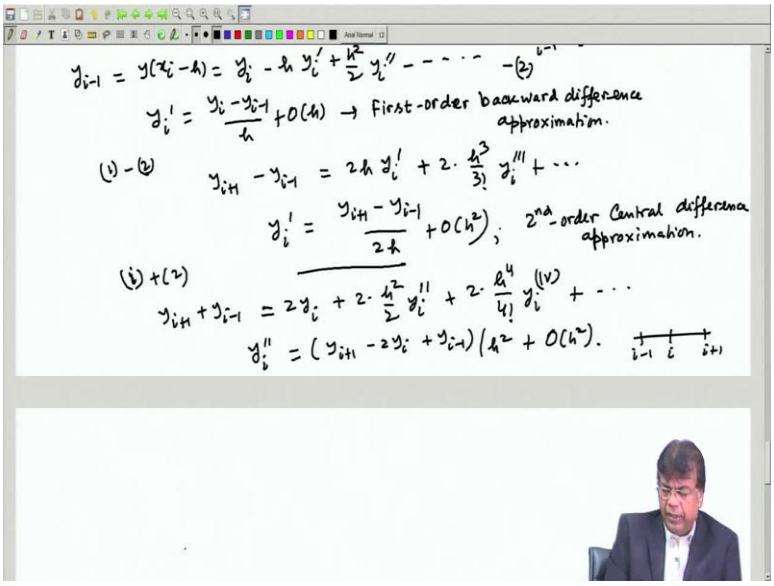

Now, for our difference because at the grid point i we are using i + 1, so that means we are going forward. Same way, if I write

$y_{i-1} = y(x_i - h) = y_i - h\, y_i' + \frac{h^2}{2} y_i'' - \ldots$ -(2) Now, if I write from here

$y_i' = \frac{y_i - y_{i-1}}{h} + O(h)$

So, this is also ordered h, first order. Now, if we subtract (1) –(2), what I get is

what do you get? This term y_i get cancelled and here it is all the odd terms will be present. So if you are subtracting the next term will be

$y_{i+1} - y_{i-1} = 2h\, y_i' + 2\frac{h}{3!}\, y_i''' + \ldots$

and so on . So, if I write an approximation

$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$

because the term whatever we are dropping we are dividing by h. So, h^3 whatever the term is the least order term, so, that becomes h^2. So, this is called the second order central difference approximation. So, we get two types of approximation for the first order derivatives, one is the forward difference approximation that means, using 2 grid point i, i + 1, another is the backward difference using i, i - 1.

The third one which is order 2 is that the grid point i is in the central location and we are using i - 1 and i + 1. So, that is we are calling as central difference approximation. Another thing also, if we add (1) + (2) so we get another representation of the derivative that

$y_{i+1} + y_{i-1} = 2y_i + 2\frac{h2}{2}\, y_i'' + 2\frac{h4}{4!}\, y_i^{(iv)} + \ldots$

this is becoming all the odd terms we will go. So, y_i'' this is a second derivative.

So, now, if I write the y_i'' an expression like this

$y_i'' = y_{i+1} - 2\, y_i + y_{i-1} \backslash h^2 + O(h^2)$

So, this is $O(h^2)$ approximation of the second derivative and this is called the second order central difference approximation. This again constitute a central difference, because here we are using the grid points i, i – 1, i + 1 this is a representation of the second derivative using 3 grid points i + 1, i, and i - 1 and it is evident that second order accurate. So, now, we have a relation of the derivatives at any grid point x_i with the function values at a finite number of grid points x_i , y_{i+1} ,y_i , y_{i-1} .So, this representation is called the finite difference approximation.

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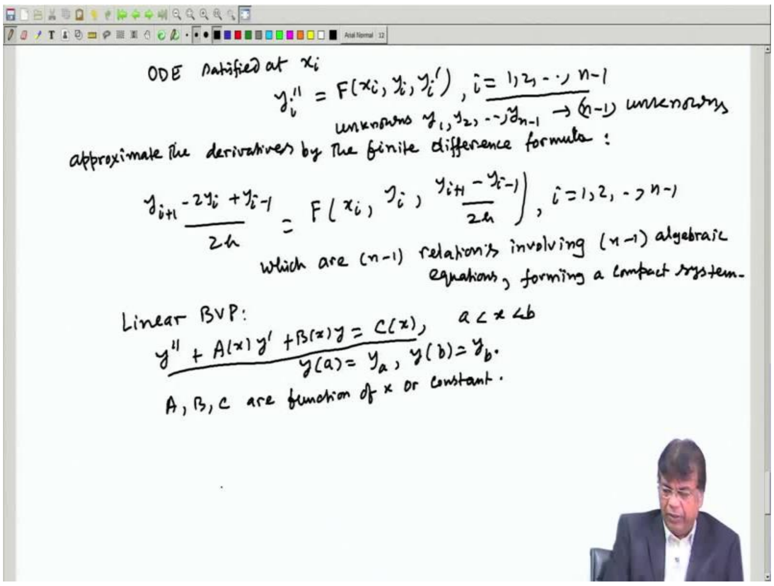

Now, if I replace, our ODE when satisfied at x_i. So, we have the equation

$y_i'' = F(x_i, y_i, y_i')$

Now, if I replace or approximate so this is i = 1 to n - 1 because we need in equations, unknown are $y_1, y_2, \ldots, y_{n-1}$ that is n - 1 unknowns. So, and if we use this equation for i = 1 to n - 1, so n - 1 unknowns involving n - 1 equations.

Now if we approximate so far there was no approximation we have just satisfied the given differential equation at the grid point x_i. Now, if we approximate the derivatives by the finite difference formula, so now, we have option there are several options, y_i' one can replace by first order forward difference or backward difference, obviously, we will go for the central difference, because that gives you a second order accuracy, higher order accuracy.

So, if we replace the approximate derivative by finite difference formula. So,we get

$$\frac{y(i+1) - 2\,y(i) + y(i-1)}{2h} = F\left(x_i, y_i, \frac{y(i+1) - y(i-1)}{2h}\right), \quad i = 1, 2, \ldots, n-1$$

So, this is. Whichever n - 1 relations involving n - 1 algebraic equations depending on whether it is the linear or nonlinear, so, should that means, which is forming a compact system, the compact system is formed that means, you have same number of equations and same number of variables.

If our boundary value problem the ODE is another linear boundary value problem. So, that means, a linear if it is the given ODE is linear, so then we can write these in this form

$$y'' + A(x)\, y' + B(x)\, y = C(x) \;,\; a < x < b$$

$$y(a) = y_a \,,\; y(b) = y_b$$

we take the simple from that means only the function values are prescribed at the two end. So A, B, C are function of x or constant or function of x or constant it cannot be depend on y or its derivative. So, if it depends on y or its derivative, then this is no longer a linear one, so degree will become higher than one. So, that cannot be treated as a linear boundary value problem or linear differential equation.

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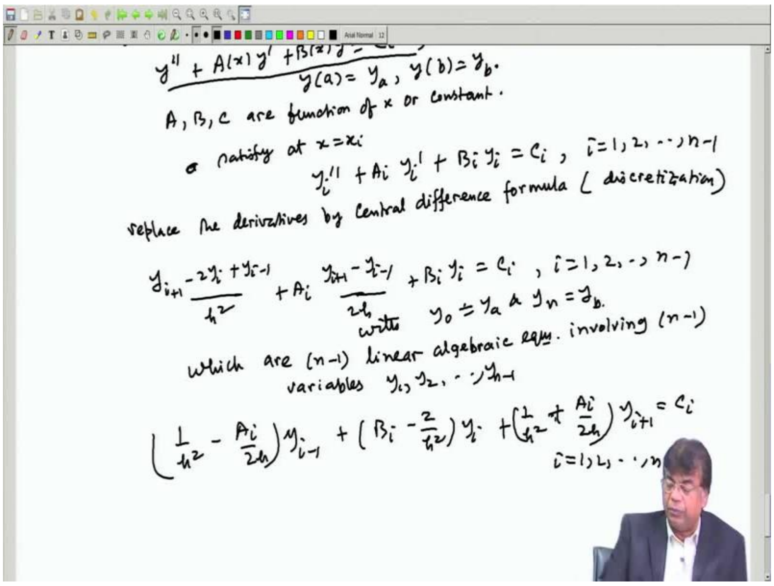

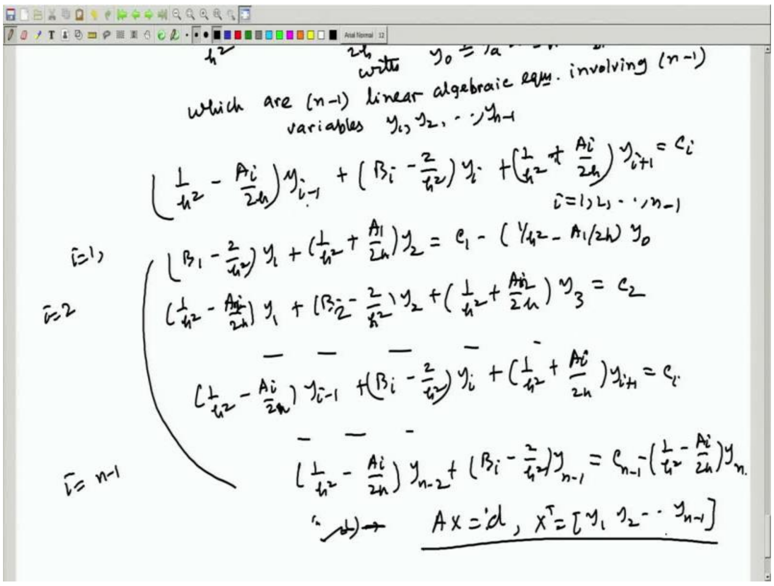

So, any linear ODE can have this form. Anything other than that is non-linear. So, that means, if A or B or C are involving y or its derivative or C is involving non-linear function of y, so we get a non-linear situation. So if it is a linear, so let us talk about first linear. So, satisfy at x =x $_i$, we get

$y_i'' + A_i y_i' + B_i y_i = C_i$, i =1, 2,…, n - 1.

Now, replace the derivatives by a central difference formula, so this procedure is called (discretization) .If we do the discretization replace the derivative for central difference formula, so, if we do the discretization what we gets is

$\frac{y(i+1) - 2y(i) + y(i-1)}{h2} + A_i \frac{y(i+1) - y(i-1)}{2h} + B_i y_i = C_i$, i =1, 2,…, n - 1

So, obviously, y_0 and y_n, so with $y_0 = y_a$ and $y_n = y_b$. So which involves (n – 1) equations linear algebraic equation that is very important linear, all these ways are appearing in a linear fashion algebraic equation involving (n – 1) variables $y_1, y_2,\ldots,y_{n-1}$, because y_0 and y_n, unknown. Now, if I write in a proper form so,

$$\left(\frac{1}{h2} - \frac{A(i)}{2h}\right)y_{i-1} + \left(B_i - \frac{2}{h2}\right) y_i + \left(\frac{1}{h2} + \frac{A(i)}{2h}\right) y_{i+1} = C_i\ ,\ i = 1, 2, \ldots, n-1$$

So this forms a system of equations given by this way. And if I put i =1 so this become y_0 so that is given so and get transferred to the other side. So i =1 I get

$$\left(B_1 - \frac{2}{h2}\right) y_1 + \left(\frac{1}{h2} + \frac{A(1)}{2h}\right) y_2 = C_1 - \left(\frac{1}{h2} - \frac{A(1)}{2h}\right)y_0$$

Then I put i =2, so what I get i =2?

$$\left(\frac{1}{h2} - \frac{A(2)}{2h}\right)y_1 + \left(B_2 - \frac{2}{h2}\right) y_2 + \left(\frac{1}{h2} + \frac{A(2)}{2h}\right) y_3 = C_2$$

and likewise we go. So,

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$$\left(\frac{1}{h2} - \frac{A(i)}{2h}\right)y_{i-1} + \left(B_i - \frac{2}{h2}\right) y_i + \left(\frac{1}{h2} + \frac{A(i)}{2h}\right) y_{i+1} = C_i$$

And the last one put i = n – 1 , this is general. So if I put i = n - 1, so this the last one will become y_n which is again known. So what we can do is we can transfer to this side,

$$\left(\frac{1}{h2} - \frac{A(i)}{2h}\right)y_{n-2} + \left(B_i - \frac{2}{h2}\right) y_{n-1} = C_{n-1} - \left(\frac{1}{h2} + \frac{A(i)}{2h}\right) y_n$$

it is given to be this is value is given. So this involves algebraic system , (n – 1) equations, (n – 1) variables, so one can write in a matrix form A X = b and there is a pattern. So A X = b or AX = d and what you find, that this X we call as all this y_1, y_2, X transpose is vector.

$$X^T = [\, y_1, y_2, \ldots, y_{n-1} \,]$$

One good pattern is that A is a tri-diagonal all the terms are not involve. So we stop here and we discuss more in the next class.