Engineering Mathematics - I
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Lecture - 42
Vector Spaces – Basis and Dimension

So, welcome back and this is lecture number 42 and we will continue our discussion on Vector Spaces again.

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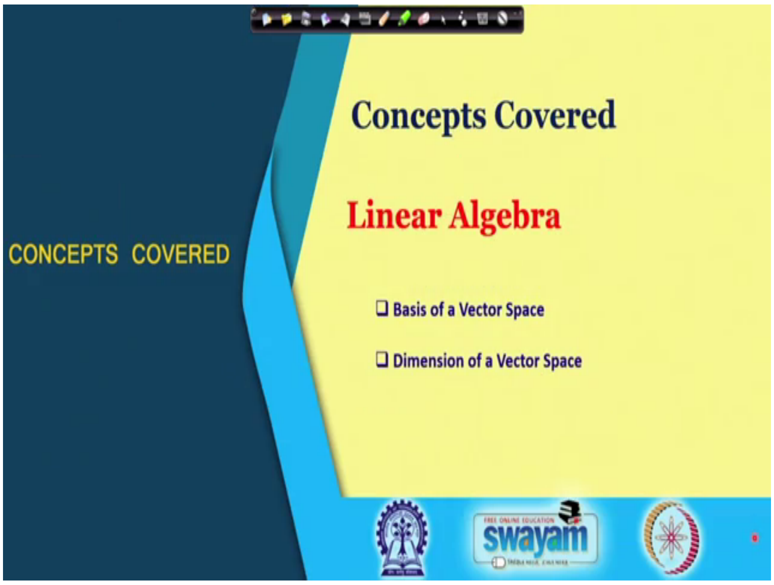

In particular today will be talking about the basis of a vector space and also the dimension of a vector space.

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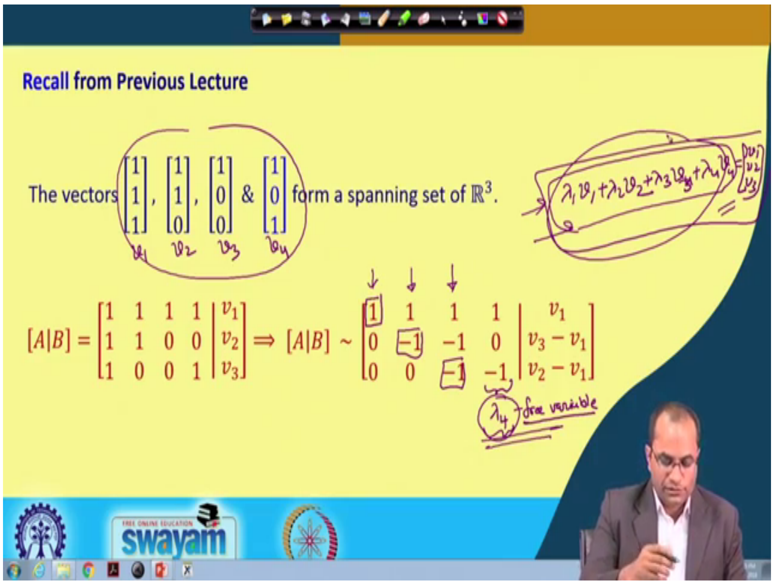

So, just to recall from the previous lecture; so, we have seen that these vectors the set of these 3 vectors are 1 1 1 and 1 1 0 1 0 0 and 1 0 1, these 4 vectors form a spanning set of R 3. The meaning was that any element of this vector space R 3 we can write as a linear combination of these 4 vectors, this is what we have seen in previous lecture with the help of this augmented matrix which we can get out of this linear combination equal to this v 1 v 2 v 3.

And from this augmented matrix when we reduce to this rho echelon form we observe the following. What we have observed that there are these 3 pivots in column 1 we have pivot, in column 2 also we have pivot, column 3 also has a pivot while the column 4 does not have a pivot and this is what we call the free variable. So, the lambda 4 in our combination when we have taken with lambda 1 with this vector the given vector v 1 lambda 2 this given vector v 2. So, we call this v 1 v 2 v 3 and v 4 and this lambda 3 v 4 v 3 and lambda 4 v 4 is equal to this given vector v which we have taken here the v 1 and v 2 and v 3.

So, with these system of equations what we observe that lambda 4 is free variable; is free variable and meaning that we can assign whatever value we want to this lambda 4. So now, we have the flexibility to give this linear combinations here. So, choosing a different value of lambda we have a different linear combination that will give us exactly this vector v 1 v 2 and v 3. So, this forms a spanning set because any vector any arbitrary vector we can pick from this R 3 space and we can write down as a linear combination of these given 4 vectors. What is interesting here that we have a non-unique representation because of this free variable.

So, when we have a free variable we can assign any value we want to this lambda 4 and then our representation of this given vector will also change. So, this linear combination will also change. What was also seen in the previous lecture that, if we do not consider for example, this vector the fourth-one if we do not consider this vector; that means, we will not have this column here, this column we can delete now. So, in that case this was a situation and what we observe now for this case that we have the every column has a pivot and there is no free variable.

Meaning when we do not have a free variable we will get basically the unique representation, the unique solution of this system. And this is what we will discuss today that having these 3 vectors we have a spanning set having this 4 vectors, that also form a spanning set. Then we will come up with the idea now that how many actual minimum vectors do we need so, that we can span the given vector space.

So, for instance in this case we have observed that taking these 3 vectors we can span the whole vector space and also the moreover the main point is that we have also the unique representation of the vector form R 3 if we take these 3 vectors. But, if we have taken the 4 vectors still it is forming a spanning set, but we are not getting a unique representation. So, keeping these facts in mind now we will continue with the definitions of the basis and the dimensions.

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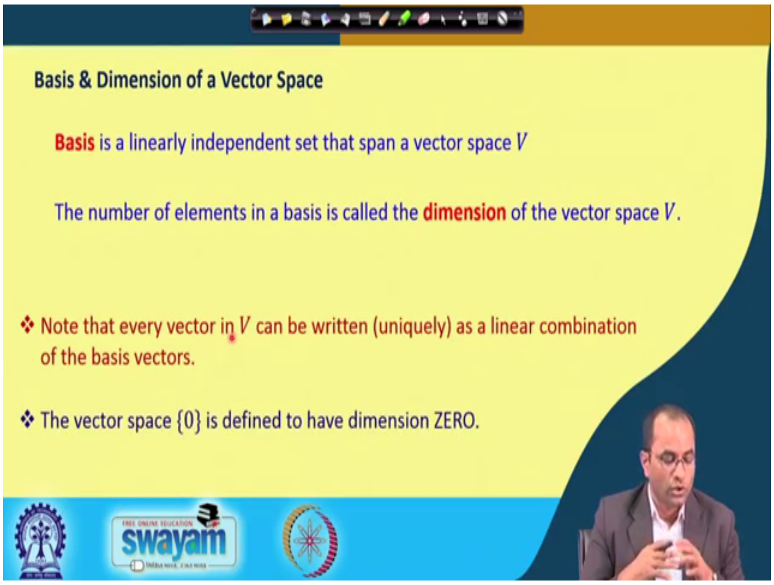

So, the basis is a linearly independent set that span a vector space. So, the simple definition it is a spanning set because it should span the whole vector space V, but what is in addition that the linearly independent set. So, the basis is nothing, but spanning set which has all linearly independent vectors that we call basis. Like in the earlier example we have possibility in this spanning set taking all those 4 vectors or taking only those 3 vectors.

And but if we take those 3 vectors we can easily figure out their those vectors are linearly independent, but having those 4 vectors in the set the set becomes linearly dependent, that also we can observe with the theory we have already developed in previous lectures. So, those 3 vectors they are linearly independent and for those vectors we can call that they will form a basis not the 4 vectors because 4 are also spanning set, but that will not be the basis of R 3 in that example.

So now, the dimension so, the number of elements in a basis is called the dimension of the vector space. So, here the number how many elements are there, how many elements are there in that basis that is called the dimension of the vector space. So, these two numbers are the basis that is the set of the vectors and that is that is span the given vector space and this number here which is which tells about the dimension of the vector space both are very important.

Just a note here that every vector in V can be written uniquely as a linear combination of the basis of vectors and this fact also we can explain from the previous example itself, that when we have the base is there; the basis means you are linearly independent vectors. And when you have linearly independent vectors there will be no free variable when we talk about that linear combination of the given vectors to represent a vector from this V.

So, that representation will be also unique whenever we have the basis our spanning set is having linearly independent vectors than the representation will be also unique. And the vector space 0 so, this is the special case when we have only one element that is 0 and we define this dimension as 0.

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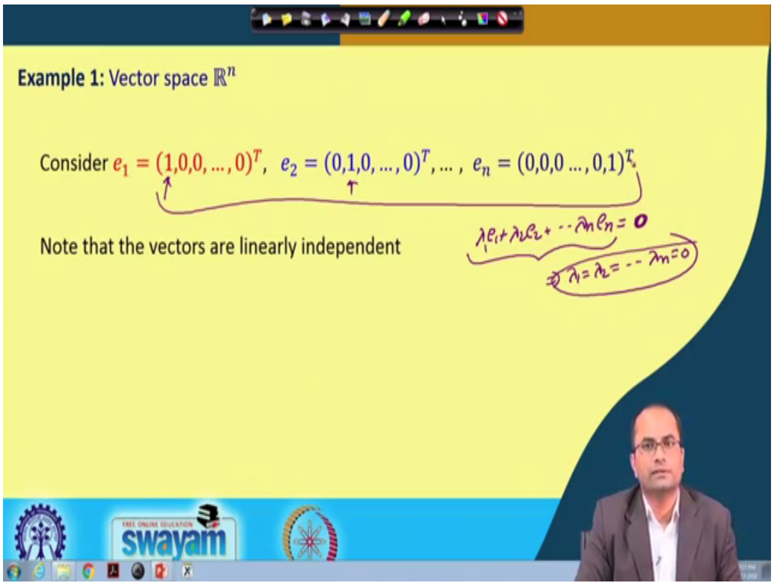

So, let us take the example here the vector space R n which we have already studied. Now, we will talk about what are the basis of R n and what is the dimension of this vector space R n. So, we consider these vectors now very special vector which we call the standard basis of R n. So, consider these vectors e 1 is denoted when the first component is 1 all other components are 0, e 2 is the first component 0 the second one all other 0 and in e n we have this n th component as 1 and all other components are 0.

So, in this way we have defined these elements these elements are from R n of course. So now, we will notice here that these vectors are linearly independent; first that we can easily check out and we have I think checked before as well. So, we have to just see that lambda e 1 lambda 2 e 2 and so, lambda n e n when we set to 0 vector then we should get out of this that lambda 1 is 0, lambda 2 is 0 and all are this 0's which we can easily see because of this structure here itself we will get from this lambda 1 0 from here we will get lambda 2 0 and so on. So, this trivial to see that this vector this set of vectors form a linearly independent set.

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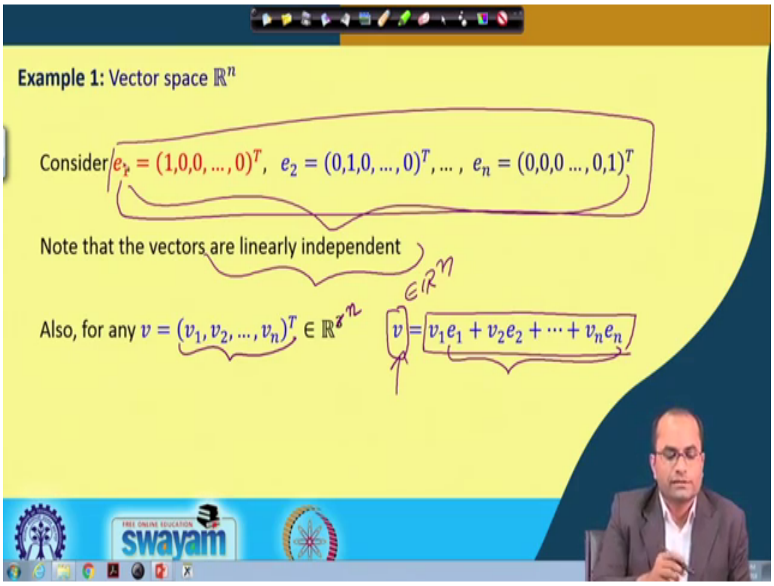

So, all these vectors are linearly independent. What else we have to observe? So, we have to we are going to find the basis and the dimension of the vector space. So, for that we need spanning set basically that can span any that can represent any vector from our vector space in the form of this given vector. So, whether here it is possible or not we will check now. So, given these vectors here n vectors are given and now any element if we take from this R n so, any this is R n.

So, this is the element of this R n. So, if we take a general any vector from this R n and which we are calling here v 1 v 2 v 3 v n then we can see that we can represent this vector with the help of these given vectors e 1 e 2 e n. The idea is again simple in this case and that is for these are called the standard basis. So, this v can be represented as the v 1 times this e 1, the second component v 2 times e 2 because of this special structure here it will multiplied by v 1.

And no where else we will get anything at this place because all other vectors have 0 as first component; for the second v 2 only the e 2 has 1 at the second place all others are 0.

So, naturally we will get just v 2 from this one at the second position and so on. So, we can represent or we have already represented here any vector v from this R n as a linear combination of these given vectors e 1 e 2 e 3 e n.

So, it satisfies all the properties of the basis we have this is a spanning set because, we can span any vector of the given vector space in the form of as a linear combination of the given vectors and also these vectors are linearly independent. So, they form the basis so, we got a basis for this R n, this set of vectors here this forms a basis for R n.

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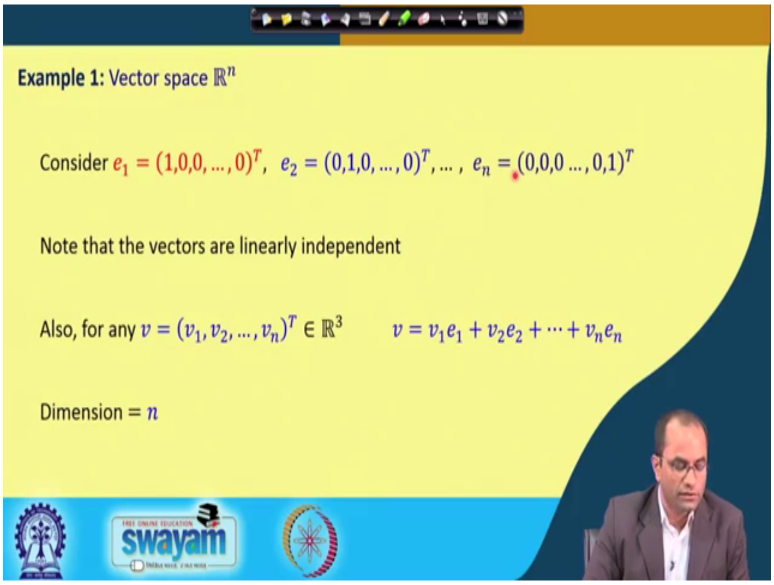

And the dimension now it is a number of these elements in this set. So, here we have e 1 e 2 e n there are n elements and we got therefore, this dimension here of this vector space is n.

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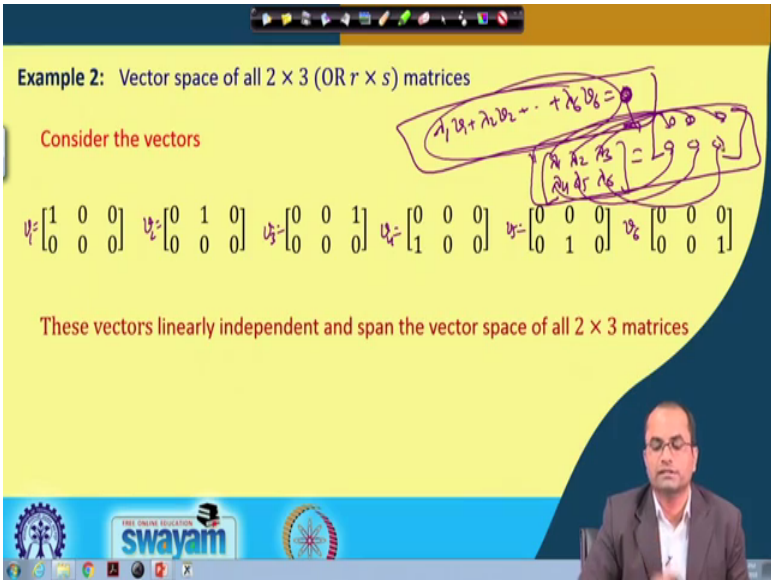

This example where we are talking about the vector space of all 2 by 3 or in general also we can discuss like for r by s matrices. So, for simplicity let us take this 2 by 3, but the idea we can extend for r by s matrices as well. So, here the vector space of all 2 by 3 matrices we are talking about and if we consider these vectors here, the 1 at this position and all other positions are 0; similarly now at the second position is 1 all others are 0.

And so on we continue with this we get these 6 matrices where this 1 is sitting here at the first position only in this matrix, in the second case 1 is sitting here and all others are 0 and so on. With this special structure again these are the standard basis for this vector space of this 2 by 3 matrices we have already seen that this format a vector space in the last lecture. So, here we are talking about or finding the basis and its dimension.

So, these vectors are linearly independent and span the vector space of 2 by 3 matrices. Why they are linearly independent? Again the same argument which we had in the previous lectures. So, if we call them vector v 1 v 2 and this v 3 here this is v 4 and v 5 and v 6 and again the same thing we do that, we take here this linear combination v 1 v 2 with this lambda 6 and v 6 is equal to 0, this is 0 matrix here.

So, this is nothing, but the 0 0 0 0 0 0 when we compute this lambda 1 v 1 lambda 2 v 2 and lambda 6 v 6 we will get matrix here with lambda 1 this is lambda 2 lambda 3 lambda 4 and lambda 5 lambda 6 is equal to 0. And now comparing these entries there we will get lambda 1 is equal to 0 lambda 2 is equal to 0 lambda 3 is 0 lambda 4 is 0 lambda 5 is 0 lambda 6 is 0. So, they are linearly independent vectors so, these are linearly independent vectors. The second we have to check that they span the vector space of this matrices 2 by 3.

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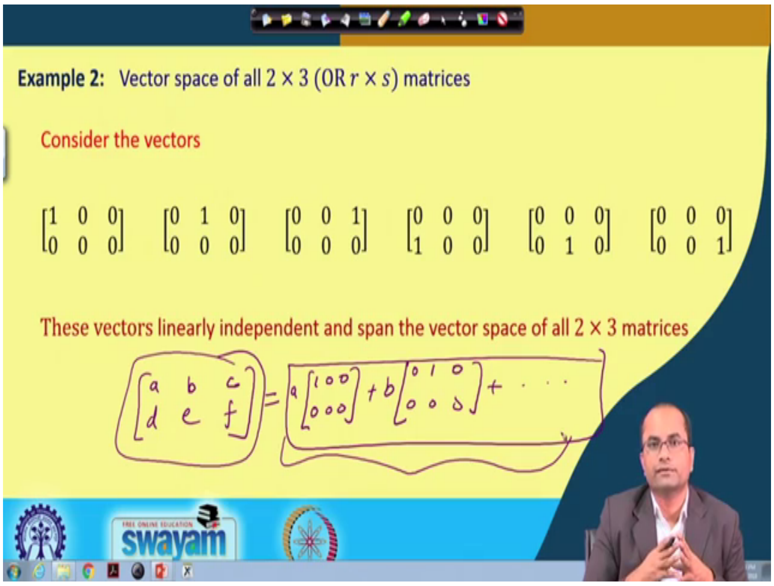

So, naturally they do because if we consider any vector any matrix from this 2 by 3 matrices or the elements we call vectors only. So, anything we take here for example, 2 by 3 so, this is a general matrix from this space 2 by 3 and with the help of this given vectors we can easily expand in terms of like a should be multiplied to the first one now and this b is should be multiplied to the second one and so on. If we continue this so, that is the representation now of this general vector from this vector spaces and that shows that we can represent any vector from this vector space in terms of these given vectors which are 6 in number.

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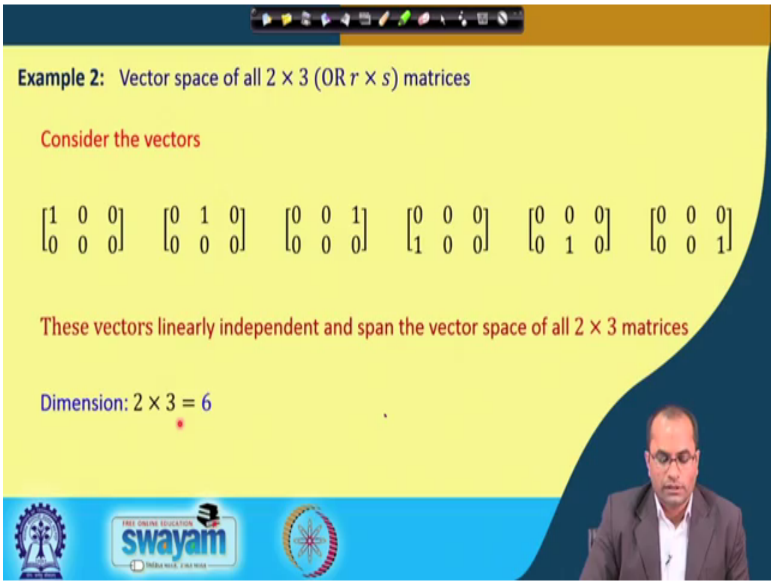

So, what is the conclusion that these vectors form a basis for the this vector space and the dimension is nothing, but the product here 2 by 3 or we have this 6 matrices. So, the dimension is 6 which we can generalize for r by s matrices also the idea is same.

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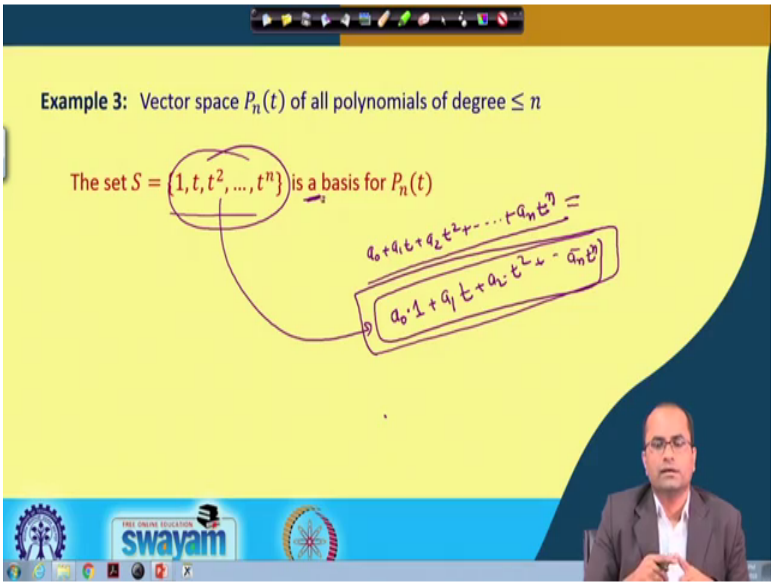

Another example where we are talking about the vector space this P n of all polynomials of degree less than equal to n; again this example we have seen that this form a vector space and now we are talking about this set here 1 t t square and so on t n. So, these n plus 1 polynomials rights these are n plus 1 polynomials, we have taken and we claim that this is a basis actually for P n t.

And why this form a basis? The reason is again clear we have to check that these are linearly independent which we can check as we have done for other examples lambda 1 times 1 lambda 2 times t lambda 3 times t square and so on. This will be set to 0 and this is only possible when all these lambdas are 0. So, naturally these are linearly independent vectors of P n t. The second we have to check that any polynomial from this P n t we take can be represent that polynomial with the help of these vectors.

So, again here if we take any polynomial that say this a 0 a 1 t a 2 t square that is a general polynomial we have in this space P n t they are the polynomials of the degree less than or equal to n. So, for instance we have taken this general polynomial and with the help of these given polynomials we can easily use this. So, a 0 with be multiplied by this 1 and this a 1 with this t here a 2 with t square and so on. So, that is a linear combination of these given vectors. So, we can represent any element of this set or this space here P n t as a linear combination of these given vectors.

So therefore, this set forms a basis for P n t again which we will also come later on to more clarification, the basis are not unique. So, this is a set which form a basis we are writing a basis. There can be other set which can also form the basis like in the example of the first example which we started with we had those 3 vectors which form the basis and they were not the standard basis. So, we can also choose for example, 1 0 0 and 0 1 1 for R 3 and this 0 1.

So, that will also form the basis for R 3 and the example we have seen those 1 1 1 that was 1 vector another one was 1 0 and the third one was 1 0 0. So, that was not the standard basis, but they were also the basis. So, basis are not unique we can have a different sets, the main properties are the elements must be linearly independent that is one property and the second one should be that they should span the whole vector space.

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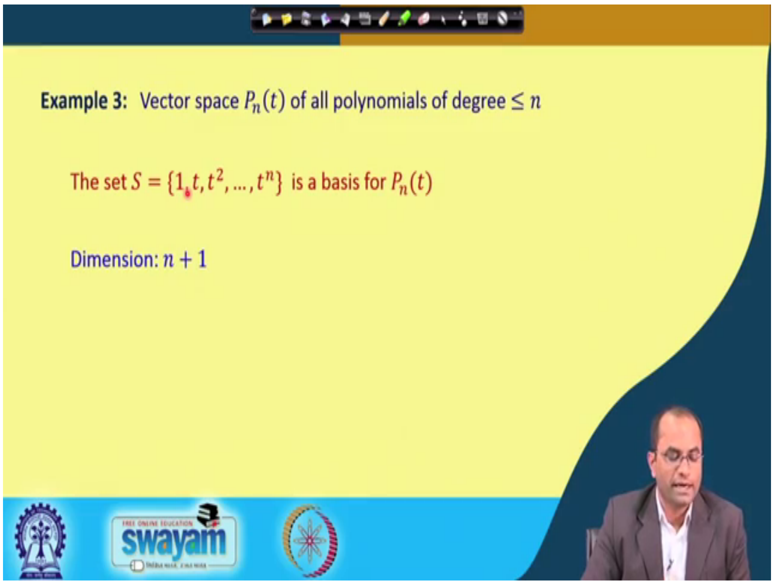

So, here we have seen that these are the basis for this P n t and the dimension here the number of elements in this basis which is n plus 1 at present. So, we have the dimension n plus 1.

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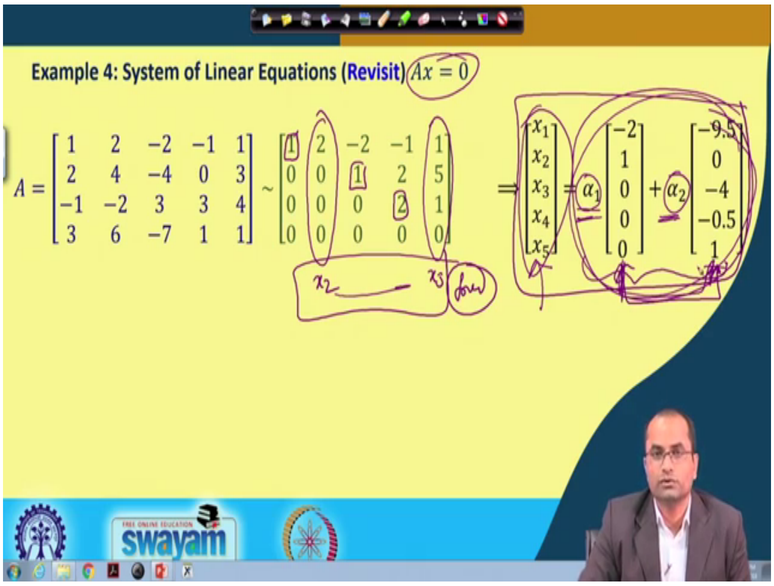

Another example where we will revisit the our problem here A x is equal to 0, the system of homogeneous linear equations and we know that the solutions set of a homogeneous linear system also forms a vector space. So, here the A was this one this example we have already discussed before with this A and we can reduce it to the this echelon form loaded used echelon form and which tells us that these x 1 x 2 x 3 x 4 by choosing because, we have to we have free variables here.

So, this pivot here there is a there is a pivot, but this column does not have a pivot here also it does not have a pivot. So, we have to choose this x 2 and this x 5 as free variables by choosing these two as alpha 1 and alpha 2 we can compute all other x 1 x x 1 x 3 and x 4.

And when we combine we got this equation here that x 1 x 2 x 3 x 4 are nothing, but alpha 1 times this minus 2 1 0 0 0 and alpha 2 times this vector. So, what we observed that with the help of these two vectors we can generate the whole set whole solution set of this A x is equal to 0. Because, we need to just keep on changing these alphas the numbers here and every for any alpha here whatever we choose this is this solution of A x is equal to 0.

So, all possible solution of this a x is equal to 0 we can generate using these two vectors and these two vectors looking at them we can easily identify that these are the linearly independent vectors which we can formally also prove we know with the help of this linear combination set to 0 and that will imply that those coefficients lambda 1 lambda 2 is equal to 0.

So, these two vectors are linearly independent and their linear combination we can generate any vector of the solution set. So; that means, these two vectors form a basis because, that is a property of the basis. We can generate any element of this solution set with the help of these two or as a linear combination of these two 2 vectors and these two vectors are linearly independent. So, they span the solution set and these two vectors are linearly independent and therefore, they form a basis. So, these vectors can the vectors that generate the solutions are these and this and these vectors are linearly independent.

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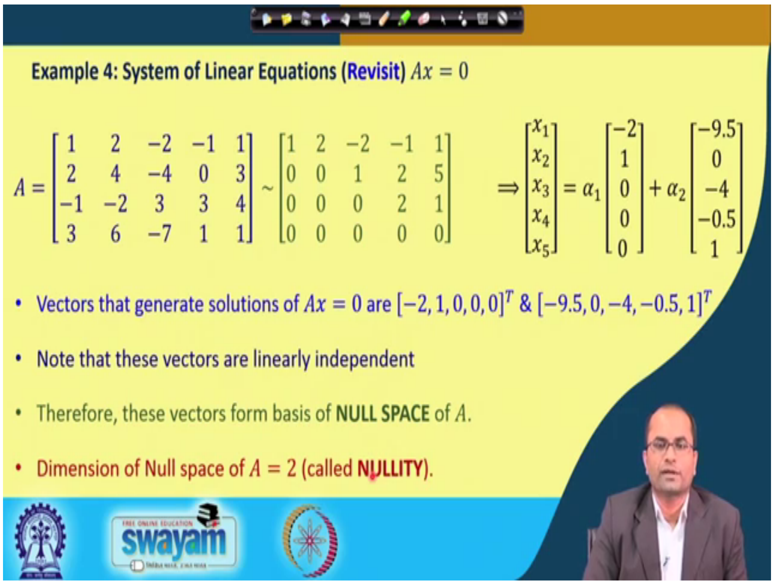

Therefore, these vectors form a basis of the null space here remember so, we call this solution set there was a special name which we call null space. This is a vector space which is called null space. So, these vectors form a basis for the null space of A and the dimension of this null space of A in this particular case its 2 which is again has a name is called nullity. So, nullity is nothing, but its a dimension of the null space of A.

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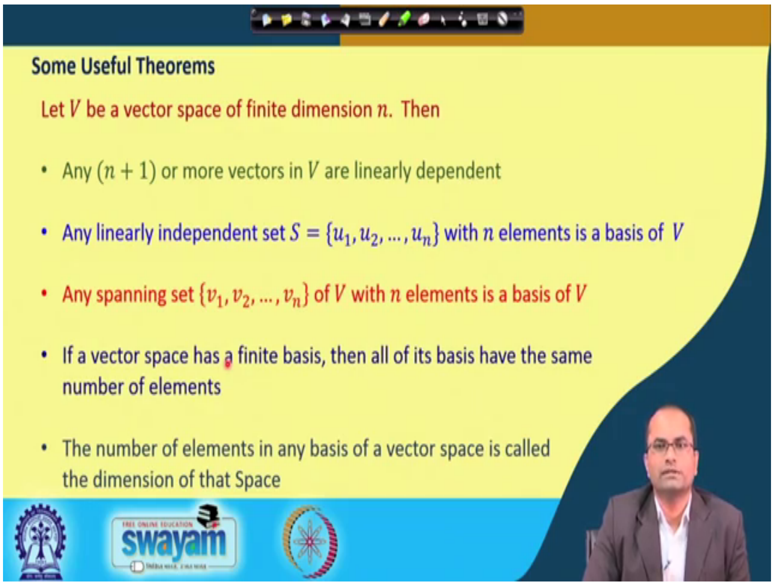

There are some useful results based on the observations what we have regarding these basis and dimension. So, let V be a vector space of this finite dimension n. So, the dimension of the vector space is n, then any n plus 1 or more vectors in V are linearly dependent. So, that is a another nice results we are not going to prove all these results, but one can with the help of some simple examples one can at least get some idea that how these results are true.

So, here any n plus 1 in the in our example also which we started this lecture today we had those 4 vectors, but they were linearly dependent and that was a spanning set ah, but it was a linearly dependent set. Therefore, it was not a basis so, their vectors were 4 while we have seen that taking those 3 vector first 3 vectors that form a linearly independent set and also a spanning set. So, that was the basis so, the vectors in the basis were 3 there for r 3. And so, if there are 4 vectors which is span r 3 they cannot be they cannot be basis because, 4 vectors from R 3 they have to be linearly dependent this is the result here.

So, any n plus 1 or more vectors we take in V they must be linearly they must be linearly dependent. So, for a vector space whose dimension is n we cannot get more than n linearly independent vectors. They will be linearly dependent if we take any n plus 1 or more. Any linearly independent set when n is a basis of V with n element this is a basis. Another beautiful result that we do not have to worry about picking up the elements for the basis we can take any n elements, but that set should be linearly independent.

So, any set which has n linearly independent vectors that will be a basis because for the for the basis these will automatically will be the spanning set. We pick any n vectors which are linearly independent that will be the basis and any spanning set v 1 v 2 v 3 v n with n elements. So, again the same thing spanning set with n elements so, that will be also the basis. So, if a vector space has finite basis then all of its basis have the same number of elements, that is another beautiful result that if we have the n number of elements in the basis.

So, whatever you have different different set of basis, but they will have the same number of elements because that is the n that is a dimension. So, whatever basis we take the dimension is n of the vector space then there has to be n elements in the set. And the number of elements in any basis of a vector space is called the dimension which we have already discussed.

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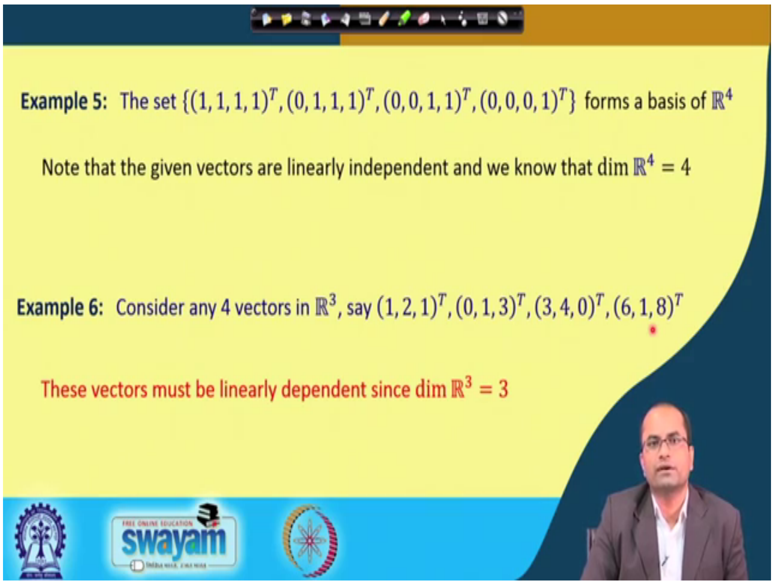

So, for instance we take this example of 4 vectors and they forms the basis of R 4 the set this forms a basis of R 4, while note that the give vectors are linearly independent that we can easily check they are linearly independent whenever we have such a special structure. So, all 1 and then these three 1's two 1's and 1's so, they form a basis which we have seen for instance in the case of this R 3. So, this forms a basis because these are linearly independent and we know that the dimension of R 4 is 4.

So, we do not have to check we have to check that they are linearly independent and they are 4 in number because, the dimension of R 4 also we know. So, if we have 4 linearly any 4 linearly independent vectors from R 4 not only this 4 vectors we can pick any 4 linearly independent vectors that will form a basis. So, here are the given vectors they are linearly independent and then they it has to be a basis because, the dimension is 4 and we have these 4 linearly independent vectors. Consider this 4 vectors in this R 3; so, if we consider these 4 vectors are from R 3 and we know the dimension of R 3 is 3.

So, when the dimension is 3 these vectors must be linearly dependent, they cannot be linearly independent. Because, if they are linearly independent then they can form a basis also, if they are linearly independent and they span the R 3 then they can be also basis, but basis will have only 3 elements not more than 3 elements. So, here we have 4 elements. So, definitely they are linearly dependent which we can conclude from the dimension of R 3 which is 3. So, this cannot have the basis cannot have 4 elements in its set ok.

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So, the conclusion is that we have the dimension which is the maximum number of linearly independent vectors in a vector space V and the basis is a set of linearly independent vectors that span the vector space.

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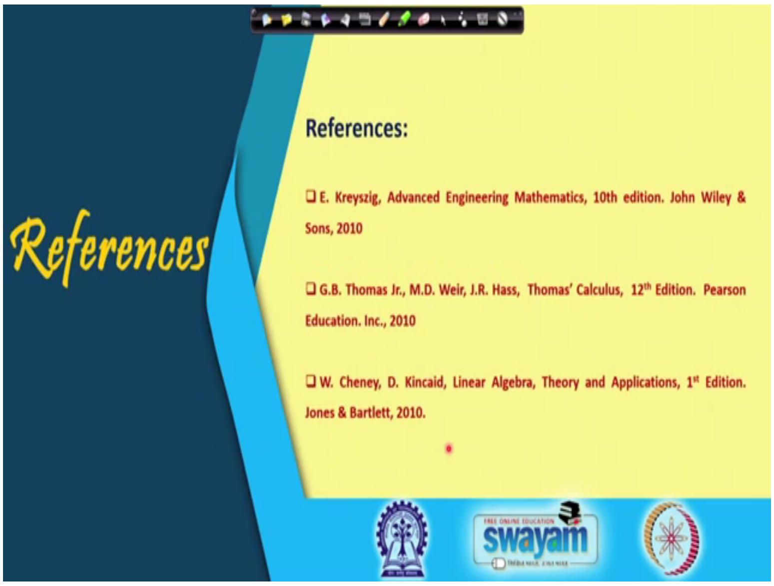

So, these are the reference here used for preparing the lectures.

And thank you very much.