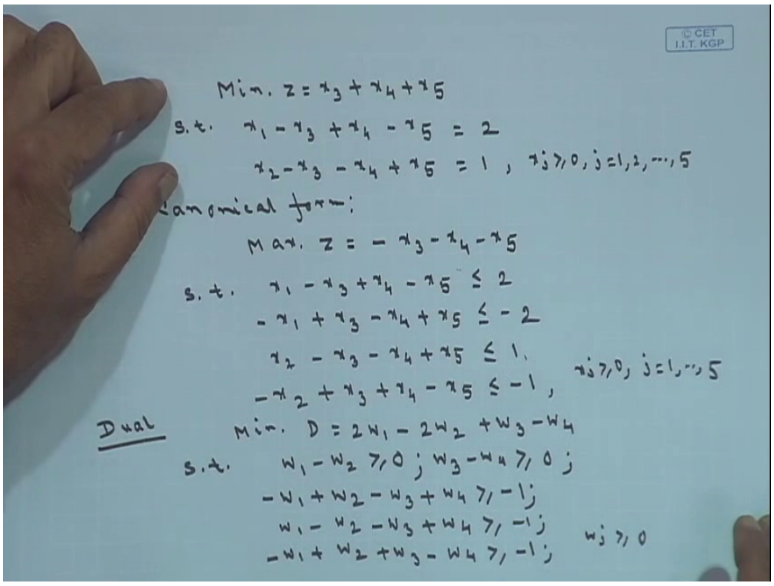
Constrained and Unconstrained Optimization
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 19
Introduction to Duality Theory- II

Now let us continue with the previous lecture, where we started the dual problem of a LPP we have done one example. Let us take another example whenever one primal is given how to first convert into it is dual problem.

(Refer Slide Time: 00:41)



Let us see this problem minimize z equals x 1 plus x 3 sorry x 3 plus x 4 plus x 5 subject to x 1 minus x 3 plus x 4 minus x 5 equals 2, and x 2 minus x 3 minus x 4 plus x 5, this is equals to 1. And of course, xj greater than equals 0, if you see the problem is not in the canonical form, because the problem original problem should be a maximization problem and the constraint all constraints should be of less than equals type.

So, this 2 inequality equality constraints, I have to convert it into inequality one will be greater than equals another will be less than equals then again greater than equals inequality. I have to convert it into the equality type. So, the canonical form of this one dual the canonical form will be and another thing, if you see note here since I have 2 constraints which are of equality type therefore, dual will have 2 variables which will be unrestricted in sign. So, first write down the canonical form of this problem. In canonical form of this problem means I have to make it a maximization problem. So, maximize means it will be of minus since you already know max of z equals min of minus z.

So, minus x 3 minus x 4 minus x 5 subject to this. I will write it as greater than equals left hand side greater than equals 2 left hand side less than equals 2 and then the greater than equals 2, I will make it less than equals 2 by multiplying by negative sign. So, this will be x 1 minus x 3 plus x 4 minus x 5 less than equals 2 minus x 1 plus x 3 minus x 4 plus x 5, which will be less than equals minus 2. Similarly, x 2 minus x 3 minus x 4 plus x 5 less than equals 1 and minus x 2 plus x 3 plus x 4 minus x 5 which is less than equals minus 1 and of course, your xj is greater than equals 0 where j lies between 1 to 5.

So, now your problem is in the canonical form. So, from the canonical form now you can write down it is dual, let me write down it is dual here itself, that is it is dual will have how many variables. In that case it is dual number of variables of the dual depends upon how many constraints are there. Since in the canonical form we have 4 constraints. So, therefore, there will be 4 variables in the dual problem. So, I can write down anything say minimize, d equals what I will do minimize d equals this v values, I have to take because this v value will be the go to the go to the objective function as coefficients of the dual variables. So, it will be 2 w 1 minus 2 w 2 plus w 3 minus w 4, subject to what will happen? Subject 2 first now I have to take the constrains one after another for the variables I will take the coefficients corresponding to the variable x 1 first x 1 is appearing only in 1 and 2 cases.

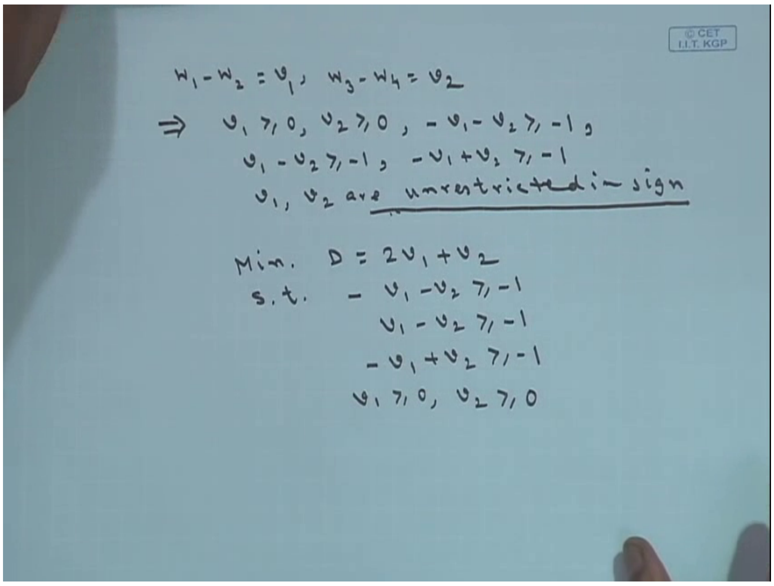
So, it will be w 1 minus w 2 this is greater than equals x 1 is absent here. So, therefore, coefficient is 0. So, w 1 minus w 2 greater than equals 0, this is one next I have to take the constraints with respect to the variable x 2 for x 2 wherever it is coming x 2 is coming on 2 places here and here. So, it corresponds to w 3 and w 4. So, it will be w 3 minus w 4 that is greater than equals again with respect to x 2 the that is no coefficient x 2 is not present in the objective function. So, coefficient is 0 w 3 minus w 4 greater than equals 0 then for x 3, I am having at all places.

So, it will be minus w 1 plus w 2 minus w 3 plus w 4 which is greater than equals the coefficient of x 3 in objective function is minus 1. So, that it will be minus 1 then the next one would be coefficients corresponding to the x 4, that is it will be w 1 minus w 2 like this way I have to write down w 1 minus w 2 minus w 3 plus w 4, this is greater than equals again coefficient of export in the objective function which is minus 1 and the last one is x 5 which is present in all 4 constraints.

So, it will be minus w 1 plus w 2 plus w 3 minus w 4 and the coefficient corresponding to this the x 5 here it is minus 1. So, therefore, this also will be minus 1 and your wj greater than equals 0. So, this is your original problem, now in this one the original problem was this and this you are writing the corresponding dual. So, please note this one the problem which was given to me initially that was a minimization problem and that was of equality type constraints.

So, at first I have written it is canonical form whenever I am writing the canonical form, I am making the objective function as maximization problem. Whereas all the constraints whatever we have taken all will be of less than type in equality only. Then we have written the corresponding dual variable dual problem in dual, there will be a how many there will be how many variables it depends on how many constraints we are having original problem. For this particular case I had the 4 constraints therefore, there will be 4 variables which I am denoting this and I have written the dual.

(Refer Slide Time: 08:06)



In this problem if you see now in this particular problem I can assume that w 1 minus w 2 equals v 1 and w 3 minus w 4 this is equals say v 2, I can assume this thing.

So, from here your wj is are greater than equals 0 and in this if you write down w 1 minus w 2 is greater than equals 0. So, from here if you see first one, is w 1 minus w 2 I am assuming this is equals v 1 w 3 minus w 4. This I am assuming v 2. So, from this constraint I can write down this is v 1. So, this implies your v 1 greater than equals 0 v 2 greater than equals 0, this I am telling from here v 1 greater than equals 0 w 3 minus w 4 is v 2. So, v 2 greater than equals 0. Similarly, the next one will be minus v 1. If you take minus common, then it will be w 3 minus w 4. So, it will be minus v 1 minus v 2 greater than equals minus 1.

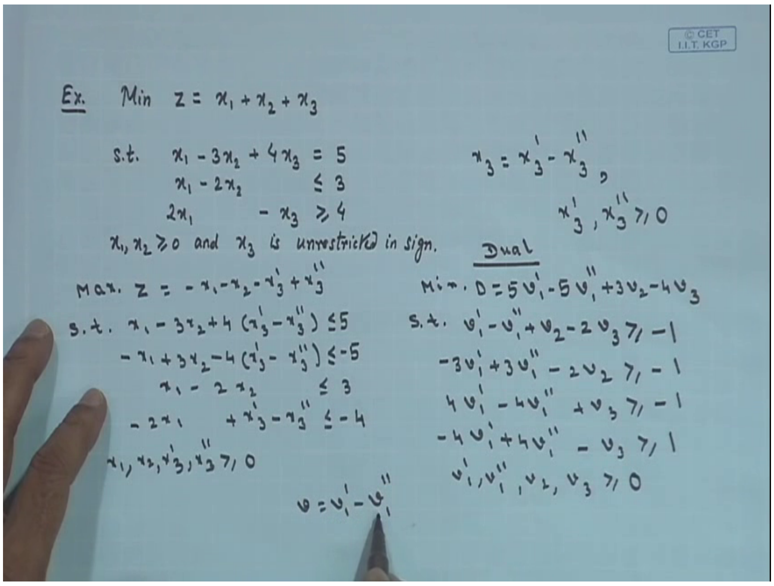
So, next one will be minus v 1 minus v 2 greater than equals minus 1. Similarly, the next one will be next one, I can write down v 1 this will be minus v 2. So, v 1 minus v 2 greater than equals minus 1 v 1 minus v 2 should be greater than equals minus 1. And the last one is you're this one. So, minus v 1 plus v 2 greater than equals minus 1. So, minus v 1 plus v 2 greater than equals minus 1, and whenever I had assumed v 1 as w 1 minus w 2. So, I cannot predict what will be the value of v 1 means v 1 can be positive can be negative can be anything. Similarly, v 2 also I cannot say anything because w 3 w 4 both are greater than equals 0, but from there I cannot assign or I cannot predict what would be the sign of v 2.

So, basically what happens v 1 and v 2 are unrestricted in sign. So, v 1 and v 2 are unrestricted in sign. So, if you remember we have told originally, there was a theorem that if I have 2 equality type of constraints then in the dual problem there will be 2 variables which are unrestricted in sign. So, v 1 v 2 are unrestricted in sign this we can tell, but here this condition, if you see this unrestricted in sign this is redundant what is the reason for this particular problem the reason is v 1 already we have told greater than equals 0 v 2 is also greater than equals0.

So, this will not be there for this particular problem because of these 2 constraints this is redundant. And we will not use it therefore, the minimization problem in terms of v 1 v 2 we can write down minimize d equals from here 2 w 1 minus w 2 was there to 2 v 1 plus v 2 2 v 1 plus v 2 subject to from here, minus v 1 minus v 2 greater than equals minus 1 then v 1 minus v 2 greater than equals minus 1, third one is minus v 1 plus v 2 greater than equals minus 1 and v 1 greater than equals 0 v 2 greater than equals 0.

So, like this way we can convert a problem into it is dual problem we will do one more problem. Then we will go for the solution of the dual problem. So, let us take this problem.

(Refer Slide Time: 12:30)



Minimization z equals x 1 plus x 2 plus x 3. Here the beauty is I have all types that is what is equality type one less than equals type one greater than equals type and 2 variables x 1 x 2 greater than equals 0. And what is x 3 is unrestricted in sign. So, in the dual what will happen since I have one equality type constraint therefore, in the dual there will be one variable which will be unrestricted in sign. Similarly, since in the original problem you have one variable x 3 which is unrestricted in sign therefore, in the dual problem there will be one equality type problem equality type constraint.

Now, I have to write down the this one into first canonical form. Since your variable x 3 is unrestricted in sign, therefore, x 3 can be replaced by can be written by x 3 minus x 3 dash double dash which we have done earlier itself where both x 3 dash and x 3 double dash are greater than equals0. So, that all the variables x 1 x 2 and x 3 dash x 3 double dash are greater than equals 0 variable.

So, this problem first we are writing it into the canonical form that is maximize z equals minus x 1 minus x 2 minus x 3 dash minus x 3 double dash. So, it will be minus x 3 dash plus x 3 double dash subject to x 1 minus 3 x 2 plus 4 into x 3 dash minus x 3 double dash less than equals 5. Then minus x 1 plus 3 x 2 minus 4 into x 3 dash minus x 3 double dash, which is less than equals minus 5. The next one will remain as it is x 1 minus 2 x 2 less than equals 3. And this I have to change the greater than equality type into less than equals type. So, minus 2 x 1 plus x 3 dash minus x 3 double dash which is less than equals minus 4.

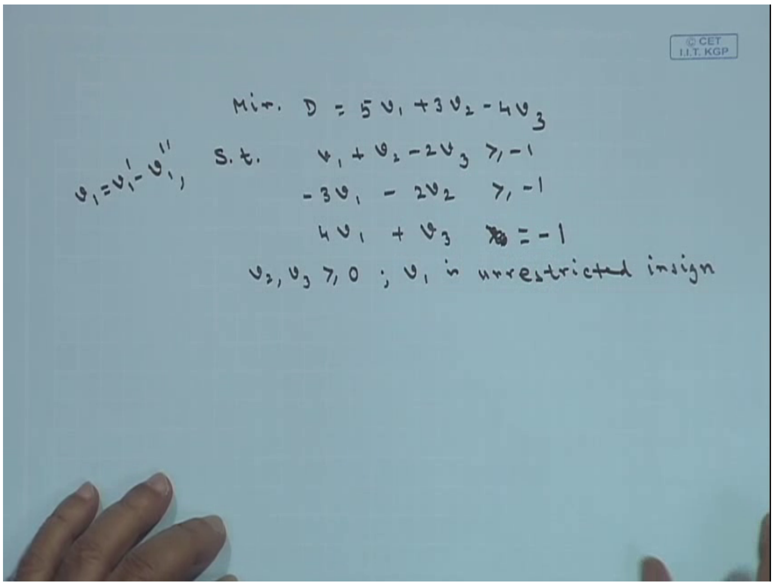
So, minus 2 x 1 plus x 3 dash minus x 3 double dash less than equals minus 4. And of course, x 1 x 2 x 3 dash this one x 1 x 2 x 3 dash x 3 double dash are greater than equals 0. So, this is the problem in the canonical form, now once I have converted into the canonical form I can write down it is I can write down the corresponding dual problem. So, what will be the dual of this. So, in the dual there will be how many variables since I have 4 constraints. So, there will be 4 variables. So, minimize say minimize d equals the first one is 5. So, I am writing 5 v 1 dash say minus 5 v 1 double dash then plus 3 v 2 minus 4 v 3.

So, I have 4 variables v 1 dash v 1 double dash v 2 and v 3. So, corresponding to the constraints I am writing 5 v 1 dash minus 5 v 1 double dash plus 3 v 2 minus 4 v 3 subject to what subject to I have to take the coefficients corresponding to x 1 first. So, I will start with this one subject to v 1 dash, minus v 1 double dash then here it will be v 2 v 2 is coefficient is one that is v 2. Then for the last one minus 2 is the coefficient. So, minus 2 v 3 will be there.

So, basically first constraints correspond to v 1 dash variable second constraints correspond to the variable v 2 dash, similarly third constraint corresponds to v 3 v 2 and fourth one v 3 v 3. So, this is greater than equals minus 2 v 3 corresponding to the problem x 1 coefficient in the objective function is minus 1. So, this is greater than equals minus, 1 similarly we have to do it for x 2 x 3 and others. So, what you will get minus 3. I am now writing directly minus 3 v 1 dash plus 3 v 1 double dash minus twice v 2 because x 2 is appearing here, v 3 is will not come this is greater than equals minus 1 next one is 4 v 1 dash minus 4 v 1 double dash plus v 3, which is greater than equals corresponding to x 3 dash it is minus 1. And the last one will be minus 4 v 1 dash plus 4 v 1 double dash minus v 3 which is greater than equals coefficient of x 3 double dash is 1. So, it is one and where v 1 dash v 1 double dash v 2 and v 3 all are greater than equals 0.

So, this is the problem now if I assume that v equals v 1 dash minus v 1 double dash say, let us assume that v equals 2 v 1 dash minus v 1 double dash, then this particular dual problem I can write like minimize.

(Refer Slide Time: 18:44)



D equals 5 v 1 plus 3 v 2 minus 4 v 3, just I am replacing on this dual problem I am replacing v 1 dash minus v 1 double dash by v by v 1 not v, but by v one. So, here I will obtain 5 v 1 3 v 2 minus 4 v 3 subject to what subject to v 1 plus v 2 minus 2 v 3 greater than equals minus 1 minus 3 v 1 minus 2 v 2 greater than equals minus 1 4 v 1 plus v 3 greater than equals sorry 4 v 1 minus v 3. If you now see this one this is 2 are greater than equals type.

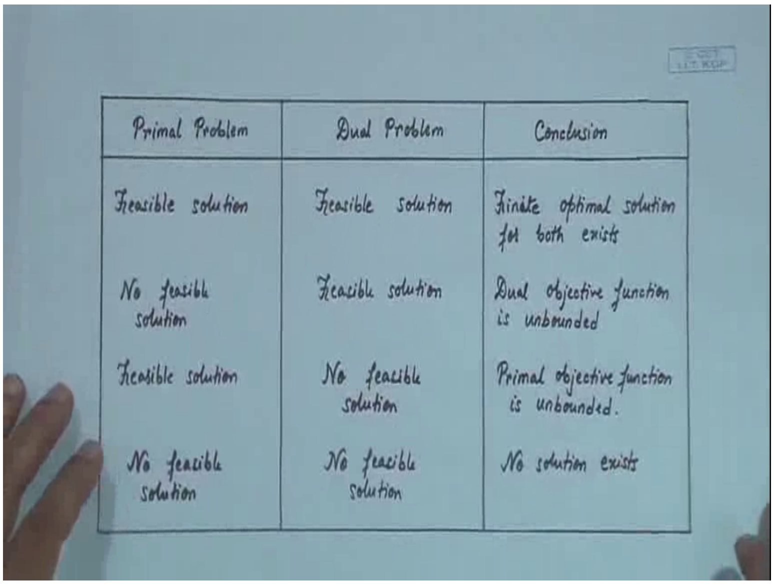
So, I can make one as less than equals. So, then both will be same and if I change the value then these 2 can be written as an equality type constraint from here, it is very easily we can do it this is greater than equals is there if I multiply minus on both side of the second equation. Then this equation will be converted into less than equals it will be 4 v 1 dash minus 4 v 1 double dash plus v 3 which is less than equals minus 1. And the first one is for v 1 dash minus 4 v 1 double dash plus v 3 greater than equals minus 1. So, this less than equals type and greater than equals type I convert it. And I can write one equality type constraint. So, this will be 4 v 1 plus v 3 which is equals minus 1 because ultimately this converted to minus 1.

So, you seen in this case your v 2 and v 3 are greater than equals 0, what about v 1 we have told what is v 1 v one is equals to v 1 dash minus v 1 double dash where v 1 and v 1 dash v 1 double dash both are greater than equals 0, but we cannot talk anything about v 1 because v 1 can be positive can be negative or in other sense v 1 is unrestricted in sign v 1 is unrestricted in sign. So, if you remember in the theorem already we have told if we have one variable one equality type of constraint then in the dual one of the variable is the one of the variable will be unrestricted in sign.

Since we have only one equality type constraint. So, one variable is your unrestricted correspondingly this one. Similarly, if you have one variable is unrestricted in sign in the original problem, then in the dual problem the constraint there will be one equality type of constraint and which is true from here if you see we have one equality type of constraint.

So, I hope now you have understood how to solve how to convert a original or primal LPP into the corresponding dual problem.

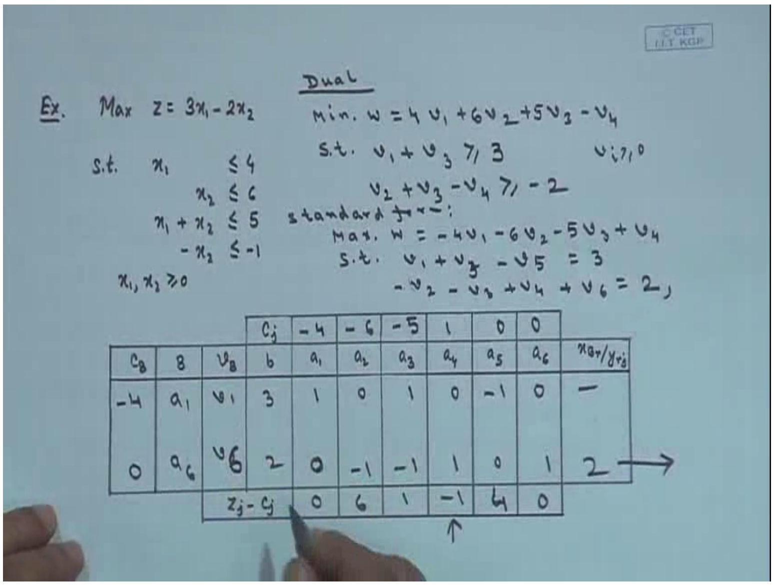
(Refer Slide Time: 22:25)



Now let us see the other one. I have a primal problem, I have a dual problem and some conclusion if a primal problem has a feasible solution dual problem has the feasible solution then finite optimal solution for both will exist. So, if the primal problem has a feasible solution dual as a feasible solution, then finite optimal solution for both will exist if primal has no feasible solution whereas, dual has feasible solution in that case dual objective function will be unbounded please note this one. If primal has no feasible solution dual has feasible solution, then dual objective function is unbounded. Similarly, if primal has feasible solution dual has no feasible solution in that case primal objective function is unbounded. And if both the primal has dual both the primal and dual has no feasible solution then the conclusion will be for the problem no solution exists.

now let us see how to solve a problem.

(Refer Slide Time: 23:37)



How to find out the solution of a dual problem will come to this one later. So, this is your original problem. It is dual let us write down what is the dual of this problem, first dual of the problem will be minimize there are 1 2 3 4 constraints. So, 4 variables will be there. So, w equals 4 I have to consider the coefficients b. So, 4 v 1 plus 6 v 2 plus 5 v 3 minus v 4 minus v 4 subject to v 1 plus v 3, v 1 plus v 3 this is greater than equals coefficient of x 1 in the objective function that is 3 then v 2 plus v 3 minus v 4 v 2 plus v 3 minus v 4 this is greater than equals coefficient of x 2 in the objective function that is minus 2 and; obviously, this is your problem will be this one.

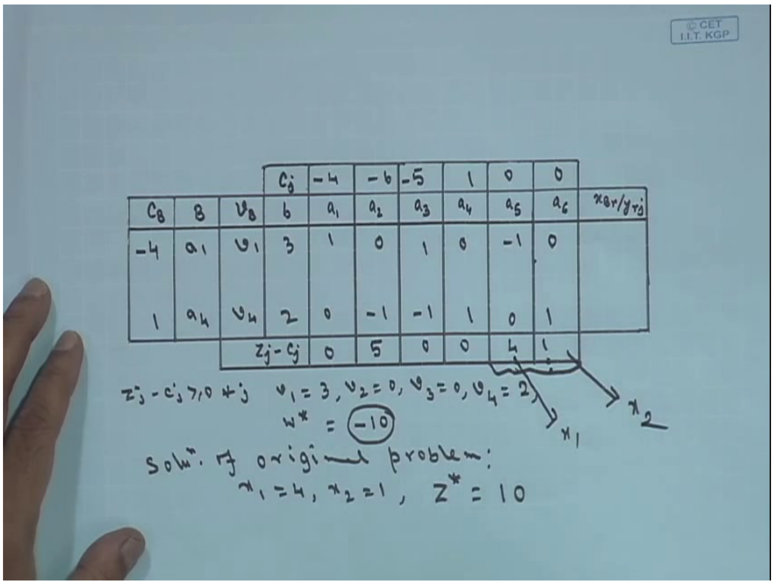
So, in standard form of course, your vi greater than equals 0 for i equals 1 to 4. Now since it is greater than equals, I can write it in the standard form because we have solved if you remember maximization problem only and constraints are less than equals type. So, directly I am writing these into the in standard form. So, this is the dual of the original problem the standard form. I am writing of this dual problem maximize say w equals I am making a maximization problem. So, minus 4 v 1 minus 6 v 2 minus 5 v 3 plus v 4 subject to using once are making less than equals, I have to add the slack variables.

So, v 1 plus v 2 sorry v 1 plus v 3 minus v 5 this is equals 3 and minus v 1 minus v 2 minus v 3 minus v 3 plus v 4 and the slack variable plus v 6. So, this is equals 2 this is equals 2 v j greater than equals 0. So, here your one is greater than equals type that is either is in r here added, I will come to this later I have just subtracted the surplus variable I have not added any artificial variable over here because for this case v 1 and v 6 will form the basis if I make other side 0, then v 1 will be 3 and v 6 will be 2.

So, therefore, the basis v 1 and v 6 will come. So, in the which basis v 1 and v 6 will be coming here it is a one and a 6 the coefficient values are minus 4, minus 6 minus 5 a 4 it is one other 2 variables are not there. So, it will be 0 v 5 and v 6 it will be 0. So, your cb will be minus 4 and it is 0 v values are 3 and 2. Now write the rows which is similar as you have done earlier 1 0 minus 1 0 and this is 3 2 is there, this is 0 minus 1 minus 1 1 0 1 zj minus cj, if you calculate 0 6 1 this one will be minus 1. Next 2 or next one will be 4 and this is 0.

So, this is the most negative entering vector is this one entering vector will be this thing. So, 3 by0. So, you will not get anything here you will get 2. Therefore, your departing vector will be a 6. So, your entering vector is a 4 and our departing vector is a 6. So, from here we are making the next table. So, in the next table you will have the v 1 and v 4 v 1 and v 2, will be a 6 will be coming over here. So, I am sorry this I am made a mistake this will be v 6 only not v 2, v 6 corresponding to this v 6 is going out. So, impress.

(Refer Slide Time: 28:44)



Of v 6 v 4 will come. So, it will be a one a 4 coefficients will remain same minus. 4 minus 6 minus 5 and 1 0 0. Here this is your pivot element. So, basically it will remain same this is one and this is 0 already it is there. So, as such there will be no change only zj minus cj value will be changing this is minus 4 this is one.

So, it will remain same 3 one 0 one 0 minus 1 0 this will be 2 0 minus 1 minus 1 1 0 and 1. If you calculate the zj minus cj value, now it is 0 5 0 0 4 1. So, if you see zj minus cj is greater than equals 0 for all j. So, you have the optimal solution what is the optimal solution v 1 is present here. So, v 1 equals 2 sorry v 1 equals 3 v 2 is not present. So, v 2 0 v 3 is 0, and v 4 again it is 2 and what will be your w star your w star will become this 1 minus 12 and this one. So, w star will be minus 10, basically and I have to take the minimization. So, in unit price it will be equals to 10, if I want to find out for the what is the solution of the primal problem then you take the last 2 columns here because these 2 will correspond to the first one will correspond to the variable x 1 and second one will correspond to the variable x 2.

So, here there was only 2 variables therefore, this first one will correspond to this first variable and second one will correspond to the second variable. So, solution of original problem. Also I can obtain from this last table that is you take the last columns depending upon the number of variables. So, here it will be x 1 equals 4 and x 2 equals 1 and z star since it is minimization of this. So, the value will be this. So, please note that in the dual original problem in the solution of the original problem, this will correspond to x 1 and this will correspond to x 2. So, this is by this way we can find out the solution of the problem. So, in the next class we will actually start the dual simplex method because by the method which we have solved it is almost similar to your normal simplex method, but in dual simplex we have certain added advantage. So, that we will discuss in the next class.