Linear Algebra
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Lecture – 19

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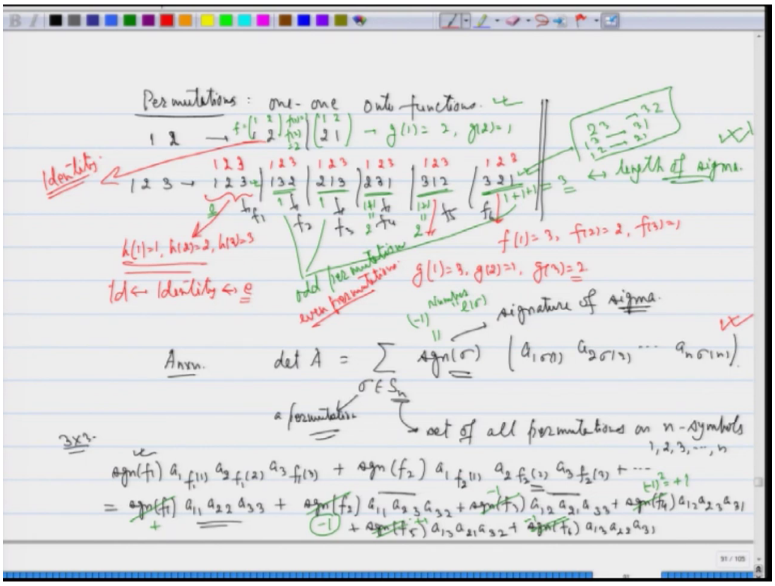

Now, let us look at the definition from another point of view, what is called the permutations. So, let me look at, what are permutations? So, for us permutations are nothing but, one-one onto functions. So, for example, if I am looking at just two objects 1 and 2, its permutations are 1 2 itself and 2 1. If I have got three objects 1 2 3 their permutations are 1 2 3, 1 3 2, 2 1 3, 2 3 1, 3 1 2, 3 2 1, alright.

So, basically even though I have got six permutations, the idea is that they come from functions one-one onto functions. So, what is the function? This function is 1 is going to 1 and 2 is going to 2. So, 1 is going to 1, 2 is going to 2 here. So, I have a function f, which is this which is f of 1 is 1, f of 2 is 2. This corresponds to the function 1 is going to 2 and 2 is going to 1. So, this is another function say g. So, that g of 1 is 2 and g of 2 is 1, fine.

Similarly, I can have here 1 1 1 sorry 1 2 3 I just leave it as it is, 1 2 3, 1 2 3, 1 2 3. So, this corresponds to the function f. So, that f of 1 is 3 f of 2 is 2 and f of 3 is 1. This corresponds to another function g, such that g of 1 is 3 g of 2 is 1 and g of 3 is 2, alright.

And, we can talk of fine composition of functions. So, I can get one permutation from another permutations and so on. This permutation has a special name what is called in this example it was this one, which is called the identity. So, this is the identity function f of 1 is going to 1, f of 2 is going to 2.

Here also this function h of 1 is 1, h of 2 is 2 and h of 3 is 3. So, these permutations we write it as generally Id as such, so identity so therefore, Id. Many books they use the word e to talk of the word identity, fine. So, I have to be careful about this. So, with this idea the definition of permutations is as follows of determinant is as follows.

So, a is n cross n, then determinant of A is look at all sigma belonging to S n. What is sigma? Sigma is a permutation. What is S n? Set of all permutations on n symbols 1 2 3 till n and then there is a notion of what is called signature of sigma, this has the word signature of sigma or permutation that I am looking at, permutation times a 1 sigma 1, a 2 sigma 2 so on till a n sigma n, fine.

So, I have this expression this looks complicated, but let us write it in terms of 3 by 3 matrix. So, in in terms of 3 by 3, alright. I will have there is one permutations let me write it as a f 1 this as f 2, f 3, f 4, f 5, f 6 then I have got signature of f 1 into a 1 f 1 of 1 a 2 f 1 of 2 a 3 f 1 of 3 plus signature of f 2 into a 1 f 2 of 1 a 2 f 2 of 2 a 3 f 2 of 3 plus so on.

So, we would like to compute what is this nature of f 1. So, let me just leave it as it is for the time being. Now f 1 of 1 is 1. So, it is a 11, f 1 of 2 is 2 so it is a 22, similarly a 33 plus signature of f 2 a 1 f 2 of 1 f 2 of 1 if I look at f 2 of 11 is going to 1, so it is 1 itself a 2, what is f 2 of 2, f 2 of 2 is 3 and what is f 2 of sorry f 2 of 3, f 2 of 3 is 2 so, a 32, fine.

Similarly, it will be signature of f 3 into a 1 let us look at this 1 is going to 2 a of 21 a of 33 plus signature of f 4 into a 12 a 23 a 31 plus signature of 4 I have written f 5 a 13 a 21 and a 32 plus signature of f 6 a 13 a 22 a 31.

We have the terms as such, all the terms which are supposed to be there, there are six terms, all of them they are there only the sign is a problem plus or minus sign and to get the plus and minus sign, we compute certain things here alright.

So, let us try to compute these numbers. So, let me use a green thing here. So, let us count here, how many elements here in the denominator are not in their proper orders. Proper order means increasing order. So, here 1 2 3 is in everything is fine, that 1 is less than 2 is less than 3. So, there is proper order there is no change. So, I have got 0 here, no change.

If I look at here, 1 is smaller than 3, 1 is also smaller than 2. So, there is no problem with 1. If I look at 3 2 the order is changed, it should be 2 comma 3, 2 is smaller than 3, but it is 3 comma 2. So, there is a one issue that I have here, fine.

Let us look at here 2 comma 3 2 1 3. So, 2 1 is in the wrong position in the sense that it should have been 1 2. So, I have got 1 there, then 2 3 is fine no problem, 1 3 is also fine there is no problem.

Let us look at here, order of 2 and 3 is fine, such 2 is indeed a smaller than 3 there is no problem as such. If I look at 2 1 this is something which is wrong, it should have been 1 2, but it is 2 1. So, there is a sign change, there is a problem there.

Similarly, there is a 3 1 here again there is a problem. So, I have 1 plus 1 which is 2 fine, let us look at here 3 is supposed to be greater than 1, but is in opposite direction. So, I have 1 here because of 3 1, 3 and 2 there is also an issue with me 3 and 2 fine, it should have been 2 and 3. So, again I have an issue with this. So, I get 2 here.

Let us look at here 3 is bigger than 2, but it is coming in the opposite direction it should be 2 comma 3, so I have 1 here. Then, 3 1 is also in the wrong direction, so again I have 1 here and I have 2 1 which also has a wrong direction. So, I have got 3 here. So, I have got these numbers new numbers which are 0 1 1 2 2 and 3.

So, signature is nothing but, look at minus 1 to the power that number alright, the number and what is that number? We generally write it as l of sigma, what is called length of sigma or the number of changes number of interchanges that you are doing alright, number of wrong positions in some sense, alright.

So, the right position is that 1 has to be before 2, 2 has to before 3 and so on, when I am looking at the rows. But, here it turns out that they get changed. For example, 3 2 1 it should be 2 3, but it is; it should be 2 3, but it is 3 2 here.

Similarly, it should be 1 3, it is 3 1 here, similarly it should be 1 2, but it is 2 1 here alright. So, there are three interchanges.

So, when I look at this. So, if f 1, f 1 comes with a 0 interchange. So, minus 1 to the power 0 which is 1. So, this will be replaced with the plus sign. If I look at f 2 there is a 1 here. So, minus 1 to the power 1 will give me a negative sign.

So, this will get multiplied by minus 1, f 3 will also similarly get multiplied by minus 1 replaced by minus 1, f 4 there are two of them. So, this will get replaced by minus 1 to the power 2 which is same as positive 1, f 5 also will get replaced by a plus sign and this will get replaced by a minus sign, is that ok?

So, this is what I want you to understand, fine. So, the sign changes are coming from you right the permutations and then look at what are called the length of the sigma and then compute it. They are also called what are called odd permutations and even permutations.

So, odd means, look at this 1 1 and 1 here odd things. So, their number of interchanges is 1 number of interchange is 1, number of interchange is three; these are odd numbers. So, these are called odd permutations and the others are called even permutations, because the number of lengths are 0 2 and 2, so they are called even permutations. So, you have notion of what is called odd permutation and even permutation.

So, those who are interested they can read it separately, but what I am trying to say is that, if you want to prove most of the results in determinants it is this definition which is helpful, the other definition the one that we gave initially which is called the expansion of the determinant expansion on the determinant using the first row alright.

Is not the actual definition, it gives us easy way to compute things fine, but that is not the right one; fine, because you cannot prove most of the results using that idea, but we are not going to prove it. So, we will just state the important results here and then proceed, fine.

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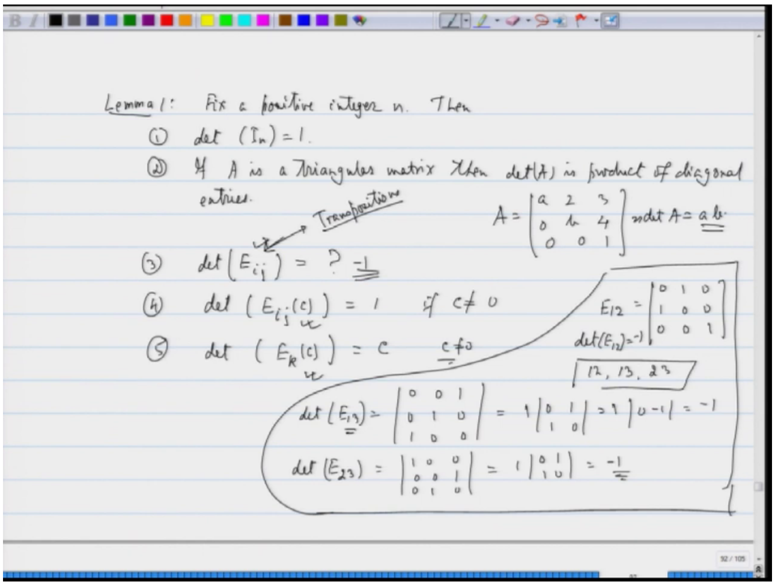

So, some lemma results which are important for you to know which will help us in computing things. So, lemma 1 fix a positive integer n. Then 1, determinant of identity is 1 this you can compute directly no problem, because you are looking at a unit cube in n dimension, fine.

If A is a triangular matrix, then determinant of A is product of diagonal entries, fine. So, what we are saying is that, if A looks like a 2 3 0 b 4 0 0 1; imply determinant of A is a times b, fine. This is for upper triangular, similarly you can look at the lower triangular part and do it yourself.

3, now let us look at the, what are called elementary matrices. So, determinant of E ij alright is something, so I would like you to see that, what should be the determinant of this? Forth, determinant of E ij of c is 1, if c is not equal to 0. 5, determinant of E k of c is c, again c not equal to 0; because our these transformations were coming from there, fine.

So, E ij would like you to understand here. So, that things are meaningful. So, let us look at A 1 2 A 1 sorry E 1 2, E 1 2 was interchanging the first and second 0 1 0, 1 0 0, 0 0 1. So, the determinant of this will be this times this which is 0, so I do not have to worry about, but it is this time this; so, it is minus 1.

So, determinant of E 1 2 is minus 1 what about determinant of. So, there are things that I need to do 1 2, 1 3, 2 3 these are the columns or these are the rows that needs to be interchanged. So, E 1 2 is fine, I have computed.

What about determinant of E 1 3? So, it is same as E 1 3 interchanging the first and third. So, 0 0 1 this row remains the same, this row is this. So, determinant of this is nothing, but, 1 times 0 1 1 0, which is 1 times 0 minus 1 which is minus 1.

What is determinant of E 2 3? This is the first row remains as it is, second and third are being interchanged. So, this is nothing but, 1 times 0 1 1 0 which is same as minus 1, fine. Is that ok?

So, the idea is that all these interchanges here I am computing it for 3 by 3, in general it is a bit difficult to compute here in this way, to easy way to understand them is again using permutations, where we will see that each of these interchanges are what are called transposition.

So, E ij's are called transpositions or permutations of length 2 what they are, alright because you are interchanging only i and j and the rest of the elements are fixed and therefore, they give you minus 1 as the determinant, fine.

This is very important, computing it the way we are using it here, fine, is not the right way to do it as I said earlier also it is coming from permutations.

So, for us we just remember that these are the things that I have as per determinant is concerned. Now, using these things I would like to write the actual properties of determinant, alright. So, let me write those data properties.

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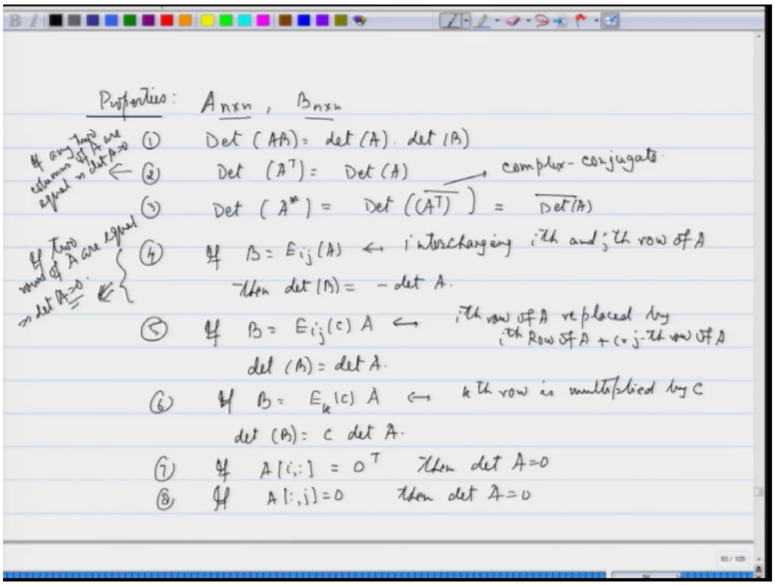

So, what are the properties? I cannot prove them at this stage. So, I am just going to state them properties 1.

So, A is again n cross n for me, A is n cross n, 1 and B is also n cross n, then determinant of A times B is determinant of A into determinant of B you can first prove them for elementary matrices and then proceed, but I will write them also for us. 2, determinant of A transpose is same as determinant of A. 3, determinant of A star is nothing but, determinant of A transpose and then looking at the bar.

So, this bar comes which is the complex conjugate. And therefore, this complex conjugate will always be there with us and determinant of A transpose is same as determinant of A should be determinant of A itself alright, or you just look at the expression in the expression you just have to put a bar and therefore, that bar will remain the same, fine.

So, I will like you to remember them. I would also like you to remember these things which are important they are the one, which relates with the elementary matrices. So, if B is equal to E ij of A alright, fine. So, this is interchanging ith and jth row of A, fine. Then, determinant of B is minus determinant of A.

5, if B is equal to E ij c of A. So, here this corresponds to ith row of A replaced by ith row of A plus c times jth row of A. So, then determinant of B does not change, determinant of B is same as determinant of A. 6, if B is equal to E k c of A, so, kth row is multiplied by c, then determinant of B is equal to c times determinant of A.

7, if any row alright of A is 0 or this is 0 transpose, then determinant of A is 0. Similarly, if any column is 0; if any column is 0 then determinant of A is 0. Now, this idea, where is that interchange here this one, this can be used to say that this will imply that, if two rows of A are equal implies determinant of A will be 0.

This, with this part will imply that if any two columns. So, if any two columns of A are equal implies determinant of A will be 0.

So, you can use these ideas to build up your own concepts and your own important results. I am not spending much time because they are not very important for us, determinant will not be computing so much in general, we will be computing only when we come to eigenvalues eigenvectors, but in general this is important to give us some ideas of how do we proceed with things, fine. So, you should know them.

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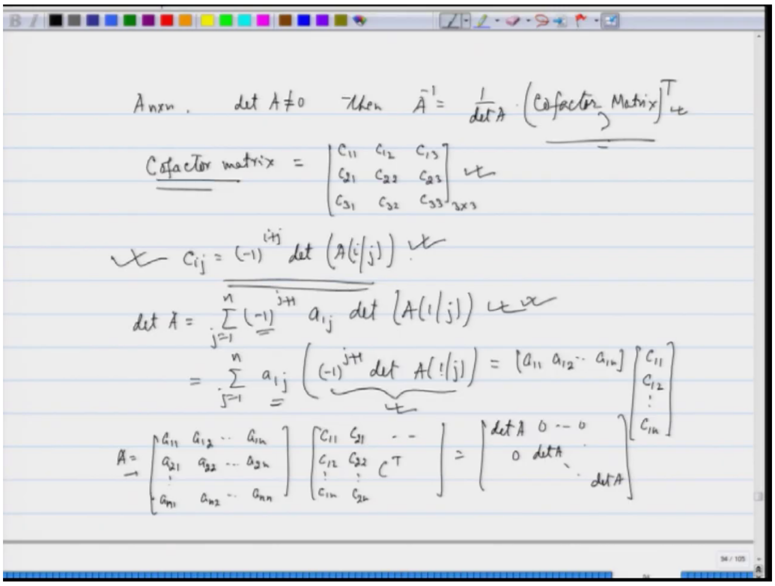

The next now, what is called trying to find the inverse of A. So, idea A is n cross n and determinant of A is not 0, then there is a notion of then A inverse is 1 over determinant of A into cofactor matrix transpose alright. So, what is this we will try to understand.

So, we are trying to find A inverse using determinant. So, if you remember what we had said was in the very beginning of the class today no in the in the previous class that determinant is not 0 if and only if A is invertible alright. So, we are just relating it that is the idea, fine.

So, let us look at what is the cofactor matrix, determinant we have already computed. So, cofactor matrix c cofactor matrix shows cofactor matrix alright. So, what we do is this is supposed to be a matrix of the same size as that of n.

So, for 3 cross 3 I am going to write it as it is, so it will be C 11 C 12 C 13 C 21 C 22 C 23 C 31 C 32 C 33, fine. And, important here it is cofactor matrix transpose, is that ok?

Now, what is C ij? C ij is nothing but, look at minus 1 to the power i plus j and look at determinant of from A remove the ith row and jth column, fine. This is what it is, C ij is this, fine. So, if I look at.

So, our definition of determinant was determinant of A was summation j equal to 1 to n minus 1 to the power j plus 1 times a 1 j into determinant of 1 and j this is what the idea was, fine.

So, this is same as j equal to 1 to n, a 1 j into let me put this here. So, minus 1 to the power j plus 1 determinant of A of 1 j, fine. So, if I look at this this product is nothing but, I am trying to multiply look at here a 11 a 12 a 1n and I am multiplying it with, what are these entries C 1 j, is not it? Look at here C ij, definition of C ij is ith row removal and jth row, jth column removal.

So, I am removing the first row and jth. So, I am trying to look at, C 11 C 12 C 1n; so, I am looking at this, is that ok? This is important for you, the determinant of A by definition is this is what we had used.

I am rewriting it this way the second way and then I look at the definition of C ij, C ij is determinant of something times minus 1 to the power i plus j, minus 1 to the power j plus 1 is already there. So, j is changing, but the row is remain same, which is 1. So, it has to be C 11 C 12 C 1n fine. So, I am multiplying this, is that ok?

So, this was the expansion along the first row alright, we have expansion along the second row and so on also and therefore, what happens is that we can check that, if I have the matrix A here which is a 11 a 12 a 1n, a 21 a 22 a 2n, a n1 a n2 a nn fine this is my A, I have my C as whatever is given look at C transpose.

So, let us look at C transpose, C transpose has entry C 11 C 12 till C 1n, then you will have C 21 C 22 C 2n and so on for me. Then, what will happen I am multiplying this row with this alright that will give me determinant of A, this with this will give me 0 why it will give me 0 just look at it.

Similarly, this second row into this entry will give me determinant of A it will give me 0. So, I will get determinant of A here and rest everywhere it will be 0, fine. So, let us do it for a 3 by 3 matrix. So, I have; so, I will just take an example to do it for you so that it is clear.

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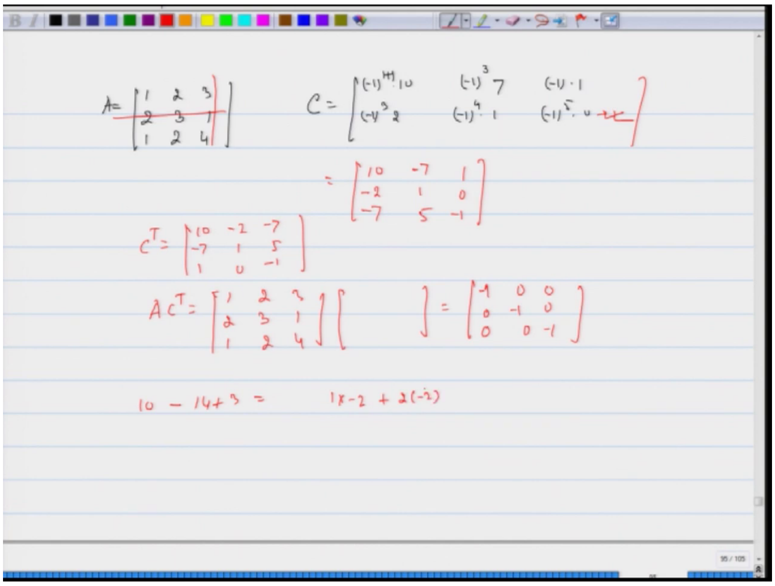

So, I have the matrix A as 1 2 3 2 3 1 1 2 4 fine. Let us compute the matrix C, the matrix C is 3 into 4 is 12, 12 minus 2 is 10. So, it is minus 1 to the power 1 plus 1 times 10, then it is minus 1 to the power 2 plus 1 which is 3 times 2.

Please have a look at it, this and this so 4, now 8 minus 1 is 7 alright, I think I should have written C 2 and I have written here alright, adjoint. So, this should be 7 here. Then, if I look at this it is 4 minus 3 is 1 so, minus 1 into 1, fine.

Similarly, have a look at it minus 1 to the power 3 times 2, minus 1 to the power 4 times 1, minus 1 to the power 5 times 0. So, let us look at this part. So, the entry 2 3 2 and 3 2 minus 2 is 0, fine and so on.

So, this matrix is supposed to be equal to 10 minus 2 minus 7 minus 2 1 0, what am I writing minus 7 1, just a minute, this copying down is always a problem 10 minus 2 7 minus 7 1 5 and 1 0 minus 1, fine.

So, C transpose will be 10 minus 7 1 minus 2 1 0 minus 7 5 minus 1. So, let us look at a times C transpose is 1 2 3, 2 3 1, 1 2 4 times this, which will be equal to I would like you to check it 1 0 0 0 1 0 0 0 1 alright. This is what the determinant was, determinant was 1, fine. So, it is minus 1 minus 1 it means alright.

Let us multiply the second with this 1 2 3. So, 1 into minus 2 plus 2 into minus 2 minus 2 1 0 so, this into this is minus 2 plus 2 is 0.

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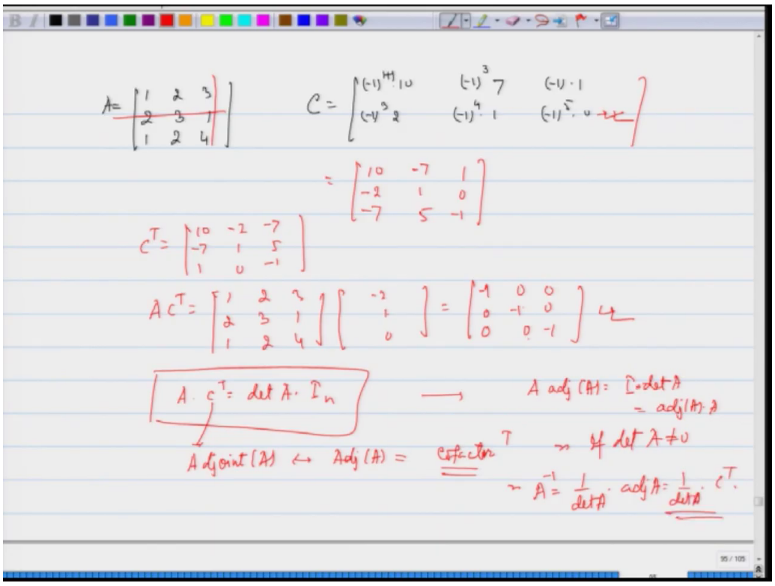

Let us look at this into this itself, again this will be 0, this way this if you look at 2 into minus 2 plus 3 which minus 1 which you what I have. This into this will give you minus 2 plus 2 which is 0, fine. So, you can see that you can compute yourself that this is this.

So, what we are saying is that it turns out that, A into C transpose is determinant of A into identity, this is what it is. So, generally this C transpose is called adjoint of A alright and in short we write it as Adj A of A which is same as cofactor transpose, is that ok?

So, from here what we get is, you can check that a times adjoint of A is equal to identity times determinant of A which is same as adjoint of A into A and therefore, if determinant of A is not equal to 0 implies, A inverse is equal to 1 over determinant of A into adjoint of A which is 1 over determinant of A into C transpose, fine. So, that is all.

Thank you.