SCIENTIFIC COMPUTING USING PYTHON

Professor Vivek Kumar Aggarwal & Professor Mani Mehra

Department of Mathematics

Indian Institute of Technology, Delhi

Lecture No. 34

Welcome, everybody, to Scientific Computing Using Python. So as in the previous lecture, we discussed other methods for solving initial value problems. So today we will continue further from that. So let's get started. So in the previous lecture, in the previous lecture, we discussed what is Euler's equals Euler's method and if we have an initial value problem like this, okay? So we saw how we will find out the solution in steps. Ok? So what will we do today? We will solve that with an example. First, we will see how to implement it. Then we will code it in Python also. So let me take an example. Like this I have the equation dy / dx = $1 + xy^2$ and y is 0 which we are taking as one. Ok? So this is our initial value problem. But what's the big deal is that y^2 is here. So what does this mean that it has become non-linear. Is it okay? So this becomes a nonlinear equation. Now let us solve this analytically. We don't know how to solve it. So what should we do? We will try to find out its numerical solution. So first we have solved this with picard. Is it okay? So what happens in Picard? That is a series which keeps on improving in each iteration, so there, that series finds out the solutions, finds out polynomial solutions towards the exact solution. Now what we have to do in Euler, we need numerical solution, so what is happening in this, now we suppose now y 0 1 is the given value to us. Ok? So this is suppose x and this is y and this is suppose one. So this is point one basically. So let me put this here, x is zero and y is 1. So our solution will start from here. Now we have to see how to solve this? So first of all we have to solve this with the help of Euler method. So the first thing we have to do is to first define h. So suppose I am told that you can take 0.1. So what does 0.1 mean? x0 is 0 which is initially given. Our x1 will be 0.1. x2 would be 0.2. x3 is 0.3. This is how it will go on for us. So this means there was a first point here. Then suppose this will become our x1. This is our x2, this is our x3 and it will go on like this. What do we do now? This has to be found out. The solution has to be found out. So now see, in our case, this is f(x), so what I am doing here is that I have written the Euler scheme here, so the Euler scheme that we have will be this, we know it, we have already discussed it and the value of n will go like this 0 1 2 3. Now we have to find out y1, so here we will get y0 + hf at x0 y0, this is ours. Now the value y0 is one given to us. h we are given 0.1. f Whatever we have, this is ours. If I put zero for x in this then this quantity will become zero. So only first will come here. So you see here 1 + 1 so the value we have got is 1.1. Now if you look at what we have here, we have the slope, now look at y1, what are our values ? f at x = 0 and y 1 so here 1 + 0 gives 1. This means that the slope here is forest. Ok? So what does slope one mean? We have to go upstairs. So we found out a tangent here and here you will see that we got a value on this and that is x1. So we took y0 plus some scalar multiple of this, we did this and we got a value and that value is the x1, so this is x1 and on this we got the value y1, now if we see, our solution has come to 1.1 y1. Now let's see what happens next. y2 so y2 is y1 + hf x1 y1 this is how we will find it out. Now what did we do? y1 is 1.1. Plus h is already our 0.1. And f is 1 plus now x is our one and y is the square of 1.1, this is it.

Is it okay? So these values have been calculated. Now we will solve all this. So this 1.1 plus 0.1 1 plus how much is this ? All this which we have, values one 1.21 will continue like this. So ultimately the value we have is 1.21 2 1. This value is 1.21.
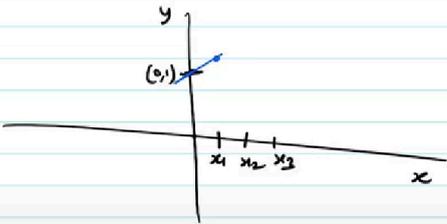

Numerical ODE

Euler's method

$f(x,y)$　　$\dfrac{dy}{dx} = f(x,y)$　　$y(x_0) = y_0$

Exp.　　$\dfrac{dz}{dx} = \left(1 + xy^2\right)$　　$y(0) = 1$

Sol.　　$h = 0.1$

$x_0 = 0$
$x_1 = 0.1$
$x_2 = 0.2$
$x_3 = 0.3$

$y_{n+1} = y_n + h\, f(x_n, y_n)$　　$n = 0, 1, 2, 3, \dots$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + (0.1)[1] = 1 + .1 = 1.1$

$y_2 = y_1 + h\, f(x_1, y_1) = 1.1 + 0.1\left(1 + (.1)(1.1)^2\right)$

$= 1.1 + 0.1\left(1 + (.1)\, 1.21\right) = 1.2121$

$f(0,1) = 1 + 0 = 1$

[8:02]

If we look at the functions that we have, these values will be the values of the slope. So what did we do in this? In this case, the values we have are of a slightly increased slope. So she went here. And what happened to him? The value came to be 1.21. So this value of ours is the value y2 at x2, so we will continue like this. From here we went to here, from here we went to here. So in this way we will continue calculating the values. Similarly, we will calculate y3. We will calculate y4. Now we have to see how much error is there in the calculation we are doing. Because we know that the error is a truncation error and it is possible only when we have a solution given. So in this case we do not have any suggestion given. So what should we do? In this case we will try to find out if there may be a truncation error. So now look, the first error will be y at x1 - y1 y1, so we have found it. We could find y at x1 only if we had a solution that was non-zero. But the solution is non-existent. Is it okay? So we will calculate this. So we know that the maximum value that this error hash 1 can have is 1/2 h² and y is double d and z we have is between 0 to 0.1. Why? Because x1 is one. So this is the thing that we will have to calculate in this case and it is giving us the maximum trunk share because there cannot be any error more than this. Ok? So now we will calculate this. So if we want to do calculations in this, then if we see, y double' second derivative is coming. So first we have to calculate this. So this is what we have already given. This is f(xy), so I will differentiate it one more time with respect to x. Now look, this is a function of two variables x and y and we have to differentiate it with respect to x. So what will we get? df /dx + df / dy dy / dx it will come. Ok? So in this case, if we see, what has this become, the function that we have with respect to x, then y² should come with respect to x, and with respect to y, it will come as 2xy and dy / dx will come completely like this, so our second derivative y' comes here, so if we get these values, then now we can calculate 1, our 1/2 h is 0.1, I squared it and its value for any is Z, we have calculated it. So we saw that its maximum value was coming at one. Ok? So we will substitute this. We have substituted those values. We know the value

of y1. The value of x1 is known. So we will substitute all the values here. Ok? So look we have y which is y1 because what do we have to find out where? Here. So y1 is what we have, which is 1.1. x1 is 0.1. So we substituted all these in it and when we calculated it, the value we got was 0.00 728. So this is the maximum truncation error and we find it only when the solution is unknown. So we did not have the session so we took the maximum amount which could not be more than this. What does it mean? The maximum time it can give exact information is up to two digits of the second. Similarly, we will take out 2 again. 2 is the second error between y at x2 - y2. So we will find the maximum in this also. So h ours will remain the same. It's just that this quantity will change. Why? Because now our x2 has become 0.2. So our ex which goes to exam will go between 0.1 to 0.2. So we will substitute it in and we will see that if we calculate its maximum value. Ok? So, we mean, whatever its value is, if we calculate it like this, then whatever value we get, we will substitute it and there we will get Apps 2. Ok? So in this way we will continue doing the calculations and the errors we have will be eliminated. So better, if we have to verify it then we can verify it very easily for those problems where solution is available. So, we will do this work with its help also with the help of code. So what this means is that now we have calculated the problems that we have faced and we can solve the same questions using picard as well. We can do the same with euler also. Now our next method is the Modified eulers Method. So what would be the Modified Euler Methods ? Now we have to make some improvement. Why? Because we know that the Euler method gives accuracy up to the order. It means that if the solution of our differential equation is linear, then it will give our exact solution, but as soon as non-linearity comes in, it will not give the exact solution, there will be an error in it. So, to improve it, we were told to call it by the name of Modified Euler method. So what will happen in this? So the method in which we are improving it is also called predictor character method. What does predictor mean? First predict one thing and then correct that value. So we also call it by the name of predictor and character method. So what will we do in this ? First we will predict the solution. Ok? So how will you predict? We can do that by any method. So how will the predictor character method work in this case? So now see we have the same differential equation dy / dx = f(x) what do we do now? First we will predict. Predictor So first I have this value h so what do I do ? First let me calculate it using the same as Euler method. So I took the Euler method and from there I calculated. So see what we did? We used to calculate yn + 1. y_(n + h) and here we go h f x and y this is our euler method. We know. So what did we do from here? If you put 0 in n then y1 will come. Ok? So whatever I calculated, I named it a star. And next is the character correction that needs to be done. What did I do in this ? Now it is obvious that we have to calculate yn + 1 only. So we wrote yn + h f Now see what is happening here? There was xn and yn. I'll write here xn + 1 and I'll write here yn + 1 star this. So what does this mean? Earlier we calculated the solution with the help of euler, I used it here and after that using the same method I found out the value of yn + 1, so whatever, now yn + 1 has been calculated here as well and calculated here as well. But if you see, these values have been corrected. It has been improved a little more. Ok? Ok? So what will happen by improving these values, now if we see it from the graph, then suppose from the graph we have our solution of this type.

Max. truncation error $\quad \epsilon = \frac{1}{2} h^2 y''(\xi) \quad 0 < \xi < 0.1$

$$\epsilon_1 = \frac{1}{2}(0.1)^2 \left[\qquad \right] = \boxed{0.00728}$$

$$y' = f(x,y)$$

$$y'' = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \right) \quad \begin{array}{l} y = 1.1 \\ x_1 = 0.1 \end{array}$$

$$\boxed{y(x_2) - y_2} \quad_{x_2 = 0.2} \quad \epsilon_2 = \frac{1}{2}(0.1)^2 \left[\qquad \right] =$$

$$y'' = \boxed{y^2 + 2xy(1 + xy^2)}$$

$$0.1 < \xi < 0.2$$

<u>Modified Euler's method :-</u> $\quad$ (Predictor — Corrector method)

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

(Predictor : $\qquad y_{n+1}^* = y_n + h\, f(x_n, y_n)$

Corrector : $\qquad ✓\, y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1}^*)$

exact sol $\quad y(x_1)$, $\quad \epsilon_1$

(x_1, y_1)

$x_0 \quad x_1$

What have we done? Suppose here is my x0, here y0 so what we did is first I calculated it and my solution suppose this came out, okay so this is the solution, suppose this is my x1, so on x1 I found this solution y1, this one is my x1 y1 and actually what was the solution that I had, this is what is y at x1 because this is the exact solution, so you will see here, if I join it to the bottom here, then this is the error and this part can be seen from here. We got this error. I can name it Epsilon One. Ok? So this is our error. Now such errors will keep on occurring in the euler and we will continue doing the calculations. And what happened to the Modified euler? Now, whatever value we calculated on this y1, what did we do here now? I used this f at x1 and y1 and y1, I named it star. Ok? So, this made him a star. So when we checked the value of this, we found out that its slope is that because this solution is going upwards, here you will see that the slope will increase a little bit. Don't go like this, you will get value like this. Only then will our solution, which is an approximate solution, keep going up. So now you see what has happened now? What improvements have been made? Earlier we had this slope. Now this is the slope we have. So this is the modified Euler method, which we mean we have to make this method. So what happens in the Modified Euler Method? Now we have to see what will be the session? So now we have made it a solution. So I write here. So now our session will be this, what will be the method yn + 1 ? yn + what should I do now? And this star star of these two, we know this will come to us, so what we did now what I did was that with this the first slope that we had is this one which is representing the green. The next one which we calculated with the help of euler is this. I have arrived. So I took the average of this slope and this slope. Ok? So look, this was here and this was the value. Suppose this was one and suppose this became two. So 1 + 2 =3/ 2 1.5 so now the solution that we will get here, I will go a little upwards and calculate it like this because my slope has increased a little bit. So now see, our solution will come here. Earlier he was coming here, now he will come here. So this means that the error has been reduced. Ok? So this is the correction has been made. So now look, our error has now become this. Earlier our error was this big. So what the Modified

Euler Method does is that first we calculate it with the help of Euler Method and then from here we take the average value of their values. So this method of ours is called the Modified Euler Method. So we also call this predictor character. So in this case we considered it as average. And this method, this is the order, the accuracy increases. This means that the error which will be there, the truncational error, will be h / 3 and this will come. So this means that if we have solutions of a differential equation and if its solution is linear or quadratic then it will give the exact solution till that point and after that if we have the solution of cubic then we will get an error. Otherwise what it means is that this becomes true for linear as well as quadratic, so our accuracy has improved. So now we will continue calculating it like this. So now take the previous example. So what about the last example that I did? Now this is the equation we have dy / dx $1 + xy^2$ y at 0 is 1. So now let's see this calculated. So if we pay attention, we calculated y1, ah 0 sorry, this was h 0.1, okay? So we calculated this as 1.1 with the help of 1.1. So we calculated this with the help of euler. Now we will calculate from this. So now we have y1 at x1. Now I am improving it again to y1. How will I do that? y is 0 that's 0. +1 / 2 What do I do now? f at x0 y0 + f at which we have calculated here. x1 y1 is that okay? So I'll make this a y star so we don't get confused. So what we did was we calculated y1. So what will happen now? Well, this was zero. Oh sorry it was one, not zero. = 1 + because y is not one. So now we will improve it, we will get 1 +1 / 2 and if we put this value, then we were getting one plus and if we put this value then we will get this, what is 1 + x1? What is 1 y1? y1 star 1 black square this will come. Ok. So what will happen in this case? The accuracy of what we have will increase. So this will be our solution. So we will solve it accordingly. We took dy / dx. So this is our work, this y1, this value of ours will come. We will calculate this right now. So y1 we have this. Now I will do y2 like this. So what will we do in y2 also? Now this y1 of ours has arrived. So I will create y2 star. What would that be? y1 + h * f at x1 y1 so from here our y2 will be formed as star. Then I'll subtract y2 from here. How much will that be? y1 + h / 2 and the values we have are x1y1 + f at x2 y2 star I will take the average of this. Ok? So if you observe this, it is happening in two steps. In the first step we are applying euler and in the second step we are correcting it and then how we are doing the correction with the help of taking that as per our method. Ok? So the value that we put, now if we see, we have helped it, so this correction, which I write correction, we can write this directly from here instead of writing this correction. I can also write this directly that we have made the correction from here. Ok? So instead of writing this directly, you can write it here that this can be written as this also.

[28:31]

So there I told you that corrections happen. But if we see the correction then we can write it directly also. So here is our correction. Ok? So we found it out. So with its help we will calculate all the values. Now let us look at this a little bit. Now look at the code we have written for pics. Picard Methods So what's in Picards ? We will start with different values and then we will keep improving it in every iteration. Ok? So the solution we are taking is this one. Well this is the exact solution. We have this exact solution. So let us first see how we will start. This is our example which we had given in our previous lecture, we have solved it. So now look what we did in the beginning ? This value came to us. This was our first iteration, y1. After that we calculated it and it came out to be y2. Then y3 came. So if you see, what we have is that in the first iteration the approximation we got was x / 3 - x + 1, then we used that in picards, so in the second iteration we will calculate x 4 / 12 with the minus sign + x ^ / 3 + x ^ / 2 like this. Then x3, then x4, then x5, so if we observe, in every iteration the approximation that we have is getting improved. First he was improving the approximation with this cubic and then with this fourth power, so if you see, this is our improvement now. So now you see here that this black one is the exact session and like this we are getting improvement. So this picard method, it takes time and in this way we will move towards the solution and our solution is exactly given to us in this case. We already knew that and we had calculated that this solution is given to us. Now let us look at it from the perspective of euler to see how euler's methods will work in it. So what are we doing now with the Oilers? Here we will input the function which is the right hand side function. t which is the t value, t span, we will input that, y0 which is the initial condition, we know that, we should know that and if the exact solution is available then we will pass it, if it is not available then we will not do it, okay, so now that we have defined it, t0 which is our initial condition at the points is given and tf is the final till where we have to go. So what should we do? first I'll generate a vector from t0 to tf+h ok? So in this way, whatever values we will get like t0 t1 t2 t3, such values will keep coming. We start the y values at y0. After that we will keep storing here because we have to see the iterations. What about the values ? What are the t values and what are the y values ? Now we have defined it. So what are Euler iterations? From here we will do y = y + h * fty and this way we will get the value. We know f and we also know h. Ok? So we will substitute the values in this and after calculation we will get the

value. So we will append the y values and we will keep writing it in it and we have defined this table and we will keep writing it here. Ok? So in the same way, whatever data we have will be displayed, we want to display our table data. After that we will calculate it and this figure of ours is a plot figure. So we will generate a plot figure of size 10 by five. So after that, whatever value we have calculated in the Euler solution, it will be plotted and if the exact solution is available then we will calculate the value of the exact solution. Now if we have the exact solution then we will find its values at every t. So that's why we want to remove the error. So look at each one of those t, I found its value. Ok? And then we'll plot it. So if the exact solution is available then we will plot it. Otherwise the error will be plotted. Otherwise we cannot calculate it. So that's why we wrote if the exact solution. So if the exact solution is available only then we will run it and we will get errors. If it is not available then it will not happen. Now look, it has come to us. So the example that we have just taken is defined as $1 + ty^2$, okay, so we are taking it in the form of t. This means the differential equation that I am taking is $dy/dt = 1 + ty^2$. And the exact solution is not available with us. Now see, t0 was 0 and tf, I suppose I have to go till one, so I took one and y0 we had one and the initial step which I took was h, I took one, okay, so now I have defined the Euler methods, here I have called the function and wrote non here, the exact solution is equal to non, so it is unknown, so if we do the calculation, then we have the solution, so we ran it, so here we have the solution of table a. Now look at the iteration, there will be 10 iterations because h is one, so t comes, it will start from 1 0, so 1234 will keep increasing like this and y was initially one given. After that look, what we calculated was 1.1. Then we next calculated 1.2121. So like this we will keep on calculating with its help and this is our approximate solution will come and see, this is the solution that has come and we have also seen it and can also verify it with that. I said that the solution will definitely go upwards because the initial slope was one. Is it okay? So the slope, if you look here, its slope is one. So it started from here. Then here we used it and calculated it. Then I came here. So our solution will keep increasing like this and we will calculate our values. So this is the thing, we want to take it to the first. I might even take it up to two. Now look, it has reached two. Ok? So, our session in this will be made in this type. So every thing depends on how far we want to take your values. So generally when we have a nonlinear equation, we do not know how the solution will behave. Now I suppose I should increase h a little more and make it two. So see, here is our solution. Ok? So what happened in this case? This solution has arrived. Now let me make it 1 6.

```python
    if exact_solution:
        error = np.abs(np.array(y_values) - np.array(exact_y))
        plt.figure(figsize=(10, 5))
        plt.plot(t_values, error, label='Error', color='red')
        plt.xlabel('t')
        plt.ylabel('Error')
        plt.title('Error Plot (Euler Method)')
        plt.legend()
        plt.grid(True)
        plt.show()

# Example usage
def example_function(t, y):
    return 1+t*y**2
    # return -2 * y + np.sin(t)


def exact_solution(t):
    return 1
    # return (np.sin(t) - np.cos(t)) / 5 + (np.cos(t) + 2 * np.sin(t)) * np.exp(-2 * t) / 5

0, tf = 0, 2
0 = 1
tep_size = 0.1

uler_method(example_function, (t0, tf), y0, step_size, exact_solution=None)  # if exact sol. is NOT known
euler_method(example_function, (t0, tf), y0, step_size, exact_solution=exact_solution) # if exact sol. is known
```

[36:24]

Suppose our session will come here. So here comes the solution. Ok? So the problem is that we do not know the exact solution in this. So we don't know what the error will be? How the solution will behave. But with its help we have calculated its solution. Ok? So now suppose I change this and suppose I take some other function. I will take this. Let's take this one. I am taking the return t - y I took it. Ok? And I will comment on this. I have commented this. Now in this case the exact solution is non. And the exact solution is coming out to be t - 1 + 2 times 2 * exponential, so np xp to the power of - t. Ok? This is its session given to us and it is obvious that this is a linear equation and we have the initial condition, suppose I take it to one only. This is the initial condition and the step size is suppose one and in this the exact solution is non, so I will take this in its place because if the exact solution is non, then we have to send that also and input it, so this will become the input. Ok. So we have the exact session. We have defined the function. Now let me run it. So look, the session has arrived. So this was first one and then it became 9. This is 8.75 so it is going down. Ok? So in this case, our solution is this, see, the exact solution is in orange and the Euler solution is this, so in this way the values that we have will be calculated. It has become negative because if we pay attention, there is a minus sign, so t - y, so that is why the values of this solution of ours are going downwards. This way I can improve it. So now look in this case we can plot the error itself. So this is our error. So such errors have increased. Ok? Because now I should make our h a little smaller. Now let's see what happens. Now see, when I did h, the accuracy increased. See how both solutions overlap. So if this is happening, then it should happen that if we keep making h smaller, then our solution that should converge, should converge to the exact solution. Ok? So this shows that the scheme we have written is consistent and our solution is converging. So this is our session and from here we got the error. So we have also seen this error that after some time the error has become almost constant. The bus arrived around 35. Ok? So like this the number of steps will keep increasing and the solution we have for now, I will keep it one. Suppose I will make it two. So let us see, let us go till two and see. So if we go till 2 then our session is going till 1.24. This will be our session. Now see, in the beginning the error will always be zero. Why? Because we started from here but after that it kept on

increasing. Ok? So in this way we will keep calculating it. You can put it here and see what our function f(xy) will be at 0 1? 0 1 will be -1. will be -1 so session will go down. So look, the solution has gone down. Then I improved it and it came on top. So in this way we can define different problems. We can calculate that. Now let me change it a little bit and see what is happening in that case. So let me change this now. So suppose I will define it in one more way. Return Now I assume I am taking y - t I assume I am taking this. Ok? And in this case, what we are getting is coming, the exact solution given to us is that this value will come, it will just become plus, control c and control v and this is our plus, so t + 1 + 1/2, well 1/2 is here 1/2, sorry this into exponential t, this is the solution. Ok? So let us now see what is happening in this case. Now there has been a change in it. Earlier it was exponential -t, now it has become exponential t. So this means that the solution will rise. The solution would be exponential. Look, this solution is coming. 1 1.1 1.2 1.3 It is increasing like this. Ok. Well, we will have to keep the initial conditions in mind also. The initial condition is 0 per 1.5. Ok? So if there is 1.5 on 0, then suppose we have to go from 0 to one. So that's it. Let's take this one. So this will become our solution. So see, the solution has been improved. And the solution is growing. See the exponential because e is raised to the power of t in this case. So now if I increase it a little then the suction will increase further. It went up to six. So such solutions are getting improved. Ok? And the error is this. So when we have the solution known then we can also find out the error. Even when the session is unknown, it means we cannot remove the error. Then we will have to show only the solution in the plot. So this is our session, it has arrived. Now we have done this work using Euler method and taken the single initial loop problem. I can do the same thing with modified as well. Ok? If it is there then everything has to be done the same in modified form. Just instead of Euler method, we have written modified Euler method. And then we created a table which is similar to what we did last time. It's just that the iteration here has changed. So first we'll take k1. Is one okay? What are we saying to him? F(ty) then I defined k2 which was our second function f at t + h and y + h * k1 which we had taken out from Euler and this we have defined f + = y plus equal to means y = y + h this is our quantity and we have taken its average divided by 2 that's all we have changed is this only apart from this everything is the same. So now this is what we do. So we had defined this example. Ok? Returns 1 + t * y² and we have the exact solution which is none. He is unknown. So I wrote it as None and defined this value. Now look, here is the solution. So if you have seen that oiler on 1.1, you would remember that that was session 1.1, if we did it with modified oiler, then the session is 1.106050 and on this one it is 1.22709, so this is our solution.

```
# Store iteration results for the table
table_data = {'Iteration': [0], 't': [t0], 'y': [y0]}

# Modified Euler's method iteration (Heun's Method)
y = y0
for i, t in enumerate(t_values[:-1]):
    k1 = f(t, y)
    k2 = f(t + h, y + h * k1)
    y += h * (k1 + k2) / 2
    y_values.append(y)
    table_data['Iteration'].append(i + 1)
    table_data['t'].append(t + h)
    table_data['y'].append(y)

# Display the solution table
df = pd.DataFrame(table_data)
print("Solution Table:")
print(df.to_string(index=False))

# Plot the numerical solution
plt.figure(figsize=(10, 5))
plt.plot(t_values, y_values, label='Modified Euler Solution', marker='o')
if exact_solution:
    exact_y = [exact_solution(t) for t in t_values]
    plt.plot(t_values, exact_y, label='Exact Solution', linestyle='--')
plt.xlabel('t')
plt.ylabel('y')
```

[46:45]

So this is how we calculated it with the help of the modified order. Similarly, now we can try the other examples we took. Ok? So I wrote the exact solution like this and defined it. I only get the euler one from here. This exact session control C control and the example that we are taking, I am taking this control C and here I have defined the control, this one is okay and the solution in this was that the initial condition in it was 1 5 on zero, now let us see this, hey, we did not send that so we have to do that with this and this one I will because our exact session is none. It's here. Ok? There is a problem with indent. So we left a space in the middle. So that is why now we have this session. So look what was happening in euler ? This is what we had calculated. So there was a little error here. Is it okay ? This is what we had calculated. We had given these very values and we had seen that this one is in the euler, let me make it one instead of two, and we had calculated this, see, here an error is coming but we did it with the help of modify, so it is exactly overlapping in this case, so it means the session has been improved, right? So, we have calculated the values that we have and our solution is coming out to be a very good session and the error has also been reduced. See, the maximum error here has gone up to 2. What was there in it was that the error was up to 6. If we pay attention here, we will change it like this, so I will make it two. Suppose if I make it two then we will have to do a little more calculation. But see in this also, what a good session it is and the error is going up to 01. So an error occurred here as well and an error occurred here as well. Ok? So according to this, we can calculate it. So this means that this calculation of ours, what we have done now is that we have solved the Euler method, single equations and modified Euler method. Now in this way we can define many methods. Ok? So what will be our main purpose? In this case, our main purpose will be to keep improving it. So what do we do for improvement? Different methods, as you might have heard, Runge Kuta method also gives a good approximation and its order of convergence is fourth order. Ok? So we can apply this. So in this case now we will define what will we do with it? Now look, I have done the same thing and we have defined Euler and Modified Euler method. So now the next method that I will discuss is called Runge Kuta method, so we have to define Runge Kuta method, we will write it as R K. Ok? So we call it RK Methods. So if we look at the RK method, we will

create different types of RK methods. First we are writing R's method of second order. So what do we have to do in second order methods ? This is how we calculate it. Basically we suppose we have to find y1 with the help of y0 so we will have to add some things to it. The K can be positive or it can also be negative. But you will have to do some calculations and add something. So K in this case is K1 + K2 /2 where what is K1? h What is f(x) not y0 K2? H f this is what we did with his help. With the help of modified euler. So we calculated this from here and substituted it here. Ok? So here K1 comes, K2 comes and we took the average of both of them and substituted it here. So if you remember that this is a second order method of R, it is the same as modified Euler method. Ok? So we did this using modified Euler method. So now we have accepted that this is absolutely correct. Now let us see how to calculate this using RK method. So now look what we're doing in R's methods ? Let me calculate this. Let's take a general form. In general because its expression is complete in general R's method of order m okay? We write that down. So we define it directly. So no matter how many methods there are, we can define it like this. So now that we have got it, let's see what we do first? Let me define k1. what is k1? h of f at x0 y0. This is what I have to find out. y1 = y0 + k This is for sure. Now let us see what k is in this case. So we found out k1, this K2, I found it, it comes out to be h f x0 + α1h and y0 + beta1 K1, we found out K1, some portion of it we have to do in K2 K3 K3, it will become hf x0 +h and here it will become y0 plus beta 2k1 + beta 2 k2, we have to take some portion multiple of it. Ok? So this is how we will define k_m. I need m order. Right now there was an order of two from here, so we went till two. In m order we will go till m. So in the end we will be left with h of f x0 + α m - 1 and here will come y0 + beta m - 1 k1 In this way, whatever comes in the last will be beta m - 1 m - 1 and k m - 1 because we calculated till m - one, so this came and in the end we should get what K, what will be K, w1 K1 + w2K2 up to wm Km so this weighted thing, we have multiplied it w1 K1 so w1 w2 w3 wm are the weights, okay so what is this w1 one w2 wm, these are the weights. These are the bests. Ok. And what's in it? Ws have a cut and quality that whatever is even will come in one. Ok? So with the help of this, we can calculate this. So if we see, if I look at this one above, then look here, this W1 is half. The W2 is also half. So both of them are equal. Ok? And from here we calculated it. So we can define a runge kuta method of any order. So now we define the Runge Kuta method which is R's method of fourth order. So what do we need to do for Fourth Order Methods ? This is for sure. We need to find y1 = y0 + k.

Runge–kutta method :- (RK)

R-K method of 2nd order !

$$y_1 = y_0 + k$$

$$k = \frac{k_1 + k_2}{2} \quad \text{when} \quad k_1 = h f(x_0, y_0)$$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$

$w_1 = \frac{1}{2}$
$w_2 = \frac{1}{2}$

Same as modified Euler's method

In general R-K method of order m is $y_1 = y_0 + k$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f(x_0 + \alpha_1 h, y_0 + \beta_{11} k_1)$$

$$k_3 = h f(x_0 + \alpha_2 h, y_0 + \beta_{21} k_1 + \beta_{22} k_2)$$

$$\vdots$$

$$k_m = h f(x_0 + \alpha_{m-1} h, y_0 + \beta_{m-1,1} k_1 + \cdots - \beta_{m-1, m-1} k_{m-1})$$

$$k = w_1 k_1 + w_2 k_2 + \cdots + w_m k_m \qquad w_1, w_2, \ldots w_m \rightarrow \text{weights}$$

[58:11]

Ok? It is not the case that after taking out the quantity, y1 will go away. Whatever y2 will be y1 + k, then it will keep repeating. But we have to define it in the first step. Now see, k1 k1 will always be the same as what we are taking out from the oiler. So x 0 y 0 is always the same. Because we will remove it from the euler. So if I put k1 in place of k here, this becomes euler. Now what will be K2? HF now look X not now what to do in this? What will be the quantity alpha1 ? See that. So let us first write alpha1 h and y not, this will be y not + beta1 K1, this will become. K3 becomes hf x0 +α2h and y0 + we write this, in this case beta2 K2 becomes this. Ok ? And our hf K4 has arrived. And we have calculated this 3h and y not + beta3 K3 directly, okay, so with its help we will calculate it. Now look at this, the K will come to us, K1 + K2 w1, sorry w1K1 + w2K2 + w3K3, this is what we have. Now we will calculate this. Ok? So if we look at its expression, its expression comes from Taylor series expansion. Ok? So if we find it out from that then we see that alpha1 is coming half. And the beta 1 is also coming in half. The Alpha2 that is coming is also half and the Beta2 that is there is also half. Alpha 3 is 1 and beta 3 is one. Ok? So we calculated this, the K that we got will be K1 / 6 + 2 / 6k2 + 2 / 6k3 + 1/6k4, after calculating this, our weights will become 1/6 2/6 2/6 1/6 and if we sum them all then we will get one, so this is our quantity. Ok? So whatever error will occur in this, now this is fourth order. This means it gives the accuracy up to fourth order polynomial. That if we have a differential equation and its solution is a fourth order polynomial, then this will give the exact session. This means that the error which will be the truncation error will start from power 5 of the order of h. Ok? So this thing of ours, we call it wave methods. So from the oiler we calculated the modified euler. Then we saw in the Runge Kutta method that if we look at the second order then it is a modified euler and after that we defined the Runge Kuta method which is of fourth order.

RK of fourth order! $\quad y_1 = y_0 + k$ $\qquad\qquad$ $\boxed{y_2 = y_1 + k}$

$$
\begin{cases}
k_1 = h\, f(x_0, y_0) \\
k_2 = h\, f(x_0 + \alpha_1 h, \; y_0 + \beta_1 k_1) \\
k_3 = h\, f(x_0 + \alpha_2 h, \; y_0 + \beta_2 k_2) \\
k_4 = h\, f(x_0 + \alpha_3 h, \; y_0 + \beta_3 k_3) \\
\\
k = w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4
\end{cases}
$$

$\Rightarrow$

$\alpha_1 = \frac{1}{2} \qquad \beta_1 = \frac{1}{2}$

$\alpha_2 = \frac{1}{2} \qquad \beta_2 = \frac{1}{2}$

$\alpha_3 = 1 \qquad \beta_3 = 1$

Accuracy upto fourth order polynomial?

$$\boxed{k = \frac{k_1}{6} + \frac{2}{6}k_2 + \frac{2}{6}k_3 + \frac{1}{6}k_4}$$

[1:02:15]

In this way we can define Runge methods of higher order also. So after that we also saw with the Python code how we can calculate its values and see the errors. So I hope you liked today's lecture and understood it, and thank you for watching this lecture. Hello.