SCIENTIFIC COMPUTING USING PYTHON

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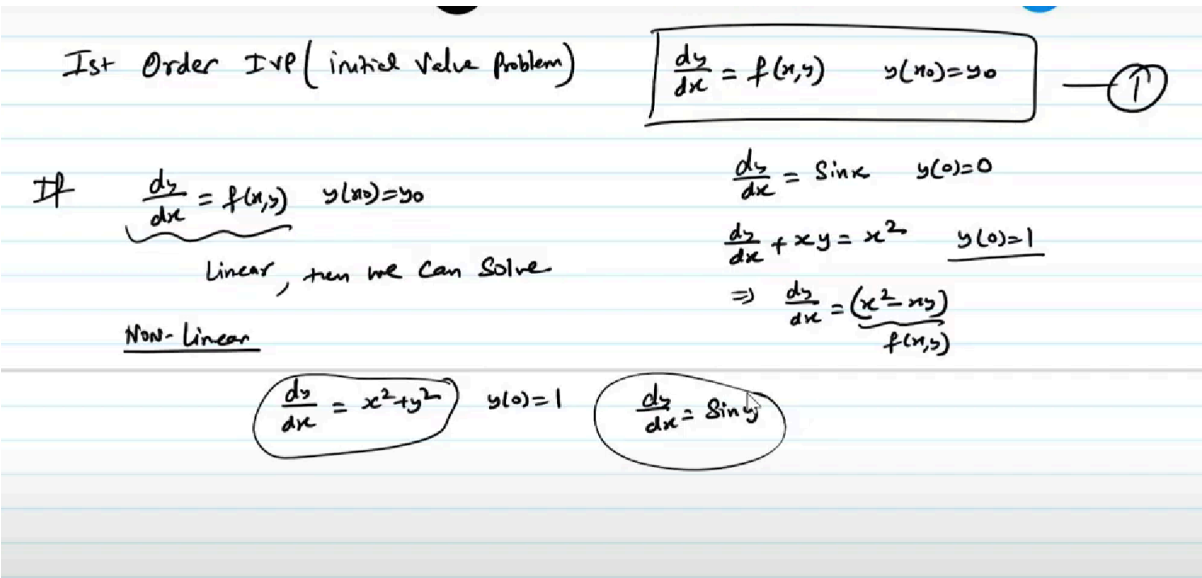
Lecture No. 33

Welcome to Scientific Computing using Python Today we are going to start a new topic called Numerical Solution for Ordinary Differential Equations Numerical Solution for Initial Value Problem So let's get started. So far we have been talking about numerical differentiation. So today's topic is that now we have to figure out how we can solve an ordinary differential equation numerically. So what do we have to do in this case? So first of all, let me just tell you what is the first order initial value problem that we write. It is dy/dx=f(x,y) equal to in general form. We can write it like this. If this is the initial condition, then in a first order case, we will call a differential equation an initial problem. If we can write it in this form, for example, you must have seen that we have written an equation like this dy/dx and I wrote is zero Suppose I wrote this, okay or else we wrote it like this dy/dx plus xy equals to x^2 and I defined y(0) suppose one, so if we write this equation like this, you must have seen this type of equation also, so we can write it like this dy/dx=(x^2-xy)/f(x,y), so this is what we have become in this case, it becomes a function and this is our initial condition, so what do we do that we have to solve this type of equation, but you mean if this is an equation, then we know that we can solve this type of equation, if we have this type of equation, I will name it equation x, if we have it in sorry xy, if this equation is linear, then we can solve it, we can calculate it easily, we have many methods, we will discuss it and in the course of differential equations, it is defined or plus You must have done it on two levels also. Now the problem that is coming is what will happen if it becomes nonlinear. What does nonlinear mean? If we get an equation of this type, dy/dx=x2+y2. Now this equation is a nonlinear equation because the square of the dependent variable y has come. So, this is this type of equation and we do not know how to solve it. Or we get an equation of this type, dy/dx=siny.



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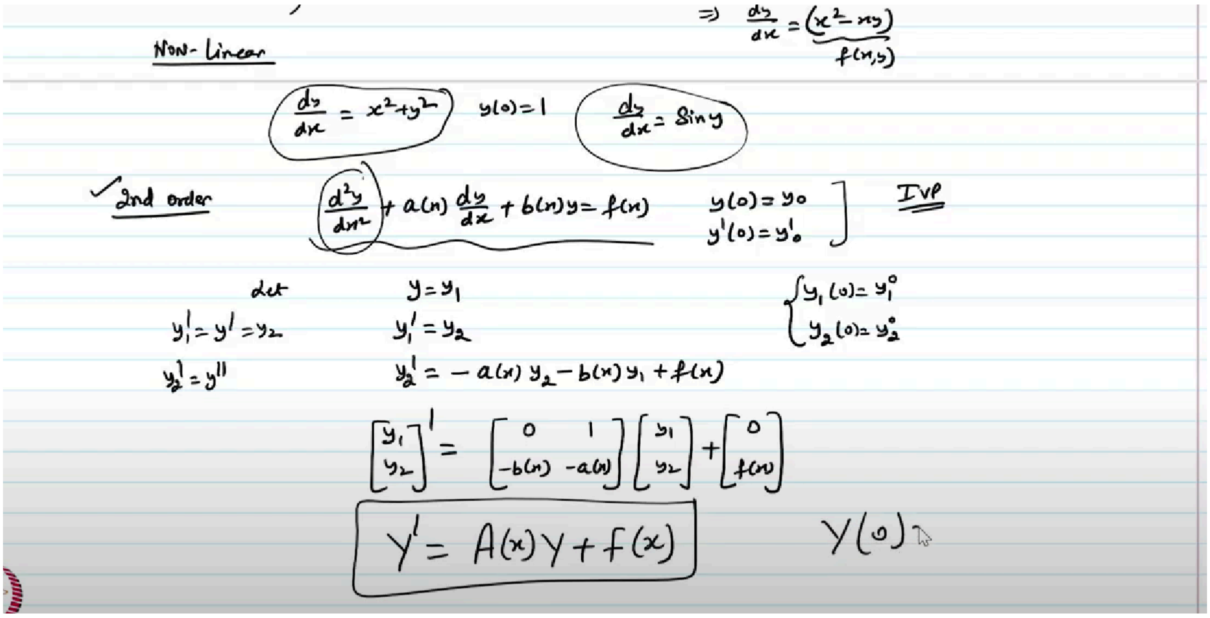
So, this system which comes, we know that this function is a nonlinear function and this dependent variable which is y has come in its form, so this also becomes a nonlinear equation. Now we do not know how to solve this also. Now, this was of first order and of second order. If we write, what happens in second order, d2y/dx2, I am writing a x now. Now I am writing linear in place of dd one, so I wrote b y = f one and here we have put the initial condition, at 0 I assumed 0 and y at zero , I defined it like this, so this is linear, but it is a second order initial value problem, we will call it initial value problem whose order is second, okay, so now it is a second order initial value problem, so what can we do with this, in any initial value problem, we can convert it into a system of first order initial problem, how will we do it, so see what am I doing with this, let's give a new name to our y1, okay and then what do I do, I take y1', I name it y now if we take y2, what will come out, see what does y' mean, wad k and we assumed y1 and what does y2 mean, y2 is so means y2'', okay Isn't it what does it mean y2''? So what does it mean to d? If we look at y2 , leaving this quantity, if I take all the things here, then it will become minus a(x). Now what is d x, the derivative of the value is right? Plus y, sorry y2 to plus b(x), sorry, the minus sign will come, y to minus b(x), the value will become y. Why, because what is y2', it will be y to, isn't it y2, what is wow, so wow comes plus and if we look at the initial condition from y, then what is it, so the value that came to me is y1, I named it y1, I will write it down, okay now what is y d y2, so this comes y2 at 0 and this comes y2, so this becomes our initial condition. So the system that we had, which was the second order initial value problem, I converted it and made it into this form, so I can write it like this, y2' is equal to this in matrix form. In this we can write where y1, y2 is unknown and if we look at it, this creature will come here minus b(x) minus a(x) plus 0 a one. So this is ours, we can represent it with capital y, this is our matrix, our A(x)Y plus f one, this is what we have, so this system of equations becomes first order and what will be the zero of y(0),



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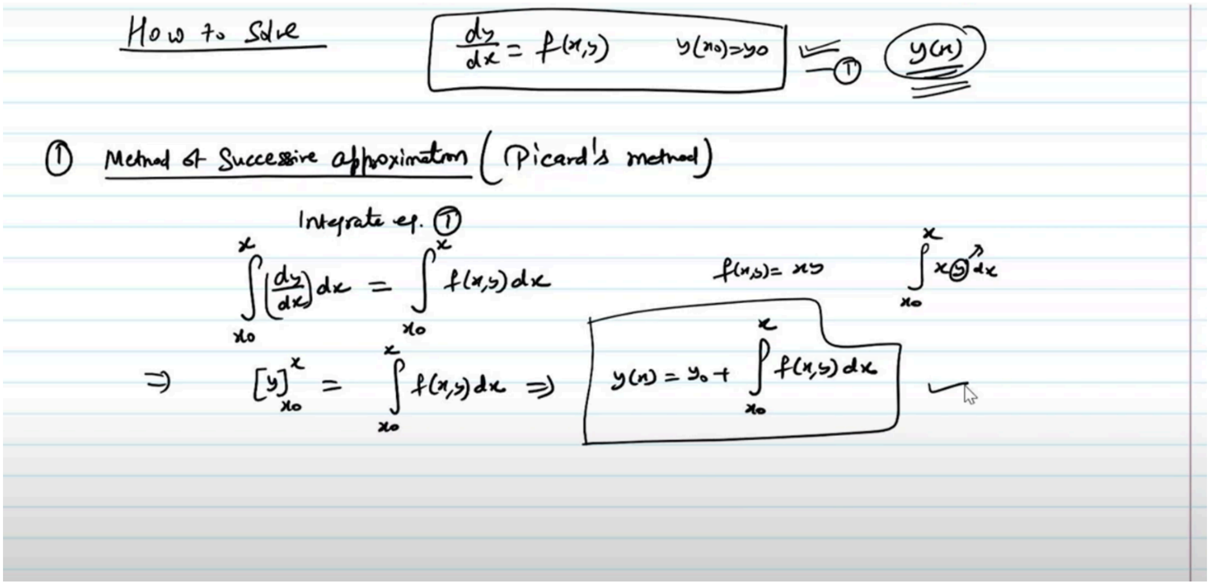
ours comes wav ji wa, so from here we can see and say that it is a system of initial value problems, where the initial condition is like this, so in this way we will keep converting all the differential equations of third order, fourth order, fifth degree, sixth degree order and in this way we will have a system of equations. Now this system of equations can be linear as well as nonlinear because depending upon whether it is linear or nonlinear, in this case I have taken linear but it can also be nonlinear, okay, so what we have This means that if we have an initial problem of any order, we can change it and write it in the form of system of first order and after that whatever methods we see on how to solve first order equation, we will apply the same on this. So now what does it mean that now our main purpose will be how to solve D by D x equal to A x y. To solve this, if I solve this, then this will also get solved. So in this way, our main purpose in this topic of this course will be how to solve this. So if it is linear then it is fine, it means we can solve it, but if it is not non-linear, then it is possible that we may be able to solve it or may not be able to solve it. So we have to find some such method on the basis of which we can solve this. So one method which we already know that we need a In successful approximations, we name it as the P-cats method. So, its solution will be any non-linear equation, if we have any solution, then the solution will be a function. Basically, its solution is y-ax. And y-ax is a function which is also differentiable, right? So, if we add y-ax in it and it is unique, then our solution will be y-ax. It can be any function. So, we also know that the function y-ax can be written in the form of a polynomial. With the help of the power series solution, we can write it in the form of a polynomial. So, our main purpose in this method of successful approximations is that we will keep improving the solution of y-ax by summing approximants. So, the methods that we name are called Picas method. So, Picas The method is that we mean how to apply it. So look, to apply it we have this equation one. So what do I do? We can integrate. So I integrated d one with respect to x. Okay, I integrated it equal to f x with respect to x. Now from where do we have to integrate. So what is the starting point x0 and what will be the solution of this? It should come in the form of x. So this means we will have to go till x. So now we can integrate this. The left one, we can easily do it. This is the derivative. We are integrating it. So this means that the integral of this will come out to be xz x equal to now we don't know what the function on the right hand side is. It can also be non-linear. Okay, it can also be a complicated function. So we don't know how to solve it. How to take the integration. If this was a function of x, then it would have been fine. If this function, suppose I have f equation, I can take it as equation. Suppose I take this, then we integrate it, if x0 to x then it will come out to be equal to d one but y is that which we do not know, it is unknown, so its solution is okay so we can write it like this, so this equation will become of x minus wa of x g, we have initial condition, so if we take it to the right side, then it will come out to be plus x from x it becomes y. Now the formula which has been formed, we have to use it to find out approximate solution now how to do it, see, we know an approximate solution of wa is 0, okay, even if it is a point given at the initial point, but we know the initial solution of wa, so now we do it, let's see how we will improve it, so to improve it what I do now, look I am converting it and writing it in an iterative form, okay so what will we do with it, an iterative I will write it in form and what will be my iterative form, look I am doing it like this, y(x)now my y is given x0 to x is also given x and I am writing this y_(n-1) okay and



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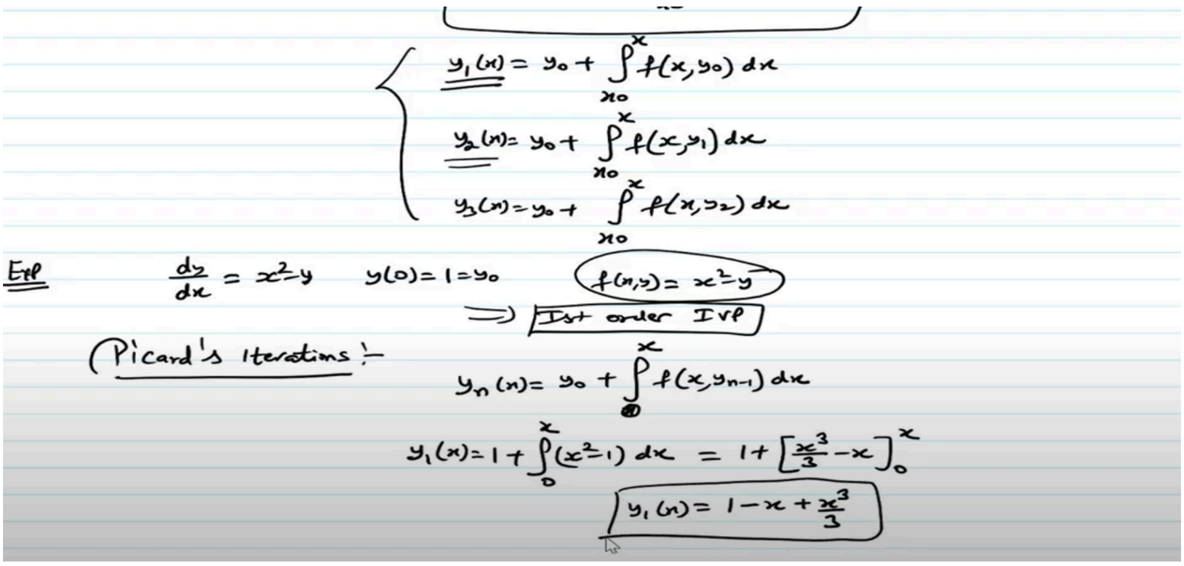
here I am coming, I have written this a so this has become our iterative form and what kind of an iterative form is this now look what I am writing, if I take a one then what will come out, see y will come out x plus x0 to x f (x,y0) one so you know if I will substract it y then it will become function of x only and if we integrate it then from there I will get y then what will happen y0 plus x0 to x here f(x,y1) will come, I am using our y1 which came out now to find a new approach, I will take out y3 in better approach mean so in this way an iteration process will start for us and we will have polynomials and solutions will keep coming and that is why we call it method of In successive approximation I take these approximants. We have been given an equation which is a body of x square minus y. So if we look at this equation, this equation is linear basically because the dependent variable is not squared or cubed, nothing, it is linear but it does not matter to us. By this method, the successive approximations order initial value problem is formed. Now see, if we want to solve this, then what we have is zero. So the x- square iteration that we have will become a x equal to was plus x which is zero, so from zero to x f x will become a minus x. Now first of all I will try to calculate what is zero, right? So from plus zero to x f what is What is ? You will have to put the value of y in it. What is it? If we see, it will become x s s s now I can integrate it, I can do it very easily because the unknown one has disappeared from here. Now I have calculated it, so this is the beginning plus x q by 3 minus x zero to x, this came and this became mine. Okay, so I would directly write it as 1 minus x + x^3, this became ours. So here we have the first approximation of our solution.



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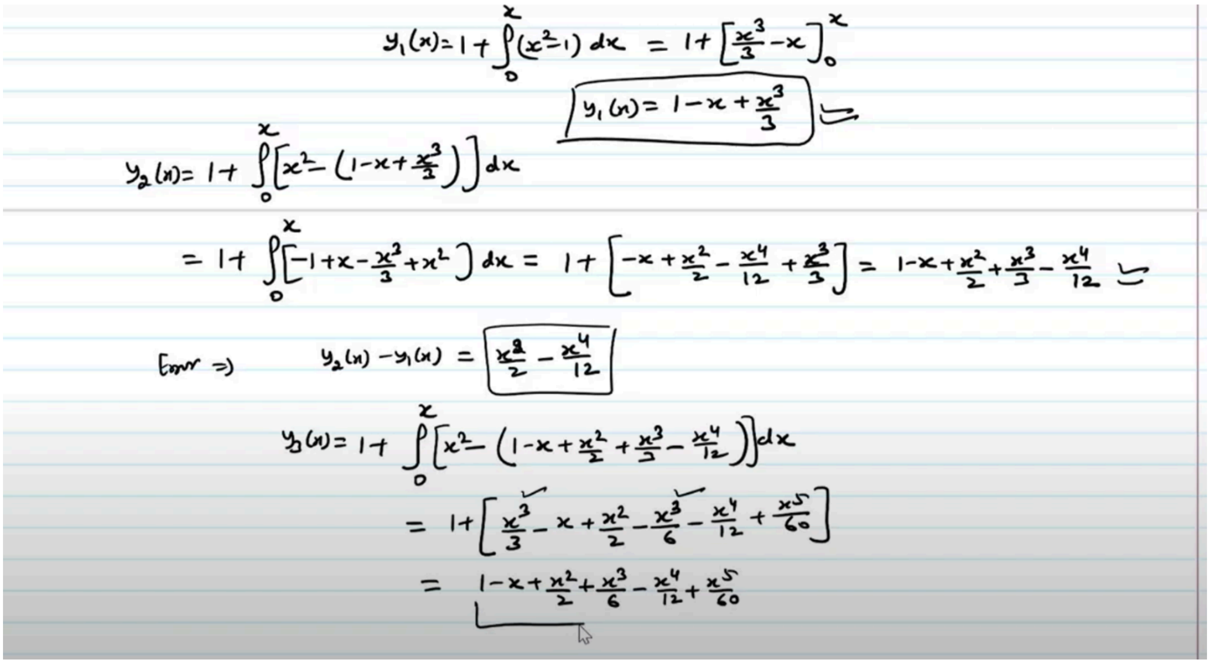
We do not know what the solution is yet, but this is its first approximation. Okay, so this is the first approximation. Now let us see the second approximation, whatever will be the second approximation, I will substitute it, so this will come as 1 + 0 to x minus. Now whatever will come in place of y1, y will come. What has come in place of y1, y has come. Minus x^3 by 3, y will come, I calculated this. So here we go plus 0 to x Now I can write it directly like this 1 plus x minus x to the power of 3 plus x squared x I will integrate this okay so what will happen after integrating this value plus see I am writing it directly this will become minus x plus x squared to the power of minus x by4 plus the power of x by 3 yah I will take this a little from one so here we go 1 minus x plus x squared to the power of 3 minus x to the power of 4 by 12 so this was cubic and it went up to the fourth power here we have a new approximation okay so now in the successful approximations I will write the approximation in the mention of approximations let me write here this what we have calculated what will be the error the error which will be the first one we are taking is approximants summation and this has come okay so if I want to calculate this then the error that we will get will be y 2 one minus y one this It will come if we minus this then 1 minus x and xq if we see then y will be left with x k x square by minus x to the power of 4 by y this is the error that we have here what will we do so we will find out what will happen plus zero to one ex square minus now if we substitute this then it will come 1 minus x x^3 by 3 minus x^4 by 12 done okay right this will become we can calculate it so it will come plus now I am doing it directly see x cube by 3 will become minus x plus x square by minus x cube by 6 minus x^4 by 12 plus x ^5 by 60 if we calculate this then it will come 1 minus x x^2 by 2 minus now this x two is coming so y x cube and y x cube if we see this then y will come plus will come plus x cube by s minus x to the power of 4 by 12 plus x to the power 5 by 60 we got this so this is our error now what is the error in this, second third it is error y3 x minus what x so if we minus y from this so look minus x is the same till here and this power of x will be left we have x by s minus sorry plus x^3 by 6 y ok so if we minus y3 then we will get this value so in this way we will keep finding out errors and our approximants and so we can solve this very easily by taking linear term integrating factor we will take ok so the integrating factor of this is to the power of x its quote is so the power of x is its integrating factor so if we solve this then the solution that we get is x minus 2x p 3 - 2e to the power minus x this will come out so this is its exact This



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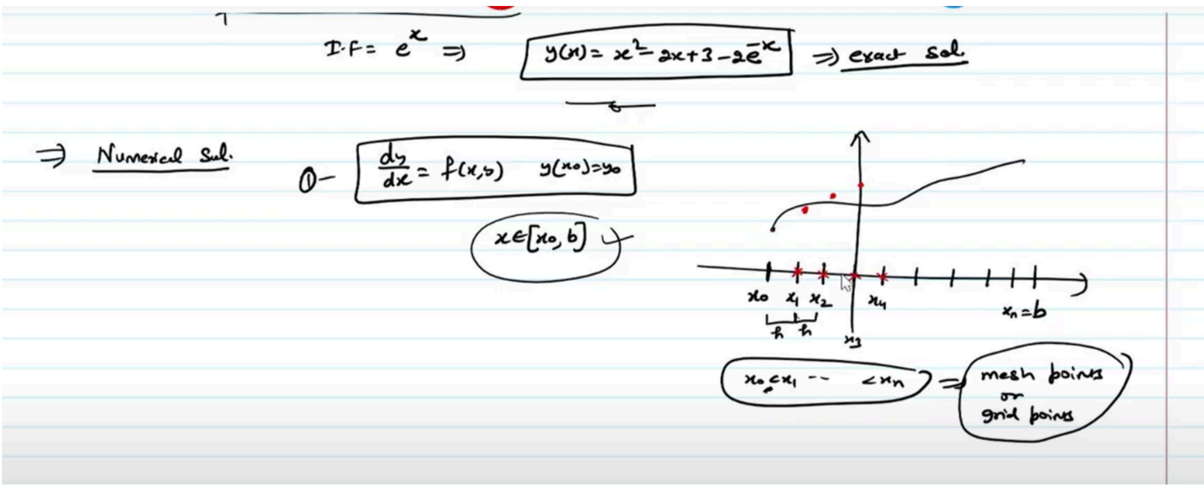
is the solution and this was the exact solution and these approximations are the approximations at the solution and they converge like this. So the method that we have done is a logical type analytical method in which we get a successful approximation solution. We are getting a polynomial in this case. Now our next main task is to calculate this by using the numerical derivative that we had taken earlier. So our main purpose in this is that we have done it till here. Now our next task is numerical solution. Numerical solution means we have this equation dy/dx = f(x,y). We have to solve this. Now see, if you look at the solution, this is a differential equation. Its solution starts with x . Here, there can be one or zero. Anything can happen if it is with respect to time. So t will come from 0 but x can be of any value so we have taken our x0 we have given some value of y0 that is here any point after that whatever happens its solution can be anything. So the value of x depends on how far it is going so what do we do when we are finding the numerical solution we are assuming that our x belongs from x0 to some value b so what do we do convert it into a finite interval first. So what do we do now the solution will start from x0 and will go till b we have defined this we are not taking it till infinity we will go till any finite value after that what will happen our solution now we have to calculate now we do not know what the solution is so we have to calculate the solution numerically what will we do numerically first of all we will divide it from x=a to x = b from x=z to b This will become xv, this will become my exact, this will become x3, this will become x4, like this and in the end the x a will become b, so what did we do, we divided it in equal partitions and we gave value to the spacing between them which is equally spaced. Okay, so now the value that we have is xz lesson x b lesson x a which is this, we name it, we call it match points and grid points, which we call nodal value till now in numerical integration in difference, now because it has been related to differential equation and this is the domain or the domain of this differential equation, if we are partitioning it, then we name it match points or grid points. So what do we have to do now, our main purpose is that now we will calculate it in every step, what are we doing on value, we have given value here, now we will do value in approximations here, with the help of this we will calculate here It can be any value, we don't know what it is. So what we will do is we will keep moving beyond each match point and keep calculating the values and after that we will combine the values together and then we will have an approximate solution. So it depends that in each interval, in each iteration we have to



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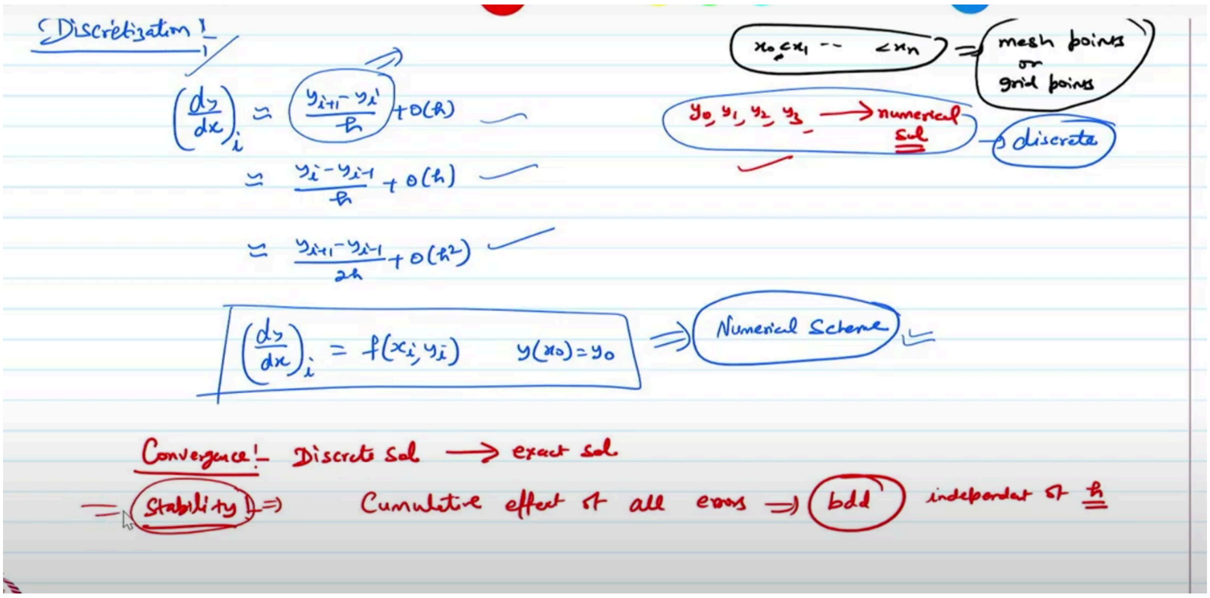
find the solution. So first we have found it at x0, we know that. Now our purpose will be to find it at x1, then we will find it at x2, then we will find it at x3, then we will find it at x4. So the solution that we have will come at these mesh points. Now we don't know what the solution is in between, but we will calculate the solution above the match points. And h is the mesh wide. So you know that if h is very small then the approximates will be that much y0,y1,y2… all right, so we will do this Let's say how to calculate this on numerical scale, how do we calculate this, so the method for that, we have some things in it, there is a concept that we call discretization, what does discretization mean now if we see in this initial problem, then our function in it is y x, which is a continuous function but the solution we get here is discrete, so what does it mean that we have to write our initial problem in discrete form, so this process is called discretization process, so what happens in discretization, our derivative is dy/dx, okay, what will we do, if we try to write it on a rectangle grid point, then that will be equal to now suppose we write it like this y(i+i) -yi by h plus order of h, so we know that we did that the derivative can be calculated by the forward final difference operator, we have read this earlier, I will call it backward I can also do it with the help of this and we had seen that this order of h is the numerical difference of approximants so with its help we can find out the derivative and this is the derivative or the value of the derivative is at the point i, okay and with our right difference we will discretize the derivative into approximants with the help of this derivative, what did our function on the right side become, then it became $f(x_i, y_i)$ okay and $y(x_0) = y_0$ we already know it, so the starting point is always known, okay so now we have this one way with the help of discretization and this is our process, we will call it that our numerical scheme will be formed depending upon how we are doing the derivatives with approximants or we can do it with something else as well, okay so with its help we will have numerical schemes which we will discuss, different schemes will come depending upon which derivative we applied and Depending on what is its order of accuracy, so we will also calculate its order of accuracy and if the solution is non-existent, then we can also find out the error, so after this is done, then in whatever numerical scheme there will be, we should always try to calculate some terms, first of all we will check the convergence, so what is the meaning of convergence that Our discrete solution should converge to the exact solution because in our discrete solution we have taken 0 by taking one h, then we reduced h by s, then our n came, then reduced it further, then new value came, so is our solution going towards the exact solution or not, this is what we call convergence and the next thing is stability, what does stability mean in this, now see, the solution that we found, first we found the solution, y at this point, on this punt, there was given, then we found the solution here, that means this solution is found at this point x, now from here, I found the y on x_0, so some error might have come in it, okay, there might be an exact also, there might be an error, using this, then I found this value, so in that, if there was an error in y, then there will be an error in y_2 as well, okay, there will be an error in y_3 as well, so the error is like this which If our steps evolve in it, then the error will evolve in the next step as well. So stability means that it should not happen that the error grows very fast and the solution is unbounded. So stability means that our error should remain bounded. So what does it mean that all the errors, the Cullet errors, should remain in bounds and should not grow further and should not go towards infinity. So we call this Cullet stability. So we can say that the Cullet effect of all errors should be bonded, unbounded or independent. What is our h? It should be bounded, independent of a point.



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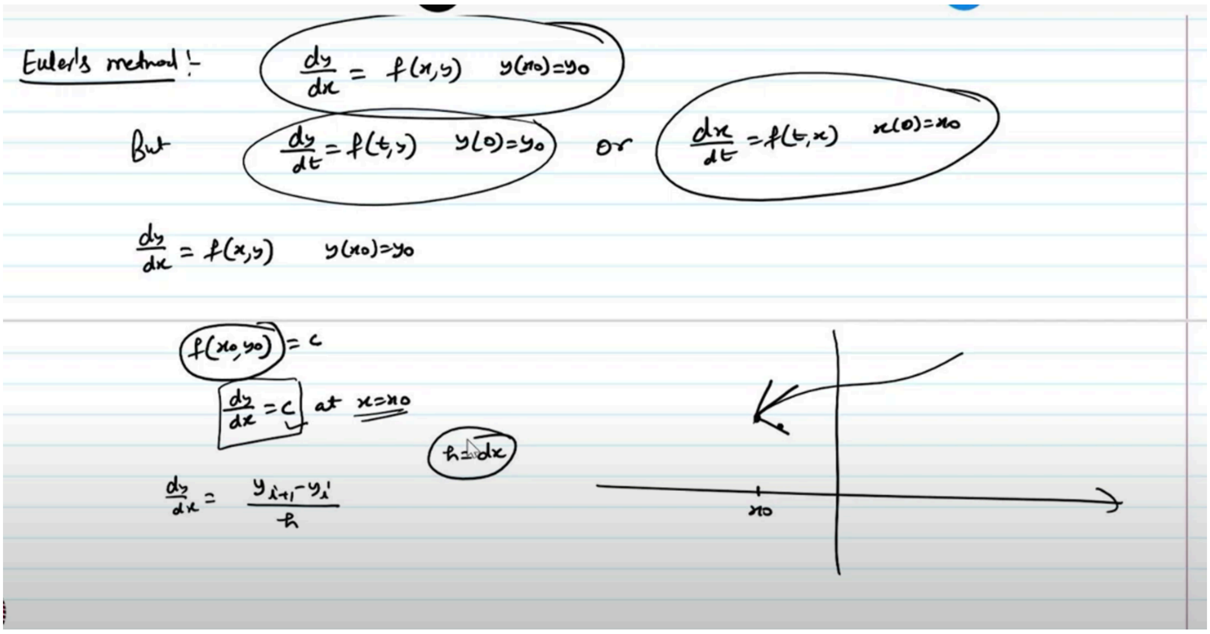
We should take it or take .0 or .001 because we will keep decreasing it. So, definitely, if we are decreasing it, then the Cullet effect of the error should remain in bounds and should not become unbounded. So, this condition is called stability. We get three things, one is discussion, one is convergence and one is stability. We keep these three in mind. So there is also discussion error because we have made discrete from continuous. So an error occurs. We call it discussion error. We already have convergence and stability. So now our main purpose is how to calculate it. So now we will discuss the first method and that method is called Euler method. What does Euler method do? Now the function that we have is the equation, which is body x is equal to y0 at x0 . So now we have taken the equation in x, but there are some equations in the form of dt, in the form of time, so we take it from zero. Basically, the time from where we started, we consider it to be zero. So it will also be given like this at many other places,dy/dx. So it can also be this, this can also be this, this can also be something. There are different givens in some books, some books have different givens, so depending on which notation we are using, okay, but our main purpose is to solve the initial problem, so look what are we doing in this case, now suppose we have a solution, this value is some x, here we have a value given, so if we look a little mathematically, we have this equation dy/dx1, at this point, if we look at x, then what will be the right hand side function, x will get some value, this is a value, suppose this value is s, if I write that dy/dx excavator and what is this body, this is the slope, the slope of the derivative at the point x, its value is s, so from here we will coindeme to know that the derivative of the solution here, the slope of the derivative, what is it, so if the slope is coming to one, then it will go like this If the slope is minus w then it will go like this, okay, if the slope is f then it will go like this, so depending upon the value of c, our solution here will be the slope, it will tell us in which direction we have to go, okay, so depending on c, so suppose our solution, whatever our c was, suppose this was also or this was, so what will we do, if we go here, then we will go in this direction, then what will we do after that, we will find out the next point, so now see what are we doing in the Euler method, which is our derivative, so what am I doing with it, forward from de approximants, I have written it y(i+1)- yi /h, that will be basically dx, so the equation that we have formed becomes f(t,x) by a equal to the right hand side, so we have put the value, okay and y at x is given to us, so if we see from here, what has this y become, it has become yi Plus h f(xi, yi) So we have a scheme and we call it Euler method.



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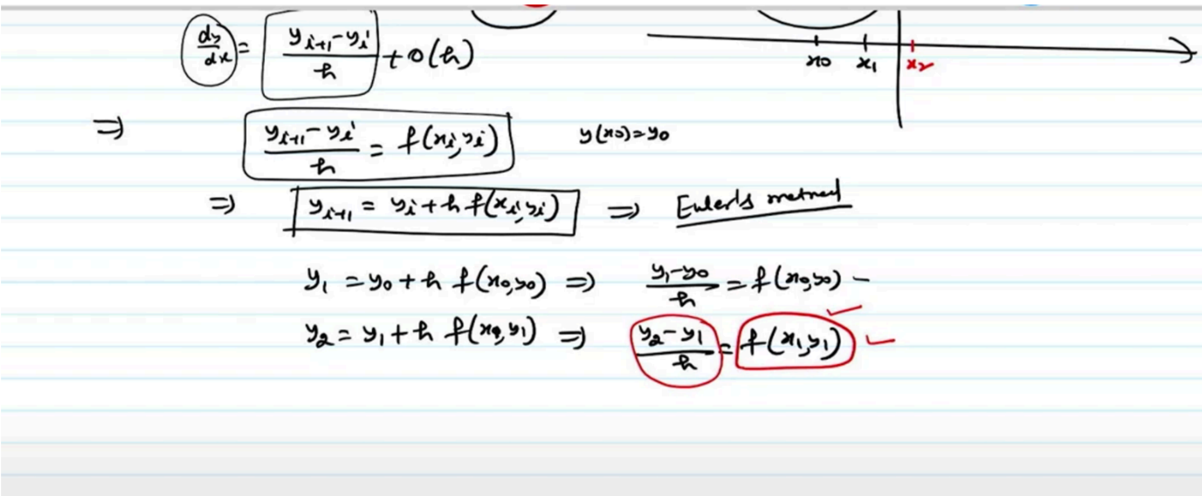
Okay, this is Euler method. Now look at the derivative, we have calculated the approximations from it. Now see how I will calculate it. Now what are we doing? If we see from here, what have we done? What is this? The slope of this function, we have made it equal to the slope of a code. So what does it mean? So x is y. So whatever value comes on x y, suppose any value y . So the value that comes, this is the value of the code. This value is y. If we see, what will happen if I put y as zero here, then it will come out. What will happen? Was plus h at x , so what will come out from y. Wow, minus van by a if f x new, so whatever value comes on f x new, we will go in the same direction and reach here and we will give it a value and name it. If this is it, then we will give it y. So this was our y ok so if we see then a tent is formed naa we have a right angle triangle and if we see the slope in it then this minus this by a so this will be formed and it is equal to this so it depends on which direction the slope is going initially so we will get a next point of the solution now we will calculate what will it be y1 -y0 by h sorry y so what has it become we have y 2 minus y1 by h = f(x,y) wav now we have the value yi the value which came y we will now substitute it in the function so now by putting it from there we will know in which direction the slope has to go so suppose here the value which is going in this direction or going a little up so what will we do, if we go like this then we will go till here this is ours suppose exact so what has come here the value of t has been found out we have okay that right angle triangle will be formed here so 22 minus 22 by a So here comes the slope and we have made the slope equal to it. So in every iteration, we are using the value of the function given to us and we are using our derivative, which we have, basically by the mean value theorem or approximations with the help of a code, which we have, basically by its value. So we keep doing it forward like this and we will get our solution, so we will keep going like this and the solution we have will keep coming, so we will calculate it with the help of this and see what we are doing in this



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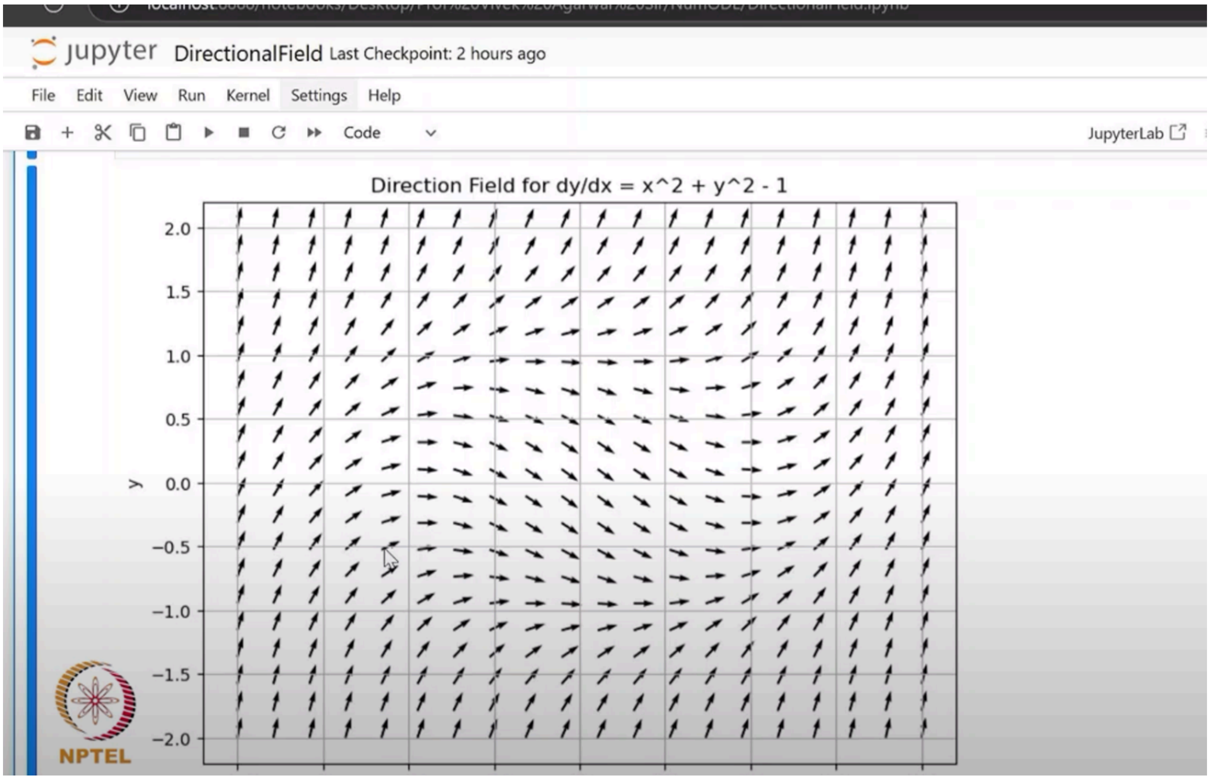
if we see this code, then what have I done, we will discuss it now, but now I have created a direction field, that if we take any differential equation, then with the help of its direction field, we can check whether the solution is going in which direction or not, so suppose here I have taken the equation y ok so we took this differential equation and I tried to calculate its direction field so when we plotted it, it came out here see the direction field is x s p s w a s m so from here we will get to know where our i point is suppose here is our initial point well it will not work on this that's okay so what will we do here depending upon the values we have so now what can we do see these arrows these arrows right hand side vector which was f x va we will put the values of x and y and the slope will come there so you see how the slope is changing here so if our initial condition is here then our solution will go like this if our initial condition is here then it will go like this if our initial condition is here then it will go like this so depending upon where our initial condition is coming we will get a unique solution and the value of slope We are taking it with the help of a function, so we took it with the help of the function on the right hand side and we got the slow values, so according to this, we need to know from the approximants field how the solution of this differential equation will behave. So without knowing the solution, what we are doing is that we do not know the solution but we can check its behaviour by calculating the direction field, then we will apply our method on it and we will solve it.



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So what we did today was that we solved a first order initial problem by approximation method and after that we started the Euler method and from the direction field, we got to know how the solution will behave. So I hope you must have understood this lecture and thank you for watching this lecture. Hello