

SCIENTIFIC COMPUTING USING PYTHON
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Lecture No. 32

Welcome, everybody, to Scientific Computing in Python. So, in the last lecture, we started numerical integration. So today we will continue further from that. So let's get started. So in our previous lectures, we discussed Simpson one third, Simpson 3/8 trapezoidal rule. So today we will go beyond that, and actually, there is a rule called Newton-Cotes' Method. So what are Newton's Cotes methods? So Newton's Cotes method is related to what we discussed about interpolating polynomials. So we can write this as the interpolating formula all these formulas in with the nodal values what is the value? That is x_i are equidistant right or let's write it as equispaced. We will do whatever words we use. And equispaced and known, okay, then it is called Newton cotes method, okay, in which we already know all the nodal values and that is equispaced, so that method is called Newton cotes method. So what this means is that in all the methods that we have used till now, even in the trapezoidal rule, we already knew the nodal values. After that we brought out Simpson and we were aware of that too. Ok? So all the methods that we have used till now related to interpolating polynomials, all of them come under Newton's cotes method. Ok? So what does this mean? What do we do at Newton Cotes Methods? There is a function $f(x)$, we have to find its integration from x_0 to x_n , we already know that. Ok? So, we do that. We can write it as $kh(a_0y_0 + a_1y_1 + \dots + a_ny_n)$ in this general form. Ok? So if we write it in general form then you can see what this kh is? That is $b - a$ which is the length which is the interval length, so from there we get to know these values and a_i 's, $a_0 a_1 a_2 \dots a_n$ this is what we call weight function weights, okay it is called weights or numbers, so we can write it in this form in general form. Now what we do is that there are some properties. So what will be the properties of Newton's methods? Which will be the first one that if we take all the weights i from 1 to n from 0 to n , I have taken all of them. If I equate them then I will always come to the first. And the second one is that now I am representing the weights like this. Let's write that let us represent weights as a_i I am already writing it. I will put n here. n means n number of weights that will come to us 0 1 2 3 up to n , so here we have written n which will be the interpolating polynomial of whatever degree, if we represent it as a_i^n then what will happen, a_i^n will be the same as $a^{n-(n-i)}$, okay so what does this mean that the symmetry will remain. What symmetric means is that a^n_0 will be the same as a^n_n . Ok? Whatever a^n_1 is, it will be the same as $a^n_{(n-1)}$. So the one here and the one here will be the same. This one, this one will be the same. This way our symmetry will be formed. So these two properties satisfy it. So if we go back and do the calculations like we did in trapezoidal or in Simpson. So if you look carefully, we can verify it in the same way in trapezoidal also. So what happened in the for example trapezoidal rule? So in the trapezoidal rule, if we have only two points, suppose x_0 and x_1 , then in that we know that this value which we have, if I take x_1 from x_0 and take $f(x) dx$, then this value which we got was $(y_0 + y_1) / h$ by sorry by 2, so this will come out to be $h (y_0 + y_1) / 2$, this was the value. So if we look at it from here, then h then our b simply means $x_1 - x_0$. Ok? So this means the k here is one. After that, the a_0 that came became half and the a_1 that came also became half.

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$$\int_{x_0}^{x_n} f(x) dx = kh (a_0 y_0 + a_1 y_1 + \dots + a_n y_n) \quad \boxed{kh = (b-a)}$$

$a_0, a_1, a_2, \dots, a_n \Rightarrow$ Weights

Properties: (i) $\sum_{i=0}^n a_i = 1$

(ii) Weights represent weights as a_i^n
 then $a_i^n = a_{n-i}^n \Rightarrow$ Symmetric $a_0^n = a_n^n$
 $a_1^n = a_{n-1}^n$

For example (i) Trapezoidal rule (x_0, x_1)

$$\int_{x_0}^{x_1} f(x) dx = h \frac{(y_0 + y_1)}{2} \quad h = x_1 - x_0 \quad k=1$$

$a_0 = \frac{1}{2} \quad a_1 = \frac{1}{2}$



(7:50)

So the sum of the two became one and both became symmetric. Is it okay? So this is how we can find out the trapezoidal roots. The second one is Simpson 1/3 rule. So in Simpson 1/3 rule the values we have are suppose x_0 to x_2 . Ok? So our b , which I am taking, suppose is x_2 and a is x_0 and $f(x) dx$, so when we calculated this, we got Simpson's one Third Rule, when we calculated this, we got that $h/3 (y_0 + 4y_1 + y_2)$. Ok? So now we have to see what will happen if we write it in this form? This is our form. Ok? So what will we do with it now? Now look, this is x_2 into x_0 . So that means if I write $2h$ it will be $x_2 - x_0$. Ok? So here I multiply it by two and divide it by two. So this comes to $2h$. Now if I write here $(y_0/6 + 4y_1/6 + y_2/6)$, this is what we have. So if we look at this, the a that we have has become two. A_0 is $1/6$, A_1 is $4/6$ and A_2 is $1/6$. Now see, if I make these three equal, I will get one, and both are symmetric about the middle values. Both will be symmetric about the middle value. So this becomes our Simpson 1/3. Similarly, Simpson 3/8. So, what will happen in Simpson 3/8 rule? In the formula that we have, our values will be from x_0 to x_3 . $f(x) dx$ and its value which we calculated that Simpson which is $3/8$ came out to be $3h/8 (y_0 + 3y_1 + 3y_2 + y_3)$ came out to be this. Is it okay? So now if we look at this, we have $x_3 - x_0$ which is $3h$. So $3h$ is already there. This means I can write it like this. $3h (y_0/8 + 3y_1/8 + 3y_2/8 + y_3/8)$. Here we go. So K here is 3. The a_0 is $1/8$. a_1 which is $3/8$. a_2 is also $3/8$ and a_3 is $1/8$.

(ii) Simpson's $\frac{1}{2}$ nd rule

$$\int_{x_0}^{x_2} f(x) dx = \frac{2h}{3} (y_0 + 4y_1 + y_2)$$

$$= 2h \left(\frac{y_0}{6} + \frac{4y_1}{6} + \frac{y_2}{6} \right)$$

$b = x_2$
 $a = x_0$
 $2h = x_2 - x_0$

$k=2$ $a_0 = \frac{1}{6}$ $a_1 = \frac{4}{6}$ $a_2 = \frac{1}{6}$

Simpson's $\frac{2}{8}$ nd rule

$$\int_{x_0}^{x_3} f(x) dx = \frac{3h}{8} (y_0 + 3y_1 + 3y_2 + y_3)$$

$$= 3h \left(\frac{y_0}{8} + \frac{3}{8}y_1 + \frac{3}{8}y_2 + \frac{1}{8}y_3 \right)$$

$x_3 - x_0 = 3h$

$a_0 = \frac{1}{8}$ $a_1 = \frac{3}{8}$ $a_2 = \frac{3}{8}$ $a_3 = \frac{1}{8}$

(11:35)

So if we sum up these three or four then we will get one and it is symmetric. So both the properties of Newton cotes methods are being satisfied. Ok? So in this Newton Cotes method, the only advantage we have is that the nodal values are already known. Now the next method that we have is the Method of Undetermined Coefficient? So, what is this Methods of undetermined Coefficient? So, look here, what we're going to do is here we have a quadratic formula. Meaning of quadrature formula. Look, if we look a little bit, let me write it down here, what does quadrature mean? So if we look at quadrature, it is an engineering term. So this means the process of determining the area in a plane. Ok? Is called quadrature. So we call this process quadrature. This is used in engineering. So what does this mean we are doing? We are also finding the area. Ok? So basically all the methods are quadratics which are finding the area even in Newton cotes method. So that is also a quadrature method. Ok? So what are we doing here? Here we are designing quadrature formula according to the prescribed specification. Ok? So what do we have in this now? If we have $K + 1$ parameters. To satisfy her, like what did we have in the past? For example, Simpson had seen that up to cubic, this is good, the error that we had there was zero up to cubic. Ok? We did the same in trapezoidal order and from there we came to know that this method is good for linear functions. Ok? Simpson 1/3 gives good results for quadratic as well as cubic and Simpson 3/8 also gives good results for cubic. Ok? So now what do we have to do in this? So what we will do in this is that specification so suppose we will have in this case suppose we have like this ah I write a to b $f(x) dx$ and put it here I put $(\lambda_0 y_0 + \lambda_1 y_1 + \dots + \lambda_n y_n)$ where y so we always know that y is a function of x okay? So we write it like this because now I write y because generally the data that we have is given in the form of x and y . Is it okay? So the data will never be given in the form of x and f . So if this is the case then this condition of ours is always satisfied. Now look what has happened to us is that if we look here, we don't even know what λ_0 λ_1 , λ_2 We don't know any value. Ok? So we have to find this out. After that we don't even know the nodal value what was x_0 ? What value is x_1 ? What value is x_2 ? What value is x_n ? We don't know this either. Ok? So what will happen in this case? If we look at it, we have $(2n + 2)$ which are unknowns. Is it okay? So, many unknowns will come to us. Why? because what did we have both y_0 x_0 then x_0 itself we don't know what will happen in this case. What will be x_1 ? What is x_2 ? We don't know anything. So let us see how to solve this. Ok? So let's try. So now we are assuming we are taking case one. In case one we take $n = 1$ i.e. Ok? So what does $n = 1$ mean? That the

values that we have, we have a function given to us from x_0 to x_1 means linear and we are writing those values here. We have the values $\lambda_0 y_0 + \lambda_1 y_1$ and they are the same. We can write it like this also. Because now we will have to use things in it. So I'll do a little bit about this because now we're doing it on the base. $\lambda_1 f_1$ These values are given. So what is f ? This is it. Let me take this so that there is no confusion. Ok? So what am I doing now? I am writing it like this $(\lambda_0 f_0 + \lambda_1 f_1)$ And this is how we know that y is our $f(x)$. Now look, we have to calculate the values in this case. Now we have to find the values λ_0 and λ_1 . Ok? That's it, let's take this, we are assuming that in this case, I will take it, let's try to direct it like this. The work we are doing right now, this is the supposition, I have written it down in this type. Ok? And I try to calculate the value of λ_0 and λ_1 . We are only going this far right now. So what will the Method of Undetermined Coefficient be? We started a formula with what we have. So whatever values we have, in this case, I am not starting it right now. I am giving an example that the method of undeterminable coefficients means that this coefficient is $\lambda_0 \lambda_1 \dots \lambda_n$, okay? We have called it unknown. That's all we're doing right now. Ok? So we have to calculate this and find it out. So how do we find this out? We suppose this is given. So let's say we have two values. x_0 to x_1 okay? So in this case, I am taking it. Now we have two values λ_0 and λ_1 that we need to find out. So what do I do? First, I take for $f(x) = 1$ a constant. So what did I do for Constant? This is written as x_0 to x_1 and $1 \cdot dx$, this comes to $x_1 - x_0$ and what is that? h and should be equal to what should that be? Now f_0, f_1 I have considered it to be one, so this will also be $1 + \lambda_1$, this will also be one, so it means this has come to us from here $\lambda_0 + \lambda_1 = h$, this has come to us, okay now what did I do after that, I considered $f(x) = x$, so from here if we see, what will come, $x \cdot dx$, so this becomes $x^2 / 2$, so $(x_1^2 - x_0^2) / 2$, this will come. Is it okay? So these values will come. This is equal to now this λ_0 now f is our x so this x will become 0. Why? because what is f at x_0 ? x will be 0. This is it, what will happen to the first one and the second one? $+\lambda_1 x_1$ So from here, look, the rule we have is $x_0 \lambda_0 + x_1 \lambda_1 = x_1^2 - x_0^2 / 2$. I can simplify this a little. I can write $x_1 - x_0$. I can write $x_1 + x_0$. y_2 here it is. Ok? So if we look at it basically, this is h . So we can write $h/2 \cdot (x_0 + x_1)$ and there we go. So now we have two equations. One is this and one is this. Ok? So now we needed two equations. We cannot go any further than this.

Method of Undetermined Coefficients! Here, we design a quadrature formula according to the prescribed specification.

$$\int_a^b f(x) dx = \lambda_0 y_0 + \lambda_1 y_1 + \dots + \lambda_n y_n$$

Unknown

$$\int_{x_0}^{x_1} f(x) dx = \lambda_0 f_0 + \lambda_1 f_1$$

$y = f(x)$

for $f(x) = 1$

$$\int_{x_0}^{x_1} 1 dx = (x_1 - x_0) = h = \lambda_0 \cdot 1 + \lambda_1 \cdot 1 \Rightarrow \lambda_0 + \lambda_1 = h \quad \text{--- (1)}$$

$f(x) = x$

$$\int_{x_0}^{x_1} x dx = \frac{x_1^2 - x_0^2}{2} = \lambda_0 x_0 + \lambda_1 x_1 \Rightarrow x_0 \lambda_0 + x_1 \lambda_1 = \frac{x_1^2 - x_0^2}{2} = \frac{(x_1 - x_0)(x_1 + x_0)}{2}$$

$$= \frac{h}{2} (x_0 + x_1) \quad \text{--- (2)}$$

So what we are doing in this is that we don't know what values will come and how. We are just using this thing depending upon how many unknowns we have. So in this case there are two unknowns. So the unknowns that will come in this will be $(k + 1)$ unknowns. Ok? So from here we will solve both of them. So we will calculate its values. We can easily calculate λ_0 and λ_1 . So we saw that from here our λ_0 which comes out is $h / 2$ and λ_1 which comes out is also $h / 2$. So from here we see that our function becomes $x_0 x_1 f(x)dx$. Now look, if I put the values of λ_0 and λ_1 , then we get $h / 2$, f_0 and f_1 and if we see, this is the same as the trapezoidal rule. It has become the trapezoidal rule, so it means the coefficient, which is the method of indeterminate coefficient, even if we go by that, even then if we have two points, then we will get the trapezoidal rule only. will go into it. Is it okay? So this becomes the trapezoidal rule. Now we have to see the error. So what should we do for the error? We found out that this method will give us the exact solution up to a linear polynomial. So this means if we go beyond that then the error will develop. So we'll see, so let me take a reminder here. R is a reminder. Where is he from and where is he from? x_0 to x_1 , $x^2 dx$ Look, now in this if I consider the function $f(x)$ to be x^2 , then error will definitely be generated. So if we calculate this minus λ_0 and λ_1 then we already know it. so $h/2$ is ok? and what will be f_0 ? This right here is x^2 , right? So x_0 squared is going to be plus x_1 squared is going to be. So this is what we will calculate, if it was linear then this value would have been zero or if it was constant then it would have been zero, but in quadratic, this value will not be zero. Ok? Why? Because this error will develop. So if I solve this, you will see that it becomes we have $(x_1 - x_0) / 3$ will come from here and $-h/2(x_0^2 + x_1^2)$ will come and if I simplify it a little bit and make it direct then this becomes $h^3 / 6$, so this reminder term has come. Ok? So now if we have to find out the truncation error, what will it be? The reminder term is divided by 2 real second derivative, we already know it. So we can take it out like this also. Is it okay? So this is the error and this function that we have, if we have the function given, then the second derivative because we have already done this. In trapezoidal also, we know from interpolating polynomials that such errors arise. So this will be a typing error. So if we see in this case, it will become $-h^3 / 12$. f second derivative z and z is between x_0 to x_1 so this is our truncation error and this was a reminder. Ok? So we calculated on this basis. So this turns out to be exactly the trapezoidal order rule. So we just had to do one.

The image shows a handwritten derivation on a digital notepad. At the top, the function $f(x) = x$ is used to derive the trapezoidal rule. The integral $\int_{x_0}^{x_1} x dx = \frac{x_1^2 - x_0^2}{2} = \lambda_0 x_0 + \lambda_1 x_1$ is equated to $\lambda_0 x_0 + \lambda_1 x_1 = \frac{x_1^2 - x_0^2}{2} = \frac{(x_1 - x_0)(x_1 + x_0)}{2} = \frac{h}{2}(x_0 + x_1)$. This leads to $\lambda_0 = \frac{h}{2}$ and $\lambda_1 = \frac{h}{2}$. The trapezoidal rule is then written as $\int_{x_0}^{x_1} f(x) dx = \frac{h}{2}(f_0 + f_1) \Rightarrow \text{Trap.}$

Next, the error is calculated for $f(x) = x^2$. The error R is given by $R = \left[\int_{x_0}^{x_1} x^2 dx - \frac{h}{2}(x_0^2 + x_1^2) \right] = \frac{x_1^3 - x_0^3}{3} - \frac{h}{2}(x_0^2 + x_1^2) = -\frac{h^3}{6}$.

Finally, the truncation error (T.E.) is given by $T.E = \frac{R}{2!} f''(\xi) = \frac{-h^3}{12} f''(\xi)$ where $x_0 < \xi < x_1$.

You can do it similarly if you want, then similarly we can find now I can calculate this also from x_0 to x_2 $f(x) dx$, I can do this and I will write here $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$, this will come out, so now we know that the variables that we have are three variables. So this means we will have to choose $f(x)$, choose x , choose x^2 and if you see the method that will be formed from what we have, then Simpson one third rule will come. Ok? So we can calculate it like this. So if this type of question comes then Method of Undetermined Coefficient, if I take any example. Let's take an example. In the example it is written find a , b and c true is that correct? a , b and c true that this is given. We are given the integration from 0 to h $f(x)dx = h (a f_0 + b f_1/3 + c f_2)$. We are already given that if we take the integration of any function, it will be equal to it. Ok? So we have to find the values of a , b and c . Now look, we have three coefficients. There are three unknowns. So this means we need three equations to solve this. Ok? So what will be the solution now? We will have to do the same in the solution. First we will take $f(x) = 1$, then I see in that, $\int_0^h 1 dx$ from 0 , so we get h , this should be equal to the right hand side, what is on the right hand side, $h a$, now see, this is one, so it will always remain one, so here it will come $a + b + c$, so from here, if we see, we get this equation, $a + b + c = 1$, the first equation is obtained. Why? Because we assume that the function is one. Because we know that these three variables, meaning they are constant, linear and quadratic, we will get exact results. If error occurs then the error will start in cubic. So it came to us. Now we have taken $f(x) = x$, so this is $\int_0^h x dx$ from 0 . So our integral of this will be $x^2/2$, so if we see, this will come and what will be this equal to h on the right side a is now x and f_0 is x , so what will happen at 0 , $0 + b$ at $h/3$ this will come to $h/3 + c$ and at h this h will come. Ok? So if we calculate from here, if it cancels out with h then it will come to $b/3 + c = 1/2$. Is it okay? So let me make this a little bit easier. So it will come to $b/3 + c = 1/2$ it will come to this. This second has arrived. Now we have to take another one. That is $f(x) = x^2$ so what happens in that case? x^2 will become dx . So if you look here, this will come, $h^3/3$ will be left and will come to the right side, $h a$, okay? So zero will remain zero. Plus b , this will come to $(h/3)^2 + c h^2$ because h was on and from here if we see, h^2 , if I take the common h will come, then look this will come to $b/9 + c = 1/3$, this second and third equation has come, so we have 1, 2, 3 three equations. So, there are three equations left. There are only three unknowns. So, if we solve this, then I will see by doing this calculation. So, it will come to $a = 0$, $b = 3/4$ and $c = 1/4$ so this means that the weight of a has no contribution. b and c contribute to this. Ok? So from here we have our calculation.

2025-03-10 - Integration Numerical (einde fflnside 1 - 7 / 6 - Scable Ink)

Exp find a, b and c st $\int_0^h f(x) dx = h \left(a f(0) + b f\left(\frac{h}{2}\right) + c f(h) \right)$

Sol

$f(x)=1$ $\int_0^h 1 dx = h = h(a+b+c) \Rightarrow \boxed{a+b+c=1}$ ✓

$f(x)=x$ $\int_0^h x dx = \frac{h^2}{2} = h \left(a \cdot 0 + b \cdot \left(\frac{h}{2}\right) + c \cdot h \right) \Rightarrow \frac{b}{2} + c = \frac{1}{2}$ ✓

$f(x)=x^2$ $\int_0^h x^2 dx = \frac{h^3}{3} = h \left(a \cdot 0 + b \left(\frac{h}{2}\right)^2 + c \cdot h^2 \right) \Rightarrow \frac{b}{4} + c = \frac{1}{3}$ ✓

$a=0, b=\frac{3}{4}, c=\frac{1}{4}$

(33:30)

Now someone will ask us how much error is there in this? Ok? So there will be truncation error in this, so we have gone up to seconds in this. So this means there will be problem in cubic. So the error that will occur is reminded by $3!$ and the third derivative. So z will be that we know x will be between 0 means 0 to h . What will happen to r now? r in this case would be 0 to the h , $x dx$ minus Now that we are left with the formula, what is the formula? h now a was zero. What b is $3/4$. So $3/4$ is the power cube of $h / 3$, right? What is plus C ? Power of $1/4 h$. Why? Because the function is our cube. So the difference between these two will be our reminder term. So if we calculate here then the error that is involved in this, the truncation error that is involved in this will come to us. Ok? So this is how we will calculate it. So this method is the method of unknown coefficients. are undeterminant coefficients. Now what we have is the main next thing, now look, the polynomials that we calculated, the polynomial that we used here was $1, x, x^2, x^3, x^4$, this one. And if we look at it, these polynomials are the standard basis that we have for polynomials. If we form the vector space of polynomials of degree n . Is it okay? So in vector space we know that if polynomials are forming a vector space of the polynomials of degree less $\leq n$ then their standard bases are $1, x, x^2, x^3$ and so on. So, the work we have done till now is on a linearly independent basis, so it is obvious that they will also be linearly independent. Now, in the next thing we are going to do is to see whether apart from linear independent we can add some extra thing, so for that we have got the method, so what we will do now is we will use an orthogonal polynomial. So this means that the polynomials we have will be linearly independent. In addition to that, they will also be orthogonal. Ok? So we are going to use this. So the name for its methods is called Gauss Quadrature Methods. So Gauss first used the orthogonal polynomials to find out the quadrature. What does Quadrature mean? To find out which area it is. So what happens? What is the Gauss Quadrature Method? When the needs are also to be determined then the method is called Gauss Quadrature Method. Ok? So what will happen in it now? So the first thing we're going to do is we're going to go to Gauss Legendre. Gauss under the quad method and we already know that the Legendre polynomials are orthogonal and the Legendre polynomials are orthogonal from -1 to 1 in the same interval. Ok? So the first thing we need to do in this is what will be our methods? First of all, our method is to take a weight function $w(x)$ function $f(x)dx$ and this integration will be from -1 to 1 and this will be equal to $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$ like this I am going and I will write $\lambda_n f_n$ now we know that in this case, the $x_0, x_1, x_2 \dots x_n$ is also unknown to us. Ok? So we don't know at what value f_0 is found out or

at what value f_1 is found out. This w is our weight function. This is always positive. Ok? And the x that belongs to us is obviously going to be -1 to 1 . So we have to use this in the Gauss Legendre. Now the question is that if we have any a to b and someone tells us to find this integral. Ok? So how do we find out in this case? Because if we have to find out the Gauss-Legendre, then we will have to calculate it from -1 to 1 . So if we have to find this out, then let us assume that we can take the weight function as one, we can take that, but what should we do with a and b . So what will we do, the interval a to b , we will transfer it to -1 to 1 . Okay, so we will convert it to -1 to 1 . So look, we will get x like this, it will come as $(b - a) / 2)t + (b + a) / 2$, like this, if I put -1 in t , then x will become a . If I put 1 in t then this will become b . Is it okay? So what does this mean? We will convert this. We will convert this interval from -1 to 1 and the function that will come here, the function that will come after getting changed, we will write it as $g(t)$ and here dt will become like this. Ok? So it will be converted and after that we will calculate it. So we can do this work easily. So now we have to see how to proceed with this. So we have done this. So let's see now. So we take different cases. So let me take case one first. What happens in case one is that suppose we want to find out the value of the function. $\int_{-1}^1 f(x) dx$ and that suppose given is $\lambda_0 f_0$ just these two values given so two values are given what does it mean in this there are two unknowns two unknowns in this λ_0 and f_0 okay so what does it mean that in the previous case we saw so if we take just one value λ_0 and f_0 in that only λ_0 was unknown but in this f_0 is also unknown. Why? Because whatever f_0 will be, x_0 is f_0 , and whatever x_0 is, is unknown in itself. We don't know what value to take. For x_0 we have to take -1 or that value in between. We don't know what to take. At first, we took x_0 to be -1 . That means we considered it as the starting point. But now in this case, we don't know. Ok? So we need two unknowns. This means we have to go to the polynomial which is the linear polynomial. So if we take $f(x)$ to be one, then from here this becomes $\int_{-1}^1 1 dx$, okay? So that x will become -1 to 1 so its value will become 2 . And what will be equal to? This λ_0 is just $f(x)$ then one, so one comes. So from here the value of λ_0 that we got is two. Now let's look at $f(x)$ for x . So let's look at $\int_{-1}^1 x dx$. Now look at x , if we pay attention, this function x is an odd function. And this is between 1 and -1 . So this is the area between 1 and this is the area between -1 . So if we look at it, it will become zero. Is it okay? So this value that will come will be zero and what will be equal to two? This $\lambda_0 x$ which is $f(x) x_0$ will come to x_0 . So from here we get the values $\lambda_0 x_0 = 0$. But λ_0 is equal to 2 . So from here the value that came out, x_0 , became 0 . So what does this mean? \int_{-1}^1 If we were to find the integral of a function, then its value would be $2 * f$ at 0 , so this means that x_0 is chosen as 0 . -1 has not been chosen in this case. Ok?

Gauss-Legendre Quadrature method!

$$\int_{-1}^1 w(x) f(x) dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n$$

$w(x) = 1$

$$\int_a^b f(x) dx \Rightarrow \int_{-1}^1 g(t) dt$$

$[a, b] \rightarrow [-1, 1]$
 $x = \frac{b-a}{2} t + \frac{b+a}{2}$

Case 0

$$\int_{-1}^1 f(x) dx = \lambda_0 f_0 \Rightarrow 2 - \text{unknown}$$

$f(x) = 1 \Rightarrow \int_{-1}^1 1 dx = [x]_{-1}^1 = 2 = \lambda_0 \Rightarrow \lambda_0 = 2$

$f(x) = x \Rightarrow \int_{-1}^1 x dx = 0 = \lambda_0 x_0 \Rightarrow \lambda_0 x_0 = 0 \Rightarrow x_0 = 0$

$\int_{-1}^1 f(x) dx = 2 f(0)$

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(45:31)

so what does that mean? that is any function we have this x_1 . This is a first. Suppose this is some function of mine. So our integral of this function is if I take out the integral, okay? Suppose our function is positive so we find its area. That area will come here. So how did we approximate it? We have approximated this. This value at zero is the value of the function f . Is it okay? So I took the value of f_0 and made a rectangle like this and now what will be the area of this rectangle? $f_0 * \text{length} * \text{breadth}$ so breadth is what we have now. What is the length? $1 - 1$ is 2 , so it means the area which has the area is $2 * f$, so we do it like this, we can do it, okay, so this approximation has come, so this approximation which has come has come from Gauss Legendre, we will do it next, what did we do in case two, $\int_{-1}^1 f(x) dx$, suppose I did it with $\lambda_0 f_0, \lambda_1 f_1$. Added one more thing. So now in this case, if you look, we have four unknowns. Which ones? $\lambda_0, \lambda_1, x_0, x_1$ because x_0, x_1 is known only then we can find the value of f_0, f_1 . So this four which we have is unknown, what does it mean that it should go exactly the same up to the cubic, means it should give the exact result up to the cubic polynomial. So if we go up to the cubic polynomial, so see what we are doing in this case, $f(x)$, if I take one of this, then I will take one, I will take -1 to 1 dx , so it will come to two and the value of that will come will be $\lambda_0 + \lambda_1$, so from here we have the first condition, $\lambda_0 + \lambda_1$ should be two, this is the first condition, after that if we take $f(x) = x$, so this comes to -1 to 1 , $x dx$, this comes to zero, we know. So this value will be $\lambda_0 x_0 + \lambda_1 x_1$ and its value is equal to 0 so here we have the second equation. Now we will also have to take $x =$ sorry $f(x) = x^2$. Ok? So -1 to 1 , $x^2 dx$ so if we take this k then it will come to $2/3$ and this will be equal to what? $\lambda_0 x_0^2 + \lambda_1 x_1^2$ OK? So from here we get the third equation. And we need one more equation, $f(x) = x$, so in that case we get -1 $x dx$. Now this x is also an odd function, so its value will be zero, and it will become $\lambda_0 x_0 + \lambda_1 x_1$, so from here we get one more condition. Ok? So these four, we get the equation 1 2 3 4 and what variable do we have? $\lambda_0, \lambda_1, x_0, x_1$ so we have to calculate and find out all four from here. Ok? So if we calculate this, then I am now directly writing that the values which will come here will come like this. From here the conditions that will come will be that λ_0, λ_1 should be the same. Ok? And the x_0 that comes will come as $+ - 1/\sqrt{3}$ and the x_1 that comes will come as $- + 1/\sqrt{3}$, these values will come and from here the values that we get will be $\lambda_0, \lambda_1 = 1$, see, we have solved this, okay, so after solving this, what we have, okay, when we calculated this, see we have got

the result $\int_{-1}^1 f(x) dx = f$ Now I suppose I am taking this $-1/\sqrt{3}$, if $\lambda_0 \lambda_1$ is 1 then it will come as $+ f(1/\sqrt{3})$, so what does this mean, we took x_0 as $-1/\sqrt{3}$, and the x_1 that we took was this. Why? You can take it the other way round too, but we will take it the other way round. So we know that x_0 is smaller than x_1 . So therefore we will choose the minus one and we will choose x_1 . So this formula of ours has come, this formula has come. Ok? So wherever there is any function, we have to take its value. Have to substitute and solve it. So see in this case, we also have the values of x_0 and x_1 . Is it okay?

Handwritten mathematical derivation for Gauss-Legendre quadrature with $n=2$ nodes:

$$f(x) = x^2 \quad \int_{-1}^1 x^2 dx = \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2$$

$$f(x) = x^3 \quad \int_{-1}^1 x^3 dx = 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3$$

$$\lambda_0 = \lambda_1 \Rightarrow \lambda_0 = \lambda_1 = 1$$

$$x_0 = -\frac{1}{\sqrt{3}} \quad x_1 = \frac{1}{\sqrt{3}}$$

$$\int_{-1}^1 f(x) dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$x_0 = -\frac{1}{\sqrt{3}} \quad x_1 = \frac{1}{\sqrt{3}} \quad (x_0 < x_1)$$

(51:34)

After that, if we want to see the error in this case then what will be the error? So what if I want to remove the error? R remainder term now it has gone up to QB so up to 4 degree so this means its error will have fourth derivative zero and z will be between -1 to 1 and R remainder term what will that be? $\int_{-1}^1 x^4 dx - [(-1/\sqrt{3})^4 + (1/\sqrt{3})^4]$ to the power of 4 this will come out. Ok? So if we calculate this, it comes to $8/45$. Ok? So what will happen to us in this error? It will be made, we will substitute it here. So I am directly writing $135 \cdot f$ to the power of 4 which is the derivative and z this will give the error. So we have calculated this. Like this we can define more methods. for $n=3$ I can also do it. What did I do in $n=3$? -1 to 1 taken. took fx , took dx . I wrote this as $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$. So you will see that in this case we have these six unknowns. So if six unknowns have come, it means we will have to go up to the polynomial of the fifth degree. So we will calculate it like this. For $f(x) = 1, x, x^2, x^3, x^4$ and x^5 we will have six equations. Ok? So if we do it in the same way, we will get six equations, and if we solve them, we will see that the x_0 that we have is minus this, the x_1 that we have is 0 and the x_2 that we have is this, and the λ_0 that comes will be $5/9$ and the λ_1 that comes will be $8/9$ and the λ_2 that comes will be $5/9$, so we will get six values of these six. Ok? So this means that we have the Gauss-Legendre quadrature formula with six numbers of unknowns. So, his values will come here.

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$$R = \int_{-1}^1 x^4 dx - \left[\left(\frac{-1}{\sqrt{5}}\right)^4 + \left(\frac{1}{\sqrt{5}}\right)^4 \right] = \frac{8}{45}$$

$$\text{Error} = \frac{1}{135} f^{(4)}(\xi)$$

for $n=3$

$$\int_{-1}^1 f(x) dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 \rightarrow 6 \text{ - unknowns}$$

$f(x) = 1, x, x^2, x^3, x^4, x^5$

$x_0 = -\sqrt{\frac{3}{5}}$	$\lambda_0 = \frac{5}{9}$
$x_1 = 0$	$\lambda_1 = \frac{8}{9}$
$x_2 = \sqrt{\frac{3}{5}}$	$\lambda_2 = \frac{5}{9}$

(54:43)

So if we look at this formula, it will become $\frac{5}{9} f(-\sqrt{3/5}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{3/5})$. Okay, so we have calculated these values. So give us any function, we will substitute it and find its solution. Ok? So we can apply these values directly. So if you look at me in this case, suppose I have an example, so before this I would like to tell you a little bit about the values that are coming x_0, x_1, x_2 or the values that came behind, what are those values? So if you look at a little bit of background, what we had was the Legendre polynomial, right? We find out Legendre polynomials in the solutions of these power series. Legendre equations are differential equations. From there we find out the Legendre polynomial. So these are the Legendre polynomials. $P_0(x) = 1$, $P_1(x) = x$, $P_2(x)$ this works out to be $\frac{1}{2}(3x^2 - 1)$ okay? $P_3(x)$ is $\frac{x}{2}(5x^2 - 3)$ like this. This is how we can calculate. So if you see, in the first question we asked, we went till x and what we got there was x_0 , which was zero. So you will see that 0 is its root. After that, when we went to the next step, we saw that when we went to the second step, the values we got were $1/\sqrt{3} - 1/\sqrt{3}$. So if you look from here, if I write and find its roots, what will come out? $3x^2 - 1$ and from here we have $x = 1/3$ square and that will come to $x = \pm 1/\sqrt{3}$ and these are the values we took. So this means that our nodal values that have come are its roots. Similarly, if I look here, what did I bring from here? $5x^2 - 3$ since $x = 0$ already. So from here, if you see the root of $x \pm \sqrt{3/5}$, then we get the roots $x = 0, x = -$ and $x =$, so you have these three roots and you have these three values. Look, this is what has come. So what does this mean? What basically are the nodal values that come up in the Gauss Legendre polynomial, which is our quadratic formula? Those are the roots of the Legendre polynomials. Ok? So similarly we can take x, p_4, p_5, p_6 and verify that the nodal values that will come in it will also be the roots of our Legendre polynomials in this case. Ok? So this is what we have now come to know that the nodal values in this case are unknown and unknown means not equally space and those are the roots of Legendre polynomials. Ok? So this remark is a very important remark that what are the nodal values x_i ? Roots of Legendre Polynomial, okay?

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$$= \left[\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \right]$$

⇒ Legendre Polynomial!

$P_0(x) = 1$

$P_1(x) = x \Rightarrow x_0 = 0$

$P_2(x) = \frac{1}{2}(3x^2 - 1) \rightarrow 3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm \sqrt{\frac{1}{3}}$

$P_3(x) = \frac{5}{2}(3x^2 - 1) \Rightarrow (5x^2 - 3) = 0 \Rightarrow x = \pm \sqrt{\frac{3}{5}}$

$x = 0, x = -\sqrt{\frac{3}{5}}, x = \sqrt{\frac{3}{5}}$

Remark! Node values $x_i \Rightarrow$ roots \leftarrow Legendre's

(59:25)

Now suppose we get an example, then let us see that if we have to find this example, then how will we do it. Now our question is to find or evaluate $\int_0^1 dx / (1+x)$, this has to be calculated. So suppose our function is $1 / (1+x)$. Ok? Now I have to solve this question using the Gauss Legendre polynomial. Ok? Find this using Gauss polynomial or formula which Cotes formula what do we need to do now? The first thing we have to do is to convert the interval $[0, 1]$ into -1 and 1 . Ok? So I apply a transformation. $t = 2x$ which is it I'm $t = ax + b$ okay? Now look what is this x that I have? 0 What is t when $x = 0$? So now we calculate this from here. Let us first give us the values of x . So when x is 0 , if we put $x = 0$ then what should b be? Whichever t comes out will be -1 . So b becomes -1 . Ok? And what will it be when $x = 1$? That will give $a + b$ and what should that be? There must be a first. So if we see from here, the values we have, a will come as 2 and b will come as -1 . This is a given. So this is the transformation that we have got. $2x - 1$ here it is. Ok? So, here we have this transformation and what is dt ? It's $2dx$. So, now look, what do we get after this formula $\int_0^1 dx / (1+x)$ gets converted? $\int_{-1}^1 dt / (t+3)$ because look, if I put 0 in x then what will be t ? - If I put 1 in x then $t = 1$ dx will be $dt / 2$ and what will come in place of $1+x$? This $(t+3) / 2$ will come in place of $1+x$ and this is what we have. And if I do this 2 by 2 then let's see what happens? $dt / (t+3)$ will become dt . So this means that we have to calculate our function. We converted it into this form. So now we know directly that it depends on which values we have to calculate. So our function $f(x)$ has now become ours, I will write it as $g(t)$. That becomes $t+3$. So if we have to calculate this $\int_{-1}^1 dt / (t+3)$. So suppose I do it with this, let me do it with three points, then it will come $\frac{5}{9} f(-\sqrt{3/5}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{3/5})$ will not come, g will come in it because I have written g and $\frac{5}{9} g$ off will come, so what has this become, our $\frac{5}{9} g$ off this, so it became $1/3$ - if I put this, I will put $8/9$, then it will come $0 + 3 + 5/9$ and it will come $1/\sqrt{3/5} + 3$ on this calculator and we calculated all of this, then our answer came 0.69312 , these values came, okay and

Example find $\int_0^1 \frac{dx}{1+x}$

$[0,1] \rightarrow [-1,1]$

$t = ax + b \Rightarrow \begin{cases} x=0 & b=-1 \\ x=1 & a+b=1 \end{cases} \Rightarrow a=2, b=-1$

$t = 2x - 1 \quad dt = 2dx$

$\int_0^1 \frac{dx}{1+x} = \int_{-1}^1 \frac{dt}{2(1 + \frac{t+1}{2})} = \int_{-1}^1 \frac{dt}{t+3} \quad g(t) = \frac{1}{t+3}$

$\int_{-1}^1 \frac{dt}{t+3} = \frac{5}{9} g(-\frac{\sqrt{3}}{5}) + \frac{8}{9} g(0) + \frac{5}{9} g(\frac{\sqrt{3}}{5})$

$= \frac{5}{9} \frac{1}{3 - \frac{\sqrt{3}}{5}} + \frac{8}{9} \frac{1}{0+3} + \frac{5}{9} \frac{1}{\frac{\sqrt{3}}{5} + 3} = 0.693122$

(1:05:07)

if we want to verify it, then if we see, $\int_0^1 \frac{dx}{1+x}$ is the value of $\log(1+x)$ from 0 to 1 and this value comes which Log is 2 and the value of $\log 2$ is 0.69693 147, this is the value. So if we observe, it is giving exact values up to four digits. Ok? So what we did was we did it with Gauss Legendre and directly we substituted the value and we got the answer. Ok? Ok? So, we can show this work with the help of the code that we have written. So this is the Gauss Legendre quadratic formula, so we have tried to define this function. Ok? We have defined the function $1/(1+x)$. The exact integral that has been found out. Then we have calculated it using Gauss Legendre form. Ok? I have taken four values of n. 1 2 3 4 it depends.

```

[1]: import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

# Function to be integrated
def f(x):
    return 1 / (1 + x)

# Exact integral value
def exact_integral():
    return np.log(2)

# Gauss-Legendre integration
def gauss_legendre(f, a, b, n):
    # Get Gauss-Legendre nodes and weights
    [nodes, weights] = np.polynomial.legendre.leggauss(n)

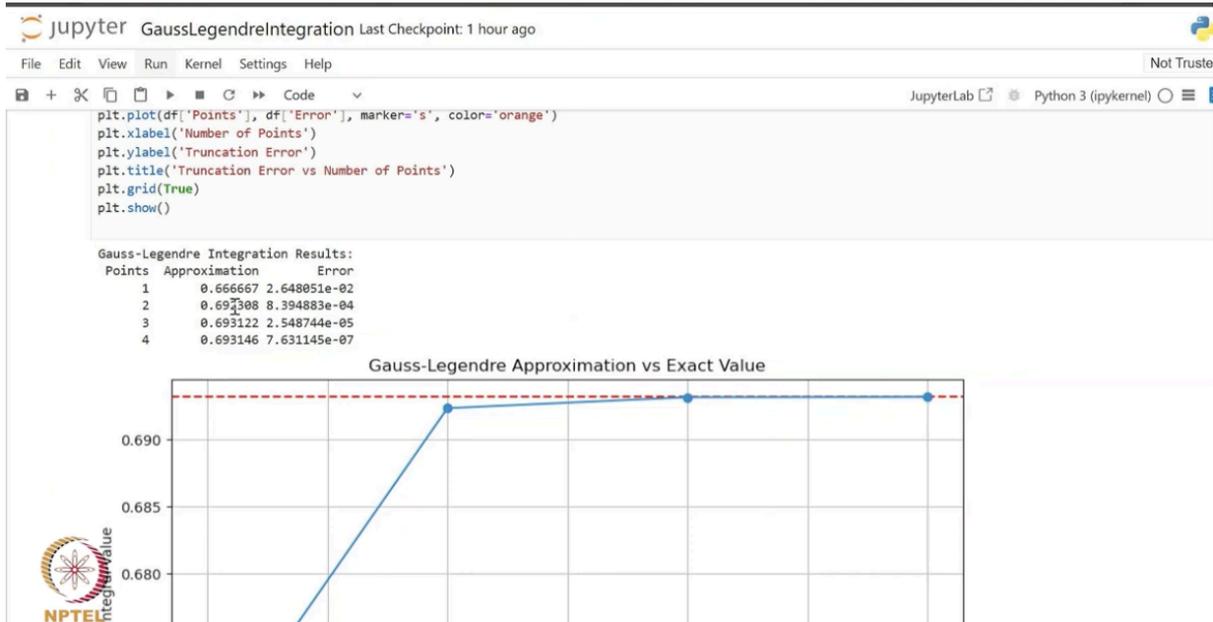
    # Change of interval
    t = 0.5 * (nodes + 1) * (b - a) + a
    integral = 0.5 * (b - a) * np.sum(weights * f(t))
    return integral

```

(1:06:16)

So we have to try and see what's coming for all four. And a is zero and b is one. We already know that. Is it okay? So we have done this in Python, so we have written the function go legendre. So what is needed in it? We need to input the value of the function. a b and n for how many n? For n1 then $\lambda_0 f_0$ for n2, so like this we will keep getting values. Ok? So after that we will calculate it and plot it for visualization. So let me run this. here comes the. So

look, these values came to us. What is this? So Gauss took under approximation. When we take n as 1 then this value will come. If we take n as two then this value came out. If we take n as three then this value comes. And this is the value we calculated just now 69312, okay? And the fourth one that is coming is 39693146. Ok? So it depends on how many mesh points we have taken out, which is n , and when we calculated it, see, first if we take one, it was coming out to be 0×0 , so for that there was this much error, then for two, that means we had $-1/3$ and $1 + 1/3$, two came out, so for that the error got reduced, for the third and for the fourth the error got reduced completely. So this exact value came to us, 69314, so this error got reduced.



(1:08:23)

We have got so many errors. Is it okay? And it's doing this truncation. So on this basis, on the basis of Gauss Legendre, we can calculate different values. Now I know that the main thing is to change the function and we will calculate its values between -1 to 1 . Ok? So this Gauss Legendre that we have done, we will calculate with its help and from there we can find out the Gauss Legendre which is the quadrature formula. And we have seen that the nodal values are basically the roots of the Legendre polynomial. So like this we have a lot of quadrants full. Ok? Chebyshev is Chebyshev methods, there are many methods, Gauss Hermite, but in our course, we just wanted to tell how we can use it, so that is why here we will go only till Gauss Legendre, but if we have to do something more than this then the students can automatically define Gauss Chebyshev, Gauss Hermite, many integration methods in this way. So I hope you have understood and liked today's lecture and thank you for watching this lecture. Hello.