

SCIENTIFIC COMPUTING USING PYTHON

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Lecture No. 28

Welcome, everybody, to Scientific Computers in Python. So today we are starting a new topic and it is called Numerical Differentiation. Given the data that we have, we know that if the function we have is differentiable, then we can take its derivative. But if we have discrete data then we will learn how to differentiate it. So let's get started. So what we want to do today is numerical differentiation. So numerical differentiation means that if we have suppose data given. So far we know that we can easily find out derivatives. We have some function given to us. Is it okay? We have a function, suppose x , then we have to take its first derivative. If we want, then we will take its first derivative. We'll take the second derivative of this. Ok? So this is how we can take the derivative. If $\sin x$ is then we will take its first derivative. We will take its second derivative. Like this we can take derivatives. We will take higher order derivatives. So these are all the things that we can do if the function is non. But if the function $f(x)$ is unknown and what we have is discrete data. We already know what discrete data means. Now what does it mean? The x that we have is the data given. $x_0, x_1, x_2, \dots, x_n$, okay, these are $n + 1$ points. Here we have the function $y = f(x)$ whose values are given. So I can write this as $y_0, y_1, y_2, \dots, y_n$ and I have written it. Ok? So y is f at x_0 , we are writing it as f_0 or we are writing it as y_0 . Now look at the thing, this data that we have is discrete data. Someone tell me to calculate f' at some point x . Using this data. So how do we calculate? Why? Because we don't even know what the function we have is. This is a discrete data given to us. So now if we want to do numerical differentiation, we will do it. So what are his methods? So in numerical differentiation, we use different types of methods. So the first method is the method based on interpolation. Interpolation means the interpolating polynomial is its base method. The second method is the method based on finite difference method and the third one is the method based on undetermined coefficient, these methods come from this. Is it okay? So the method is based on interpolation finite difference or undetermined coefficient. So, we use this. Ok?

DISCRETE am.

$$\begin{cases} x: x_0, x_1, x_2, \dots, x_n \\ y=f(x): y_0, y_1, y_2, \dots, y_n \end{cases}$$

$f(x_0) = y_0$

$f'(x)$

- ① Method based on interpolation.
- ② " " " Linear difference method
- ③ " " " Undetermined Coeff.

$f(x) = x^2$ $f'(x) = 2x$ $f''(x) = 2$
 $= \text{sine } f(x) \quad f''(x)$

[4:33]

So now basically what are we going to do in the first one? So let us take this first case. So in the first case, see what do we have to do now? What will we do in it? We'll use the interpolating polynomial that we have. So in the method based on interpolation we use interpolating polynomials that we have found out earlier. Now look, for example, let me take an example. Like we have two points. We have the values x_0, y_0 and x_1, y_1 okay? So in this we have two values given. So suppose there is a curve. There are no values here. So this is the first point. Suppose this is the second point. Ok? So I took this as x_0 and I took this as x_1 . Now what did we do? Approximated it by interpolating polynomial. So we know that if we take the Lagrange polynomial, we can approximate it by p_1 . And here I can write $(x - x_1) / (x_0 - x_1) y_0 + (x - x_0) / (x_1 - x_0) y_1$ and here we have it. Ok? So we approximated it with this one line. In this case, we had an error which we knew about. The error is our $(x - x_0)(x - x_1) / 2$ and this z is between x_0 and x_1 . Ok? Now someone tells us to approximate the value. Now this is approximating the function $f(x)$. Ok? So we know these values, now someone tells us that we have to find out a value. f - at some x this has to be found out. Or suppose we have to find out $f(x_0)$. $f(x_1)$ has to be found out. This has to be found out. Ok? But the values we get are just given by the data. So what will you do in this case? See what I do? So what do I have now, we get this polynomial. is a polynomial of first degree. It is continuous. This can be differentiated easily. So now we've taken the derivative of this. Now I'll take the derivative of this. So look what we got? I will approach this. Then the forest will come here. So this will come as $y_0 / x_0 - x_1$ plus and from here it will come as $y_1 / x - x_0$ So if we look at this, if I take this x_1 here then I can write it like this. If you observe a little bit, $y_1 - y_0$ will come by $x_1 - x_0$, right? So if we look at this, what is this? y_1 and y_0 So if I draw a triangle here, this is it. Ok? So $y_1 - y_0$ comes here and $x_1 - x_0$ comes here. So if you observe, this is the slope of this tangent. Is it okay? So this is the tangent value. So what is the tangent value? This Divided by This. And if we observe this then we know that it is equal to the derivative of the function. This is its derivative. Ok? At the point x_0 , we call it the derivative. So, if we take the derivative of this line, if we define the slope, what will happen? $y_1 - y_0 / x_1$

- x_0 is correct, right? So what do we mean by this? If we differentiate this interpolative polynomial, it will give us the approximate value of the derivative of the function. Ok?

[8:59]

Now suppose I look at this case, what will be $p_1 - x_0$? If they now see this value then it will remain constant always. Because it was linear, so we know that if we take the first derivative of linear then it is a constant. So from here we know that this will be a constant. So p' which is not x is also the same and p' at x_1 is also the same and for any x they are also the same. Ok? So in this case, our derivative will be our p_1 and we will calculate its value here. Ok? So I can now write it like this in this case if our function was a linear polynomial, an interpolating polynomial, then the derivative would be this. So, we will differentiate the error as well. Ok? So if you see the error in this case, we differentiate it so it comes to $1/2$, it comes to $2x - x_0 - x_1$ and $z y$ is behind x_0 to x_1 , so the error in this was coming from quadratic polynomial. Whatever is there in this has come linear. So we will have this error when we approximate the derivative, this error is created in it. Ok? So if we look at this, we know that the interpolating polynomial always has an error at the nodal point that is zero. But if you look into this, what is E' at x_0 ? This comes to $1/2$ and from here x_0 will come to x_0 so this will become $x_0 - x_1$ $f''(e)$ E' at x_1 look then this will come to $1/2$ and this will become $(x_1 - x_0) f''$ this will become. Ok? So what happens in this is that the polynomial at that, its derivative, at the nodal points also gets an error in it and the error that is coming is the same. Is it okay? So the error in interpreting the polynomial was zero. Ok? So with its help we can calculate it. So now if we take a linear function then we know that if any is an interpolating polynomial then we have $n + 1$ points. Ok? So what we thought was an interpolating polynomial, we can define it like this. It seems like an interpolating point, so we will define it. How will you define it? This comes of degree n because there are $n + 1$ points, so we will write it as $L_0(x) y_0 + L_1(x) y_1$, like this we will write it $L_n(x) y_n$, okay because the points that we had, we have given $n + 1$ points, so we get a polynomial. Now whatever we have, if we look at it, what is any l I am taking? That is $(x - x_0) (x - x_1) (x - x_{k-1}) (x - x_{k+1}) \dots (x - x_n) / ((x_k - x_0) (x_k - x_1) (x_k - x_{k-1}) (x_k - x_{k+1}) \dots (x_k - x_n))$ these are the values. So we have one, let's write it like this

which we have done before also. I'm going to represent a πx πx meaning we're taking the product. Ok? We represented this. This becomes $(x - x_0) (x - x_1) \dots (x - x_n)$ right? I took them all. Ok? Now if we take its derivative, then the derivative that we get, you will see that if it is a polynomial of second degree, then the first degree will become linear. This is a polynomial of $n + 1^\circ$, so the polynomial of n° will come. Is it okay? So this way we can calculate it by taking its derivative. So if you see then we will not be able to take its derivative. What do you suppose I do with this? I will take the second degree. Let me take this right now. So this is our fundamental Lagrange polynomial, we can write it like this. $\pi x /$ Now look, $x - k$ was not here. Ok?

$(P_n, L^{(n)}) = \dots$
 $E'(x) = \frac{1}{2} (2x - x_0 - x_1) f''(\xi) \quad x_0 < x < x_1$
 $\begin{cases} E'(x_0) = \frac{1}{2} (x_0 - x_1) f''(\xi) \quad \checkmark \\ E'(x_1) = \frac{1}{2} (x_1 - x_0) f''(\xi) \quad \checkmark \end{cases}$
(n+1) points
Lagrange interpolating poly.
 $P_n(x) = l_0(x)y_0 + l_1(x)y_1 + \dots + l_n(x)y_n$
 $l_k(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$

[14:42]

So let me divide this here by $x - x_k$ so that it cancels out. And here $\pi dx k$ will come, whatever we have, it becomes $L_k(x)$. Ok? So all the interpolating polynomials that we see can be written like this and accordingly, I will write it like this. So this will come i from 0 to n L_i y_i it will be found. Ok? Now the main question arises that how do we mean pie dash below? How did you take the derivative? So we can define it like this, we can just calculate it here. For example, what we have is a second degree polynomial. Ok? So we have πx suppose we have $(x - x_0) (x - x_1) (x - x_2)$ so if I do π' then if I do π' then what will be the result? Look, first derivative into derivative of the second derivative, this will come to $(x - x_1) (x - x_2)$, its derivative plus $(x - x_1) (x - x_2)$ and the derivative of this will come to 1. So we have this, if I calculate it then it will come to $x - x_0$, now we have to take its derivative, so first function derivative the second one plus second function derivative the this is what we have $(x - x_1) (x - x_2)$, this is what we have, okay, so this thing we have the derivative of πd at x_0 , now see what is πd at x_0 ? We have to check πd at x_0 . So if we see, if I put x_0 then all of this becomes zero. So we are left with $(x_0 - x_1) (x_0 - x_2)$. What is π' at x_1 ? Look at x_1 , this has become 0. This also became 0. So we are left with $(x_1 - x_0)$ and $(x_1 - x_2)$ π' at x_2 what is it? This will also become 0. So this will be left with us, this will also become 0. So this will be left over $(x_2 - x_0)$ and $(x_2 - x_1)$ will be left over. Is it okay? So we know that if we write the interpolating polynomial of second degree, which we are looking for, if we write P_2 , then what will be in P_2 , we will have $L_0 x y_0 L_1 x y_1 L_2 x y_2$, this is what we will have and what is

L0 ? L0 is $(x - x_1)(x - x_2) / (x_0 - x_1)$ and $(x_0 - x_2)$. Now if we look at this, what do we have? Now look, this is it and the π that we will get, the πx that will be formed in it will be $(x - x_0)$ $(x - x_1)$ and $(x - x_2)$ because there are three points. So I can write this here. Now look, I wrote this as πx divided by. Now, here I divide this by $x - x_0$, which is what we have done. π' at x_0 that would be this. I did this. Now here it will become $(x_0 - x_1)(x_0 - x_2)$ and that is it. Is it okay?

The screenshot shows a digital note-taking application with the following content:

- General form of the interpolating polynomial: $P_n(x) = l_0(x)y_0 + l_1(x)y_1 + \dots + l_n(x)y_n = \sum_{i=0}^n l_i(x)y_i$
- Formula for the basis polynomial $l_k(x)$: $l_k(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$
- Formula for the error term: $E(x) = \frac{n!}{(n+1)!} f^{(n+1)}(\xi)$, where $x_0 < \xi < x_n$
- Formula for the derivative of the basis polynomial at x_k : $l_k'(x_k) = \frac{n!}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$
- Derivation of the derivative formula: $n! = (x_k-x_0)(x_k-x_1)\dots(x_k-x_n)$, $n'(x_k) = (x_k-x_0)[(x_k-x_1)\dots(x_k-x_n)]' + (x_k-x_1)\dots(x_k-x_n)$, $= (x_k-x_0)[(x_k-x_1) + (x_k-x_2) + \dots + (x_k-x_n)] + (x_k-x_1)\dots(x_k-x_n)$
- Derivatives at specific points: $n'(x_0) = (x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$, $n'(x_1) = (x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$, $n'(x_2) = (x_2-x_0)(x_2-x_1)\dots(x_2-x_n)$
- Final formula for the derivative of the basis polynomial: $l_k'(x_k) = \frac{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$

[21:32]

So now what this means is that I can now write this L0 as $(x - x_0) y$ at x_0 . Ok? So similarly I can write L_x like this. $\pi x - x_1 \pi' x_1$ Like this, I will write L_{2x} as $\pi x / (x - x_2) \pi'$ at x_2 and this will be calculated and this will come out to be the result. Is it okay? So now what this means is that if there is a polynomial of n th degree, we can always write it in this form. So if we wrote it like this then see what happened now? Now the polynomial that we have is the n th degree polynomial. $L_i x y_i$ from one to n we got it. What do we have now? We will calculate this. We know how to calculate it. Is it okay? And we also know the error that comes in this, what is the error that is involved? So we know $\pi x / n + 1$ and $n + 1$ which are the derivatives of z and π from x_0 to x_n . Ok? So I take this as a suppose one. What do I do now? If we look at this, now it is a function of x , if a function of x is a polynomial, then it is differentiable. So I can take the derivative of this. So what did I do? I wrote $\pi' x$. What would that be? That will be the summation of i from 1 to n will be written here. y_i this will become. Ok? Take the second derivative, that will come, the summation i from 1 to n will become double dash, so in this way we can define the polynomials of higher degree and higher order. Is it okay? So we will keep defining it like this. Now, what will be the errors in this? So there was an error in the polynomial in the x . The error in the first derivative will be this. Is it okay? $(\pi - x) n + 1 + \pi x (n + 1)$ and its derivative with respect to x of this function. Well, we already know it. Ok? So what should we do now? Now if we look a little into these values, then these values are unknown because we do not know where he is bringing them. So its values are unknown. So that is why if it is unknown then we cannot take its derivative. But we can do this in this case. If we are not able to take these derivatives then what we can do is calculate the errors.

So this means that whatever errors will occur, we can calculate them at the nodal points here. On x_i we can calculate what were the nodal values. Why? Because what will happen on x_i ? These terms will become zero. Is it okay? Provided that these derivatives are constant, which means they are bounded, some will be bounded, if multiplied by zero then it will become zero, then the error in our derivative will become π' at $x_i / n_1 + f_{(n+1)}$ derivative of z , which is j , will fall between x_0 to x_n . Ok? So, this is what we have is we took the first degree derivative, okay? We got so many errors in it. We will get such a second degree case. Is it okay? So we will take it in second degree. So, this thing, we will keep calculating it and with its help, the errors that we have will be found out. Is it okay? So we can see on the nodes that how much error is coming. So with its help we will calculate this and find it out. Now someone tells us that the values that we have are data points. I'll take any values, I'm zero one two a three four point. Let's go, there will be more. Let's reduce one. Take this, I will remove it. Ok? So these are the pay values. y can be anything. Meaning, this is not necessary. The equipment will be spaced. There may also be many spaces. So the value of y if we assume is one. This is 3.5 and the one I have is suppose 10. Ok? If we try to keep it equal to approximately square, then someone asks us, brother please tell me its value y at 2.5 and y at 2.5, please also tell me this. So if we want to find this thing out then we will know that we have to take the help of interpolating polynomials. Ok? So what do we do? In this we will have to define interpolating polynomials. And interpolating polynomial is something we know how to define. Ok? So in this case the interpolating polynomial we have will be p_2 . what will be p_2 ? We have $(x - x_1)(x - x_2)(x_0 - x_1)(x_0 - x_2) y_0$, like this we know that we will keep this at the end, $(x - x_0)(x_2 - x_0) y_2$ will come. Ok? So now after this we will take its derivative. If we take derivatives then here we will get to know the values of derivatives. What would that be? Now this which is in the denominator is a constant and here if we take the values then it will become $(2x - x_1 - x_2) / (x_0 - x_1)(x_0 - x_2) y_0$ plus what will we have next? $(2x - x_0 - x_2) / (x_1 - x_0)(x_1 - x_2) y_1 + (2x - x_0 - x_1) / (x_2 - x_0)(x_2 - x_1)$ This will give 1. Is it okay? One white this is done. So this is what we have, by taking the derivative we got its polynomial. So now we have come to know that this which was quadratic has become linear. Is it okay? So here comes the linear polynomial. Now look what is happening in the linear polynomial? If I want to see p_2' at x_0 what would that be? If we calculate from here then it will come out that $(2x_0 - x_1 - x_2) y_0$ will come out. Now if we put x_0 here, then here we can write it, it will become $(x_0 - x_2) / (x_1 - x_0)(x_1 - x_2) y_1$ plus now if we put x_0 here then this will come out to be $x_0 - x_1$ and this will come out to be $(x_2 - x_0)(x_2 - x_1) y_2$ so this is the first derivative that we get. So if we have to put any value, we will substitute it in it and find it out. Is it okay? So this value came at $P_2 - x_0$. Just like this it will come on $P_2 - x_1$. The value will come on x_2 . And from here we came to know that she is not passing it through whatever it is. Which are our nodal points. Is it okay? Now you look at this, if it is P_2 . Now what if I want to take P_2'' ? This was my P_2 . If I want to take P_2'' , then here you will see that it will come as 2 by these values. $x_0 - x_2 y_0$ okay? 2 by $x_1 - x_0$ and 2 / there we have it. So this constant came. Look, there is a constant here. So it is true for all. So we know that if we have a second degree polynomial, if we take its second derivative then we will get a constant. So this constant came.

$x: 1 \quad 2 \quad 2$
 $y: 1 \quad 2.5 \quad 10$

$l_1(x) = \frac{n(x)}{(x-x_0)n'(x_0)}$
 $l_2(x) = \frac{n(x)}{(x-x_1)n'(x_1)}$

$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \dots + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$

$P_2'(x) = \frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)} y_2$

$P_2'(x_0) = \frac{2x_0-x_1-x_2}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{x_0-x_2}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{x_0-x_1}{(x_2-x_0)(x_2-x_1)} y_2$

$P_2''(x) = \frac{2}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{2}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{2}{(x_2-x_0)(x_2-x_1)} y_2$

[30:34]

So this second derivative is here. Here comes the first derivative. So we can calculate on any values we have. And the error of the polynomials can be checked on the nodal values in this case. Is it okay? Otherwise we will not be able to define the within value x because we have to find the second derivative if the function is given then it is okay. If not, then how will we calculate it? So if we want to see in this case, here in this case the corresponding error that I have, okay at x_0 , what will that error be? If we were to take a dash, it would come to $x_0 - x_1$ $x_0 - x_2 / 3$ meaning 6 and the third derivative of the z , which is j , will come between x_0 to x_2 . So this error corresponding to our error will come to the approximation that we are finding the derivative at x_0 , similarly we can find it on x_1 and x_2 . We have defined that above. Is it okay? So there is a problem in defining it at any x because we cannot get this value. But in nodal points, we came to know that we can calculate the nodal points with its help. That is what we have done. So from here we have calculated it. Ok? So this is how we get cubic quadratic four times, okay? We can define the fifth polynomial like this. So this method of ours is based on interpolating polynomial. So the interpolating polynomial is we take the lag or we can take Newton divided. Where the nodal values are unevenly spaced. But it may also happen that our data points are equally spaced. So for that, if we look at the nodals, the nodal points, whether they are equally spaced or unilaterally spaced, then what will happen in that case? Ok? So in that case we can use this also. Lagrange is for both. Ok? But then what we do with this is that we can also use the forward method or backward method that we had applied. Ok? So if we make this quadratic which we have made above. If we have Equus paged points, then suppose we have Equus par space point, what does it mean? That is x_0, x_1, x_2, x_n then $x_{i+1} - x_i$ will be a constant. that is h . Is it okay? So this is the width. We call this width. Whose? The width between the nodal points is h . He is a constant in this case. And here we also have $y_0 y_1 y_2$ all given. Ok? So now if we want to calculate this, then look, if we look above, what will our P_2 become? Let me calculate this. So if we do a little calculation of this, this is what I see. Ok so what will become of the p_2 that we have? $x - x_1 x - x_2$ Bye Now look what is coming down? $x_0 - x_1 x_0 - x_1$ So this becomes $-h$. Why?

Because x_0 is smaller than x_1 . and what is $x_0 - x_2$? That will become $-2h$. So now if we multiply both of them then it will become $2h^2 y_0$ plus $x - x_0 x - x_2$ now coming down to $x_1 - x_0$ and $x_1 - x_2$ so see $x_1 - x_0$ is h and $x_1 - x_2$ will be $-h$. Is it okay? Because x_1 is smaller than x_2 . So here it will come to $-h^2 y_1 + (x - x_0)(x - x_1)$ and this $(x_2 - x_0)$ so $(x_2 - x_0)$ is $2h$. $(x_2 - x_1)$ is h . So this will become $2h^2 y_2$, we have calculated this. Is it okay? So we found this out. So if we look at this, suppose I take its derivative. So now whatever values we have for P_2' , what will be left in this case? Now look what is here? Let me check at x_0 . Let me just try to calculate this. Ok? So look here, it is being divided by $2h^2$. Here I have written x_0 , then if we take the derivative of x_0 then this will come. Suppose we will use this formula. Ok? So if we calculate this, then if we see, the thing which is in the numerator will come, I can write it like this $x_0 - x_1 + x_0 - x_2$. Is it okay? So $x_0 - x_1$ is $-h$ and this is $-2h$. So if this is seen then it will become $-3h$ divided by $2h^2 y$ not this will come. Ok? If we calculate it from here, this one will become $-2 - 2h y_1$, okay? Divide by $2h^2$ is just minus $2h y_2$ or or y are we right? So I can write x as $x - x_0 + x - x_2 x - x_0 h$ and yes, x_0 will become 0 so if we do this for x_0 , the result will be $-2h$, okay $-2h$, right, so where did this come from, if we write it again then it will come as minus $y_0 - 2h y_1 / h^2$, okay and it will come to us from here, so we have calculated it, so let me write it directly, now take plus $-h$ by $2h^2 y_2$ like this. Is it okay? So if we want to calculate this, see that h has cancelled. This got cancelled because of this. This got cancelled because of this. So what can I do? I can take it for 2 hours commonly. So if we see, this will come $-3 y$ not plus $4 y_1 - y_2$ this has come, it is okay because two is coming, two is coming here also, so hence two common will also have to be taken, one this value will come from here, so it will be very easy to calculate for the 21 spaces that we have, similarly if we see, if we calculate on x_1 , then we can calculate on x_1 . Ok? So if we calculate on x_1 , then the value R that we will get will be $y_2 - y_0 / 2h$ and p_2' at x_2 , what will be there is coming to us $1/2h (y_0 - 4y_1 + 3y_2)$ so it will be quite easy to calculate then if we take it equally spaced. Is it okay?

Nodal points \rightarrow Equally spaced

$\begin{cases} x_0, x_1, x_2, \dots, x_n \\ y_0, y_1, y_2, \dots, y_n \end{cases}$ $(x_{i+1} - x_i = h) \rightarrow$ width

$x_1 - x_0 = h$ $x_2 - x_0 = 2h$
 $x_1 - x_2 = -h$ $x_2 - x_1 = h$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{2h^2} y_0 + \frac{(x-x_0)(x-x_2)}{-h^2} y_1 + \frac{(x-x_0)(x-x_1)}{2h^2} y_2$$

$$P_2'(x_0) = \frac{-3h}{2h^2} y_0 - \frac{2h}{h^2} y_1 + \frac{-h}{2h^2} y_2 = \frac{1}{2h} [-3y_0 + 4y_1 - y_2]$$

$P_2'(x_1) = \frac{y_2 - y_0}{2h}$

$P_2'(x_2) = \frac{1}{2h} (y_0 - 4y_1 + 3y_2)$

[42:25]

So in Equispaced we know that the calculation will become very easy and we will directly put its values of y_0, y_1, y_2 . So we will get the value of the derivative at x_0 . Similarly, if we check on x_1 and check on x_2 , then we can easily take the derivative. Ok? So if we look at this, if we do this then if someone asks us what will be the function y_2 at x_0 that we have? So we'll simply write that we have the value given by y_0, y_2 . Substitute this here. Its value has come. One might ask: What is the value of the derivative at x_1 ? So we will put it directly and we will find out its values. Ok? Ok? So this is for the equispaced. So what we have for equispaced and for unequispaced, we have seen that we can calculate the interpolating polynomial and then we can solve it. Now let us look at the order. So here we define a numerical differential method which is a numerical differentiation method. Ok? is set to be of order P if what would be any function means taking the derivative minus the polynomials that we defined as interpolating polynomials which is the difference between those two if less than equal to some constants $C H$ becomes P to the power of P . Ok? Where the constant C is, it is a constant independent of H . So if this is defined, then we'll say that this is the numerical differential method, okay? That is the order of P . Ok? So we can also take this R in the first derivative. We can take the second derivative. Like this depending upon which derivative we are taking. So these methods that we have are based on what we have done so far in the previous units. Now the second method we have is a very important method. That is the method based on finite differences. Now let us look at the method based on finite difference. Ok? So the meaning of finite difference is that we know that we took the forward difference and we took the backward difference. If you remember, we have taken forward, backward and central. So this is how we defined the finite difference operator. Ok? So now we will try to calculate with its help as to how we can find it out. So for example, now look, we know one thing. What am I doing now? Suppose we have the value x_0 . Ok? Then we know that $x_0 + h$ becomes x_1 . Then we have $x_0 + 2h$, so it becomes x_2 . In this way we will keep getting value. So the finite difference that we're going to use here is for equal space. Best nodal values. Ok? So for this we will calculate. So now see how to do it. Now there is a function f . I'll write that as $x_0 + h$. And suppose our function is differentiable. Whatever its derivative is, I can do it as a Taylor expansion about the point x_0 because $x_0 + h$ will give x_1 . If it will be $x_0 + 2h$ then we will calculate it like this. So I suppose I write it like this. I can write it like this. See $f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3} f'''(x_0)$ and like this we will define it. So I can write f as $x_0 - h$. It has come to us, it has come to us $f(x_0 - h) = f(x_0) - h f'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{3} f'''(x_0)$ plus it will come to us. We will calculate it like this. What do I do now? So I suppose I'll name it 1 2 See, what do we do with first? From one what am I doing? I took it to the left. So I wrote $f(x_0 + h) - f(x_0 - h)$ moved $f(x_0)$ to the left. Ok? I took this on the left. I divided everyone by h . So if we look at it, we will have this: $\frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + \frac{h^2}{3} f'''(x_0)$ like this we will get it. Ok? So if we see, if you mean if we remember the definition of derivative then you will see what is happening? If I put a limit on this as h tends to 0, then this is exactly the derivative of the function at the point $x = 0$. This is the derivative. Ok? Now what does this mean? What happens if I take H Quite Small? Ok? Now if H is small, small means very small. Very small. Ok? So what will happen in that case? Now the second degree of h will become smaller. x will become very small. So if we see what it means, from here I can write this $\frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + \frac{h^2}{3} f'''(x_0)$.

Def: A numerical diff. method is said to be of order p , if

$$|f^{(k)}(x) - P^{(k)}(x)| \leq C h^p \quad C \text{ is a constant, indep. of } h.$$

Method based on finite difference: Δ, ∇, δ (Equi-spaced nodes pts.)

$x_0, x_0+h=x_1, x_0+2h=x_2, \dots$

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \dots \quad (1)$$

$$f(x_0-h) = f(x_0) - h f'(x_0) + \frac{h^2}{2!} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + \dots \quad (2)$$

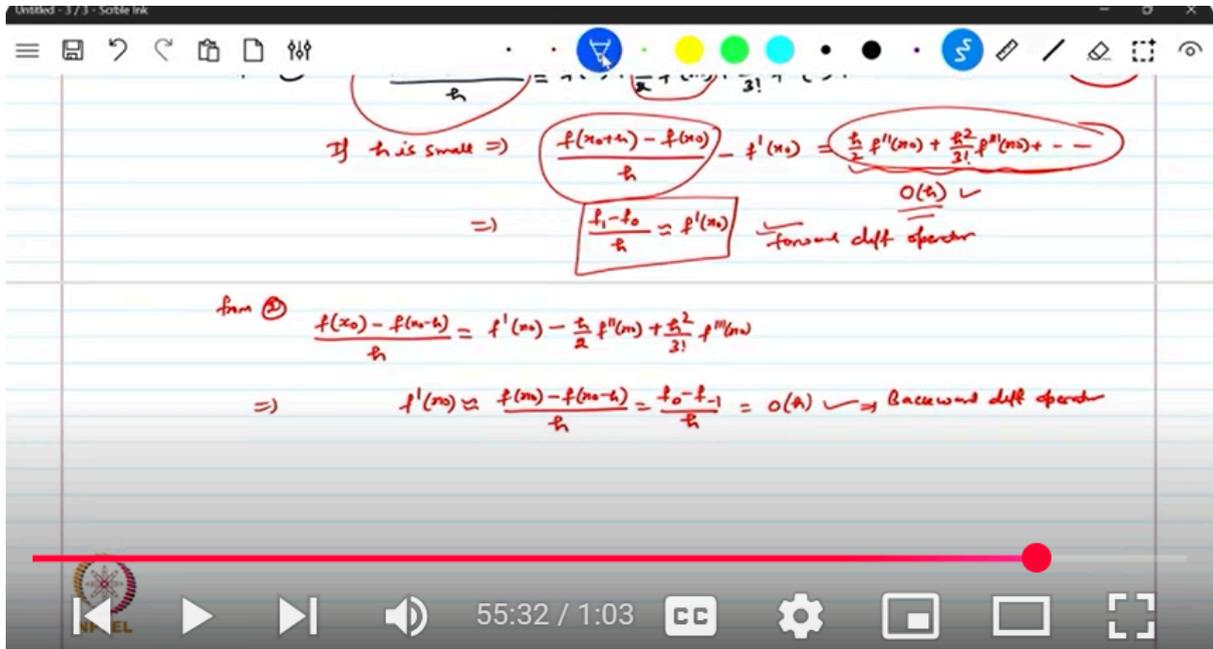
from (1) $\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{h}{2} f''(x_0) + \frac{h^2}{6} f'''(x_0) + \dots$

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[50:42]

What will happen if we see the difference between these two? If you see, these terms will be the dominant terms. Is it okay? Derivatives are some bounded values as compared to the other values. So I can write this from here, I can take h common from here for these terms. So these values that are left, we are left with $h/2 f''(x_0) + h^2/6 f'''(x_0)$ and so on. Now this function, let us give a name to this function. Why? Because whatever is there in it, we can take it common. So in this we can take it that this is of order h . Ok? This function has come to us, these values have come to us. This is of order h . Ok? It's Big O. So what does the Big O mean? Like this is a function, so let us say that h this function is of order h . What does it mean? If I take the limit, h tends to zero because h is very small. And if I take this function, then if you look in this function, h will be common. So this h has become common. So I can write it like this. $f(x_0+h) - f(x_0) = h [f'(x_0) + \frac{h}{2} f''(x_0) + \frac{h^2}{6} f'''(x_0) + \dots]$ is arrived at. So h will cancel out. So these values that will come will be some finite values. So if this finite value comes, then we call it this, then we write this type of function, this function is of big $O(h)$. Ok? So from here we come to know that our approximation of the derivative of the function with respect to this is accurate of order h , so what is this, if we see this then what will it become for us, what is $f(x_0+h)$, if x_1 is there then it will come to $(f_1 - f_0) / h$, so this will be the same as at the derivative of the function at x_0 , so what am I doing now? I am calculating the derivative of a function as a finite difference. Ok? So what happened is that we have a derivative. And its accuracy will depend on what h is and that will be the order of h . Is it okay? So the order of h will come. Like this, if I suppose I want to see it from here, from the second one, now look at this, if I do it like this for first and second, then we have found this out. And if you see this, we will call it forward. Is it okay? We took $x_0 + h$ further and minus it from x_0 . Now I can get the same thing from the second one also. What to do in from second from second? This is what I am doing. So what did I do? I took $f(x_0)$ there and I minus it and wrote by h . If you notice, we have $f(x_0 - h) / 2 f''(x_0) + h^2 / 6 f'''(x_0)$ like this. So if we do the same calculation then from here we will come to know that f' at x_0 , I can write it like this also. And if it is seen then what is it? $f_0 - f_{-1} / h$ okay? So this is going backwards. This was going forward. If we

wanted to take it out at f_0 then we had to take the next step f_1 . What's here? We have to go backwards. So this will be the order. This is also the order. That's the same way we can find it out. You can take the common and calculate it. So this thing that has come to us has come backward. Ok? So here comes our forward differential operator. Here we go, our backward differential operator. here comes the. What do I do now? I will try 1-2. Let's see what will happen? 1 - 2 Let's see this.



[55:32]

The second has to be minus from the first. So that's going to be $f(x_0 + h) - f(x_0 - h)$. It's going to be $f(x_0 + h)$. Now look, if I minus this from this then this and this will get cancelled. It will happen, right? This will be cancelled. Both of these will be added. Then it will be cancelled. Then both of these will be added. So if we look at it, what will we be left with? This $2h f'(x_0)$ this first term is left. Then $2h^3 / 6$ means $h^2 / 3$ and so on. So here h then h to power 5 then h to power 7 will go on like this. Now look how we wrote it? I am writing this. Of $f(x_0 + h) - f(x_0 - h)$, I have divided this by $2h$, taking the common from all, okay, and $-f(x_0 - h)$, I will bring the remaining amount here, so if we see, what will be left, see, I have taken all of $2h$, so what will be left here is $h^2 / 6$ cubed, right? And 2 also got gone and h also got gone, so h^2 will be left sec third derivative and all the terms, if you see, the error that we have in all the terms, basically the error between these two is this, and if we see, the error has become of the order of h^2 now because we can take h^2 common. Is it okay? So if h is small then the error is reduced even more. This is what it means, so in this we can say that the function whose derivative I took, I can write it like this. Now this is approximately the same as $f'(x_0) = \frac{f_1 - f_{-1}}{2h}$ and its accuracy is of the order of h^2 . So this means it is more accurate. So we call this the central difference operator. Now, it is taking the first derivative, so we will call it central difference, which is the first derivative central difference operator, we can define it like this: first derivative or first central difference operator. Is it okay? So if we look at this, I would like to represent it with a little graph. So we have some data like this. Ok? So at this point we have defined some x_0 . This is my x_0 and here somewhere we have x_1 given and suppose $x_0 - 1$ is given here, so what happened in the first one, we have only x_1 , suppose this

is x_0 y_0 , these values x_0 y_0 , these values are x_1 y_1 and the value that is coming somewhere here, this is $x - 1$, $y - 1$, we are just calculating. So in forward I if I want to calculate the derivative here so what do I do? I calculated this. This is divided by this.

$$\text{from } \textcircled{2} \quad \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0) - \frac{h}{2} f''(x_0) + \frac{h^2}{2!} f'''(x_0)$$

$$\Rightarrow f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h} = \frac{f_0 - f_{-1}}{h} = O(h) \checkmark \Rightarrow \text{Backward diff operator}$$

$$\textcircled{1} - \textcircled{2}$$

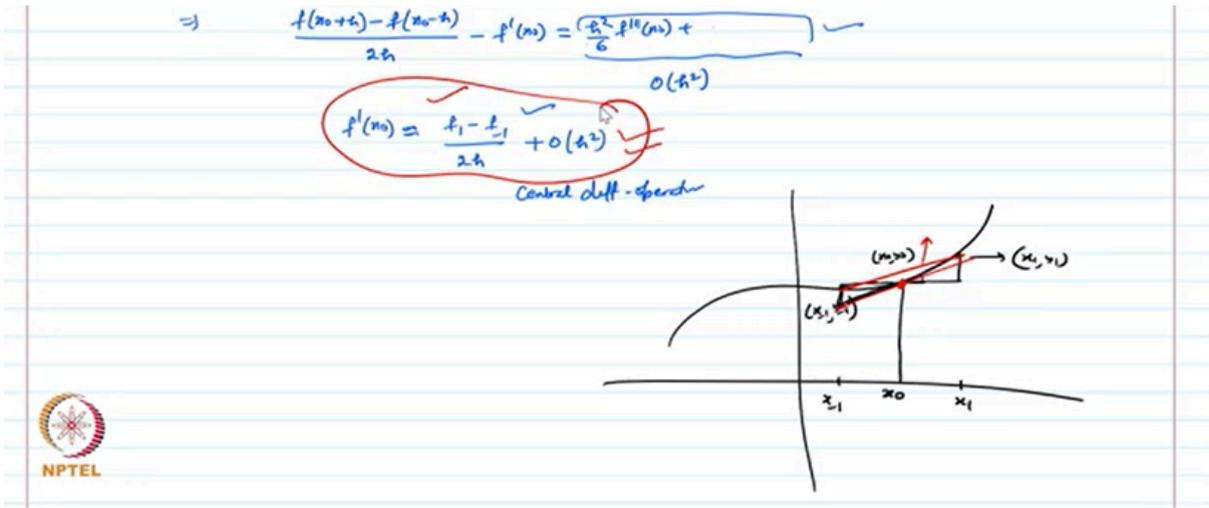
$$f(x_0 + h) - f(x_0 - h) = 2h f'(x_0) + \frac{2h^3}{6} f'''(x_0) - \dots$$

$$\Rightarrow \frac{f(x_0 + h) - f(x_0 - h)}{2h} - f'(x_0) = \frac{\frac{2h^3}{6} f'''(x_0) + \dots}{2h} \checkmark$$

$$f'(x_0) \approx \frac{f_1 - f_{-1}}{2h} + O(h^2)$$

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So this will give us the derivative at this point x_0 y_0 . The coordinate at x_0 is x not y , not, if we have to calculate the derivative there with the help of finite difference, then see, this will come, so this will be the forward difference operator, we went forward, now what do we have if we have to take it backward, okay, so look backward, now what is happening is $f_0 - f_{-1}$, so we are going here, this is this, then suppose it will be formed like this, okay so what will we do in this, the derivative will be this, this, okay, so here if we have to take the derivative at x_0 , then what will we do in this, I went to the left, went here, so what did we do, this value is minus this by this, so we got this derivative. Ok? So this derivative has come backward. And what happened with the centre now? f_1 this is it. f_{-1} this is it and by $2h$ so if we look at it, what have we done? Combined both of these. this and this this came from here. So this derivative is the derivative at this point, the derivative has been calculated from here. So if we see, this looks better because the derivative here will be this. The tangent will be this and the derivative at this point will be this and these values and these values, these lines, both appear to be parallel. This means the slope of both is the same. Right? So this means that this one which has come from Central, it gives the better result. To find out the derivative at this point. Because this is where we have to find the derivative. So we did it once from forward, once from backward, once from centre. So we felt that better things were coming from the centre. So in this we also came to know that this which is coming is also giving the accuracy of order h^2 , which is the error of ours which is off order of h^2 . Is it okay? So in area we had also written earlier that the numerical differential method, the differentiation method, is of order p , we will say this only if this happens then what does it mean that this central difference operator is of second order. Ok? For the first derivative, which is taking the first derivative.



These ones who are taking it from the forward or from the backward are of the first order. So the second order one will obviously be better because h is small. If h is small then the accuracy and error h^2 will become smaller and it will give better accuracy. Ok? So what we did today in this lecture is called numerical differentiation, that if we have given data and on the basis of that data if we have to take derivatives because nowadays in machine learning we are given data and for that we have to take many first order, second order such derivatives. So if we have to take the derivative and the function is unknown, then definitely we will have to do numerical differentiation. So today we saw on that basis how we can do numerical differentiation and I hope you understood it and thank you for watching this lecture. Hello.