

it so if I solved it so let's see what will come out from here if I take it here If I take this and this, we will ignore it because it is the product of both, so if it is zero, then it will also become completely zero, minus two, so it will never be zero, okay, so it will come from here, so now here if we see, we are getting values, okay, a summation e^{-6x} plus i from one to 4 because it will be multiplied by this, okay, so it will become $-6 + b$ from 1 to 4, if it will be multiplied by this, then it will come to $-5x$ is equal to, now both of these were negative, so we took one side, on the other side we will get summation f_i to the power of $-3i$, this will come, okay and i from 1 to 4, this will come, what will come in the other, we will take it out like this, so we have a summation i from 1 to 4, now see if it is -3 , then $-f$ will come to e^{-5x} plus b summation i from 1 to 4 and this will become -4 and if we see, what will come on the right side, we will have 1 to 4 f_i to the power of $-2x$, this is what we have. This has become a system. We will solve this system for whatever values we get. And here after solving it, the values that we get will be the values of a and b . So when we solved this, we got the values 0.6853 and 0.3058. We have solved this.

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$x = \log x$

$Y = A + Bx \rightarrow$ Linear polynomial in X

$$\begin{bmatrix} \sum 1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} \sum \log y_i \\ \sum x_i \log y_i \end{bmatrix}$$

$Y_i = \log y_i$

$A = \log_{10} a \Rightarrow a = \text{Antilog}(A)$

Ex $y = a e^{bx} \rightarrow$ Exponential fn

$$\log_{10} y = \log_{10}(a e^{bx}) = \log_{10} a + b x \log_{10} e$$

$Y = A + Bx$ $A = \log_{10} a$
 $B = b \log_{10} e$

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So, similarly, if we are given any function, how can we approximate or fit that function? Polynomial or any other function, like we were given function and we did it. similarly, we will also take approximations of different types of functions and see that if we use it, here we have some functions and let's see how can i approximate it? okay, we have some functions, we can call it a power function, we can call it an exponential function. So, brother, how will the exponential function be? So, now suppose I take an example. Now, suppose we have the data, the same x_1, x_2, x_n and n the number of data points. And on this, we have the values given f_1, f_2, f_n . Like this, we have the values given. So, we are told that it is okay. Best fit the curve. Now we have taken the curve which is equal to I will take Y , yes we can use small as well, there is no such thing, okay y equal to, now see what we are doing, I am taking

this, a x to the power of b , like this, okay, so we have the power one of x , take 2 the power of x , so we know that it is a Polynomial type, but what is in this, I can take it like this, our one can come in a fraction, the power of x can also come in this way, b can come in a fraction, but we don't know what will happen, okay, so we were told to best fit it, now we have to best fit it, okay, so we can either do two or either we will best fit it directly, okay, what did we do to this, we directly fit it or we see what to do to do with it, what do we have to do now, we have to best fit it, a x to the power of b , so, what I do, I will take the log on both sides, okay, if I take the natural log base 10, then I will take the base 10, log, okay, then I can call it base, so it will be base 10 and here will come a x to the power b and this will become our $\log a$ plus $b \log x$, we know that it is a product, so in product there are logs are added and power come forward. Okay, so what we have is this, so what did we do with it, we assumed the $\log y$ to be some capital Y , we assumed $\log a$ to be capital A plus b and we assumed $\log x$ to be capital X . Now if we see, what we have is a linear polynomial, now it is a linear polynomial and linear polynomial in capital x and capital y , so let's see, its value, we can write this linear polynomial in capital x and what is our capital x it is log, so this is that now whatever work we did earlier, we can do the same work in this, but what is the problem in doing this, what will happen in this that whatever x we will choose, now the capital x , we will have to take the log of the capital x given, we will have to take the log of this, okay, so I have taken x , so I have taken the log of this, now what values are coming to us, capital y will come from this, what is capital y it is the difference of $\log y$, okay. So if we see the $\log y$, it means this, we can write it as y as well, okay, we can write it as well, so what did we do to find it out, now look, we will have a matrix in which we will take out a and b , now we have to calculate it, so what did we do, summation one summation x_i , okay capital x_i , then summation capital x_i , the square of summation capital i , and the values that we have on the right side, now in that, we used to take the summation of f_i 's, now we do not have to take here, we have to take the summation here, of which is y 's, means $\log y$ the summation of these and here we have to take the summation $x_i \log y_i$, take this, okay, so we will solve it because now if we see what will be the error, what will be the error, what is the error, y_i minus now the values that we have given, what was y_i , it was $\log y$, so we will call it $\log y_i$, the difference between these, okay, that is the error that we have, so now what will we do, we will find out the values here. We will take the values from this system, as we found out, we got capital A and B . So the values that we got are capital A B . So, okay, we had B , so we had B and B and the same came, but here was A , it was capital A . So, if we see, what is capital A , \log_{10} base A , so from here A comes, antilog A . So, from here we have A , so small a and b come and from there we solved it, so we did it like this.

(20:04)

In the second example, suppose we have given that the same data is given, so now the function that we have is suppose y is equal to A , I write it, let me take exponential, so e to the power of $b x$, I took this. So, I want to fit the curve with the help of this, so we will call it exponential function. Now it is the exponential function. In this, A and B are unknown. In the above, we had the power function, in that also A and B were unknown. So we want to find it out like this. Now we will calculate it. So, you see, if we have to calculate it, what will we

do? In the same way, I will take its log 10 base and this will come $\log_{10} a e b x$, this will come how much will it come a plus now b is and x will come here and here will come log base 10. So just constant value will come, whatever comes, so we have now we can do it like this, let me assume it to be A A plus now I assume it to be B x this value. So you see, what we have is A and what is \log_{10} base A what is B, B will be small b into $\log_{10} a$, this is what we have. Okay, so now the polynomial that we will have, if you see, we will see that it is a linear polynomial. In linear polynomial, x is the same as what we will have given. Okay, so from there we will calculate y and log y's will come. Okay, we will take the difference of both. After that, just like we solved it above, we will solve it like this. Do it like this. We will take it, okay. So now if we have any type of function given to us, we can either direct it or we can do it by taking it like this directly or by taking log. So this is exponential log and it has power. So similarly we can get more functions. So whatever number of functions come, we can either direct it or we can do it like this. So with the help of this, we can do any values, any values. Now we have seen that we have done approximation with the least square method, we have made it linear, we have made it secondary. So we have done functions of this type. So if we want to see that we calculate it, then if we calculate it with the computer and see how it is, then what do I do, we have done linear, so we check it.(23:34)

So, we have made this code for discrete data, so discrete least square approximations. So, in this, we have used NumPy and matplotlib for plotting. Okay, we have taken values for the calculation like this. We have taken the value data, okay and we have taken the degree up to 1, 2, 3. After that we have written this code. So, we have to find the least squares approximation for the given data. So, we have the data and we have calculated it. We have defined the function here. Okay, let's say λx some c to the power of x is for i from this. Because we know that we take polynomials. In this, okay, if we keep taking the sum, then c_0 plus c_1 into x plus c_2 into x square will go on like this. So, like this, after that we will get it printed. So, what did we do? We took this data on this. x, which is our data, is 0.2, .5 .8 1. Okay, is equal to this. So, if we see, what we had calculated earlier in it, we had done x cube. In the previous one, we had taken both the values in x cube in the last example. If we see, this was, look in x cube, we had taken these values and after that we had taken this value. So, from there, c_0 c_1 c_2 . We can do it linearly as well. We can do it quadratic as well. We can also do it with cubic. We can also do x cube with 4th degree polynomial. Then we also did a continuous. In the continuous, we got this value from linear. We said that we will solve this, when the normal equation comes, then c_0 c_1 c_2 these values will come. So, we did the same thing with this and let's see what the result is. So, this is what came. The result showed that the least square polynomial degree one, c_0 , is minus 0.1 and c_1 is 0.95. So, this coefficient came. We have three coefficients in the Least square which is of second degree. So, our value of c_0 is 0.02, c_1 is minus 0.54 and c_2 is 1.5. Okay, in this way we have the Least square polynomial cubic which comes. So, we calculated this. So, if you see, see what is coming in cubic, c_0 is coming as zero. Okay, here c_0 is the value of c_1 is also zero, the value of c_2 is also zero, the value of c_3 is not coming, so why because we had taken it in respect of x cube, so we calculated it and see, this is our data. (26:31)

Now look at this data, we have given us these values of x , in this values, in the approximations case, you will see that the linear line is not passing any point, even then what is happening in this case, in this case, what is happening is that this best approximation even is not passing any of these points, still the error of this best approximation from this point is minimum, so the error will be minimum, so we will say that this is the best fit line, okay, so what did we do after that, look at the second degree, second degree, this second degree came in red color, so this second degree, if you see, it is passing this point, otherwise It is going from some other point to the last point, otherwise it is not passing anywhere, so this is a second degree polynomial. In this way, we have a third-degree polynomial in green color. So, this is a third-degree polynomial, it is passing through most of the points, right? And this is a third-degree polynomial, it is exact. So, in this case, ours has got the best fit. So, the number of points we had was five points, five points. So, if we see on the basis of five points, we have fitted the data, line fit it, quadratic fit it, cubic fit it. So, if we have to calculate the interpolating polynomial, then you will see that in that case we have a polynomial of the fourth degree, right? But in this, we have got a third-degree polynomial, and it is also passing and it is also giving the best approximation. So, we can calculate this thing. (28:56)

So if I change this data and suppose I take some other data, okay then we see in this that now I have taken this other data, the value of the data on 1, 2, 3, 4 was 2.2 2.8 3.6 4.5 5.1, okay so we have these values, so we took these values, so we have c_0, c_1, c_0, c_1, c_2 , this is linear, this is quadratic, this is cubic, we solved it, then in this way we got our data, so it is not necessary for us to pass the polynomials through the points, it is just that the errors that will be there should be minimum, so we have calculated it and from here we have fitted this degree one, degree two, degree three like this, okay so we can calculate according to it, it is not that the points that we have should be this many, 5 5 should be coming in both because once we have done this I have calculated, I will take six, I will take 7, 8 from it, okay, I have taken 8 points and added three to it. Now suppose I will take it as 5.5, okay, I will take it 7 and take it as 10. Suppose now let's see, we have this, so c_0 is the value, this value is in second degree, this value is in third degree, now if you see, we have these approximations best fit curve, so the line, which is of first degree, the red one is a second degree curve, the third one is a cubic polynomial, so now you will see in this that any line, any curve, is not passing through any point, in this case, like it was in that case, it is not passing anywhere, in this, all the curves are not passing through any point, so it is not necessary to pass through the point, and in this case, the error that we have should be minimum, now the work that we did, we did it on discrete data, now I did the same work which is continuous function I can do it for functions also, so in continuous function you saw what we did, we took x cube, so I did it in the same way and the function I did is λx and if you don't want to write the name of a function, then we write it as λx , so we did it with x cube, okay, and the interval is zero to one, why because we have to do integration also, so for integration we have an inbuilt function which is in the library `scipy`, in that it is `integrate`, now we took `degree`, after that we defined λx , okay, similarly the values that we have are polynomial and they will be defined and we are plotting from here, so what we have to do is integrate, basically we

integrated from here, we got the right hand side vector, we calculated it, from here we will have a system of equations, A which is a matrix and B vector which is the right hand side vector, this is it, okay, so according to this we did it, so I will calculate it, so let's see now c_0 which is minus 2 and c_1 .9 came almost the same in the discrete form and in this also the same came, the second degree polynomial c_0 0.05 c_1 minus .6 c_2 is 1.5 and in c_3 you will see that c_0 c_1 c_2 all are zeros and c_3 is 1 came, so if we see in the continuous form then it has come to us, so the points are not there in it at all because the function is a continuous function, we have to take, so in this case, see, the x cube function and the degree one is passing from here, the degree two is the red one and if we see, the degree three is exactly overlapping, why is it x cube because there were some values of x cube, that x cube approximations from the polynomial will get zeros, so it is exactly overlapping, in this case.(33:46)

Now I should change the space of the function, so suppose I change the function here and I have taken the function. It is exponential, okay, e to the power of x and if we plot it, then we can write it there also like this, this is the exponential function, we can approximate it with the help of first degree polynomial, second degree polynomial, third degree polynomial, so with this we have approximated it, but the condition remained the same that we have to solve it using least square method means best fit method in this case of exponential c_0 came to .8 and c_1 came to 1.6, we took this least square of second degree, so c_0 c_1 c_2 these values came and now you can see in cubic, in that, when we were cubing x , only c_0 c_1 c_2 were coming 0 and this is not happening in this, if you see in this then some values are coming because the exponential function, now the exponential function that we took, its approximations okay, the first curve that has come of 1st degree, but the curve of second degree is almost overlapping and the curve of third degree is completely overlapping, okay, so in this case this is our approximation, but we have not done this means that if we calculate it this will come. (35:56)

Now instead of exponential x , I suppose Let me take $2x$. Now let's see what happens. This value has come. Look, it has come. Now the exponential will become $2x$. So, in this way I can change the value of exponential. Let me take 5. I took 5. The values that I saw were how many came out c_0 minus 24 c_1 108. All the coefficients have changed. And if you see, the polynomial has also changed completely. The exponential is e to the power $5x$. I did not change it yet because if we see, then I will have to write it as e to the power $5x$. Let me suppose, I have written it here. I will also write here e to the power $5x$. Second, because we have to write it in the form of latex, so I will write here. This has come to us. The e to the power $5x$ has come. See, the original function which is e to the power $5x$ is the black one. So, we have passed the best fit line from here. It is of blue color. This one is of second degree. The color is okay, third. The degree is the green one. So, if we look at it, the functions are not overlapping at all. In this case, our approximations will keep increasing the degrees and our approximation will keep getting a little better. But it is not like that, it will not always happen like this.(37:56)

This is the Runge effect or what is the Runge phenomenon. He said that what is happening is that we increase the degree of the Polynomial. As we polynomial, whatever normal equation we have, because the normal equation will keep increasing, that normal equation will become

an ill condition. So what will happen with the ill condition, whatever our parameters will be, we will get bad approximations. We will get a matrix from our normal equation. Here our parameters will come and it will be equal to the right hand side vector. So this is the matrix. Now, we had a cubic first order Polynomial. So, 2 by 2 second In this case, if I take 3 by 3 cubic in 4 by 4, then if I take 10 degrees, then I will take 10 by 10 matrix of 100, then what we have will keep on increasing. It was said in it that if we see, what should happen is that as we are increasing the degree, approximation should get better right? But this is not happening so, Runge observed this abnormal behaviour, he said that if we see this thing, then as we are increasing the degree, it may be wrong, then this should be avoided, so he wrote that it can be avoided by taking. Now, we had earlier taken linearly independent which was a polynomial or a function, now we have to take that it can be avoided by taking orthogonal basis function. Just one more thing has been done in this, it was linearly independent, now it is orthogonal also, so this thing can be avoided, so this work can be done by orthogonal so see what is Runge phenomena in this, if you see, this is there, so what is happening in runge phenomena, so what did he do, he took a function $f(x)$ is equal to $1/25x^2$, okay, so this we took this function. Now if we see this function is continuous function is not a problem. We took the interval minus 1 and 1. So if we see this function, if we take the degree of the polynomial of any degree, then we should get the approximation. The best approximation. So what did we do? We took its degree, of 5 degrees, 10 degrees, 15 degrees. We changed it like this.(43:56)

So we ran it. Now what happened after running it. The Runge phenomena came this and we took equispaced. No problem. Now see in this case, the black color one is our function. And we chose the interpolating nodes from it. So what did we do now? We took a polynomial of 5 degrees. So the 5 degree polynomial is this one. It is of blue color. So we took this. So we saw that the curve here in the middle till here is fine. As soon as it goes towards the boundaries, see, it is going towards the boundaries, then there is an oscillation there. It started, then we took a 10 degree Polynomial, this is in orange color type, okay, so this 10 degree polynomial which has come, we saw that this is also in the middle, so let's go, okay, from here till here, this approximates it better as compared to degree 5, but as we go to the boundary, the oscillation will start there, so see how much oscillation is there in this area, how much oscillation is there here, how much oscillation is there here, then we increased another degree, took 15 degrees in green color, so what happened in 15 degrees, then the problem came, this approximation went till here, till here, from this point to this point and if seen is better approximation but as soon as it moved to boundary again oscillation started, okay, so in this way we calculated 15 degrees, 20 degrees, 25 degrees, then we found out that the polynomials, the oscillation started, they started moving towards the boundary, the motion started towards the boundary, so this is the Runge phenomenon, it is very important because if seen, we will say that like As we keep increasing the degree, we should get very good result, but in reality it does not happen. So this phenomenon is given to us by Runge. Now as I said that how can the abnormal behavior of the Runge phenomenon be removed, I said that the basis function should be taken. So what has been done now, it has been fixed by the Chebyshev nodes, a polynomial is basically orthogonal, so we took its roots. We took its Chebyshev nodes. On the basis of its nodes, we did not do anything else, we did the same

thing. Just its nodes value, the nodal value which is minus one to one, we took the roots of the polynomial of Chebyshev. And then we approximated it. So what happened with Chebyshev nodes it got a little better because we have taken degree less. Now see, the Chebyshev node degree 5 is this polynomial, so it is doing this quite smoothly. Then we took 10 degrees in red color. This is red color. Then took 15 degrees Chebyshev nodes. We were taking equispaced that was doing lot but as soon as we have taken this in brown colour, you can see that brown colour is very well approximating it, but in Runge that we discussed, there were fluctuations but here it is approximating well. So, we do this way that we are having problem in approximating, so we can find it out with the help of orthogonal polynomial, so you can take this function and check it, you can even take other functions, you can also take $1/(1+x^2)$, so you can check in that also what will happen, right, instead of 25, so in that also you can check what will be its values, like this is here, if I take it here, I make it one, okay, so with its help we see what will happen, so this has come, okay, so what happened in this, there is fluctuation, if we see, there is a lot of fluctuation, okay, and if we change it in this also, then the value will come, so in this way we can check with the help of different functions. (48:36)

Okay, so this Runge phenomena, it is very important, it tells that there is no benefit in taking higher degree polynomials, right? So these things mean our main purpose should be that we get our approximations even with smaller polynomials, polynomials of lower degree and best approximation. So, when we have done this work, we can apply the same Least square method in a different way and you will see that we will get the same thing. (48:56)

So, we call it in this case, so I will just explain a little bit in this that the Least square method using pseudo inverse. And it is also called Moore Penrose. So, in this case, we have now we will do the same question. So, what does the same question mean? Like we have a lot of points. We have points x_1, x_2, \dots, x_n . These are the points. We have n points. And I have approximated it with a polynomial y equals to c_0 plus $c_1 x$. So, this is what we have. We have $p_1 x$. We can write it like this. Now what do I do? Now see, its value is going up. I put c_0 plus $c_1 x_1$, okay, so its value will come capital y_1 , okay, capital y_1 will come its value, so this value is this approximations, okay, so the value of y_1 , $c_2 c_0$ plus $c_1 x_2$ this what, so this value of y_2 is equal to this, this is our p_1 at x_1 , this value is p_2 at x_2 , okay, which we calculated but we do not know the value of y , or the values that we know, so we know these values of x_1, x_2, \dots, x_n and here y_1, y_2, \dots, y_n these values are given and these values are given to us as nodal point, we put x_1 inside it and its value should be, the approximately equal to y_1 or we are assuming that it is equal to y_1 and we will go on like this and this comes this to us, now what is our purpose, to solve it, okay, we will solve it, now if we see this, we have a system of equation and see this type of formation, $x_1 \ 1 \ x_2 \ 1 \ \dots \ x_n \ 1 \ c_0 \ c_1 \ y_1 \ y_2 \ \dots \ y_n$, okay now if we see this matrix, this is the matrix The dimension of capital N cross two is how much is 2 cross 1 and how much is N cross 1. Now we have this system. We know that if such a system in which no equation is greater than the number of variables, then that type of system is called over determinate system. Over determinate system is an over determinate system in which the number of equations is greater than the number of variables because the number of

variables is only two c_0, c_1 . To find out, the value of this will be an over determinate system.
(53:20)

So if this matrix is written as capital X , I will write as c and I will write as y , then we have this system. Now how do we solve this system because this system has more number of equations than the number of variables. So, this system does not have a solution available. So, if the solution of this system is not available, then how can we solve it? So, what did we do? This matrix, I took its transposed suppose I multiplied both the sides by it, x transpose y , this is the matrix we get, x transpose x now x transpose x we know this matrix is a symmetric matrix so now if you see that if I what will happen $1, 1, 1, 1, x_1, x_2, x_3, x_n$ multiplied by $1, x_1, 1, x_2, 1, x_n$ y will go into c_0, c_1 and this one on the right hand side will come $1, 1, 1, 1$ and $x_1 x_2 x_3 x_n$ multiplied by y was $y_1 y_2 y_N$ this was we so if we multiply both of these because this matrix we have is a symmetric matrix, okay so what have we done, if we solve this then you will see that the first column will go from the first row, so it will come to us summation one, then this second one will go from the first, so it will come to us summation x_i then it will be multiplied by this, so summation x_i and this will come to summation x_i square so if you see, we have got a two cross two matrix, that is to be found because two cross N and this is N cross two so this is a two cross two matrix then. This is $c_0 c_1$ and if you look at it on the right hand side and multiply it, you will see that the summation of y_i comes and the summation of $y_i x_i$ comes, so you will see that we have the same system equation which we had derived from least square, so what did we do in this case, in this case, we did it in the form of a matrix and after that we took its pseudo inverse, right? So we call it pseudo inverse because if we look at this, now this is x transpose x in fact there is no square matrix in it, so we cannot take its inverse, so what will we do to solve it, we will multiply it by x transpose, so if we look at it now, now the c that we have, now our system has changed, okay, so now we are solving this system, so exactly our c that will come, I can take it as c head, what will it be inverse of x transpose x, x transpose y this will come, okay so what does it mean, we have a value, this is what is it Earlier, if the matrix was square or invertible, then it would have its inverse, but this is not the case in this case. In this case, we have this, so the quantity that we have got is called pseudo inverse.(58:21)

So, here we have an approximate solution. We call it the best solution, it may be that it does not satisfy all the equations. Now, the c_i 's that we get, that c_i , it is not necessary that it satisfies all of them. We will put the value, but definitely it will be close to all of them. As we saw it in that list, it is not necessary that it passes through the nodal point. What will happen is that the error will be minimum. So, in this way, we can calculate it. So, it means that if we have any over determinant system in which there are number of equations, we can always solve it like this. You can take any. The over determinant systems are We can always find out with its help. Okay, so we have calculated it. Now look, there was one more thing left for us to tell that we have calculated this, this x transpose x . So look at this, I am writing one as one. So I can write one like this. One can be written as x transpose $x c$. I took it to the left. Now what have I done, I have written it like this. I have taken x transpose common. Okay, so $x c$ minus y if we see, what is $x c$. If we calculate it, then we have capital Y . By putting the values

of polynomial, we will get values. Okay, and the value that we have been given is this. And this x transpose equals to zero. Okay, so the values that we have got, if we see, this is our error. Okay, and this matrix x transpose equals 0 so, that is why we say that this is normal. What does normal mean? The dot product between these two, this product is zero. It means x transpose which is x The vector and the e vector are normal to each other, they are orthogonal to each other, that's why the equations that come from here are normal, orthogonal to each other that's why it is called normal equation basically, so the least square approximation, basically everything is based on this, so we can do it in that way as well, we can do it in the form of pseudo inverses as well, so both of them can be done, but ultimately the near purpose will be solved that we get the best approximation, so this work is that going forward we can do it with the help of orthogonal polynomial, Chebyshev polynomial, Legendre polynomial for approximation. But in this course we will do till here.

(01:01)

we have seen that if we have to do best fit polynomial, if we have to do best fit exponential function, if we have to do best fit power function or if we have to solve over determinant system as well, then in this lecture we have seen how we can do it with the help of pseudo inverse, and ultimately the pseudo inverse and the Lee square that we did are the same thing, the solutions that will come will be the same for both, so today I hope that you liked this lecture and understood it, and thank you for watching it, hello