SCIENTIFIC COMPUTING USING PYTHON
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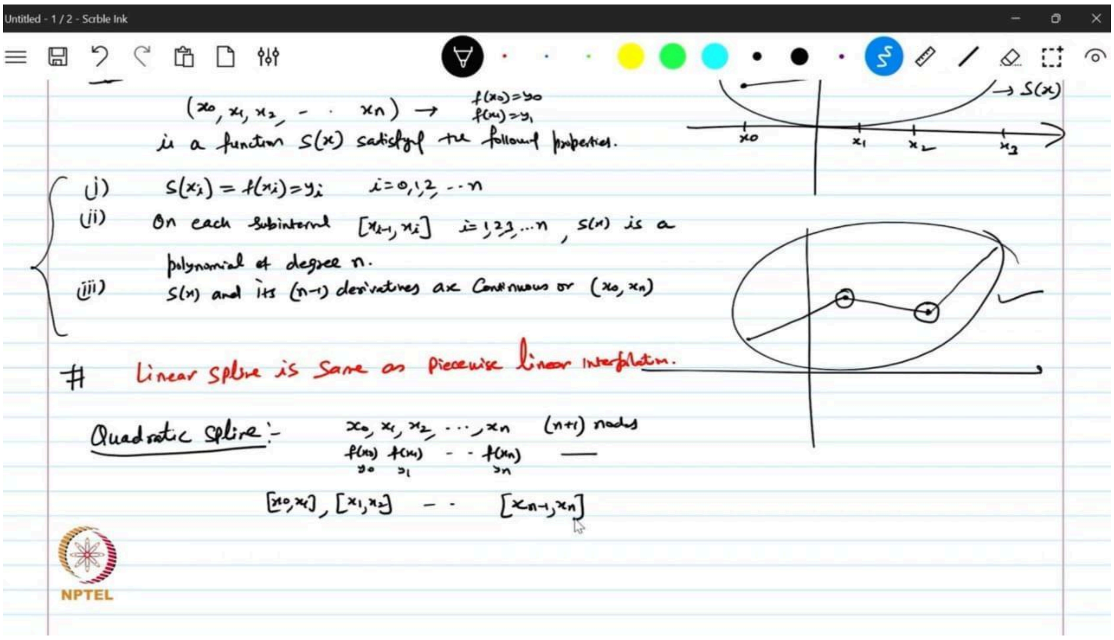
Lecture No. 25

Welcome, everybody, to Scientific Computing Using Python. So today we are going to start a topic called Splines. So let's get started. So we saw in the previous lectures that we discussed interpolating polynomials. We discussed piecewise interpolating polynomials. So now there is a new topic which is very important and that is Splines. So, where did this name Splines come from? So spline's means that like earlier there used to be a draftsman who used to do plotting , make projects, make building structures. So when there were no computers, suppose they had to draw a curve. Ok? Suppose he had to draw a curve. So what did they do to draw the curve? Like this is the one. So in this I have to draw a curve. So he used to put some claps here. He put nails etc. and with the help of thread, he fixed the plot of the thread in the form of a smooth curve. So now this thread would be in the form of a smooth curve and then they would plot it. Ok? So, he started this and said that these spline words came from there. So what do we do now ? So we also want to use a smooth curve. Like we just did piecewise interpolation. So what we did in piecewise interpolation was that we saw that our piecewise interpolating polynomial I in each sub interval was just satisfying the continuity. Continuity means that whatever polynomials we had in the past, suppose I change it now. So what we did first was we had these few points. So we saw that if we take these points in linear manner then this was coming to us. Then such a point came. Then we take the interpolating polynomial linear. So these continuous which were common nodes, which were interior mesh points or nodal points, we saw that the function at these points is satisfying the continuity. But this was our requirement till then. But now, if we want to make a smooth curve and we want to make that smooth curve in piecewise form, then what should we do? So that is the thing for us so we defined Splines. So if we observe, we can write the definition of splines. So this is our main purpose now to give a smooth curve to the splines. So a spline function will be this, definitely of degree n because it is a polynomial with nodes, it is also written like this, what is x0 x1 x2 xn, these are our nodal values of x. Ok? So explain a function of degree n with the n that this is a function because it will definitely be a function. We represent that with suppose S x. S can mean supply satisfying the following properties or conditions. The first thing is that s at each one our values will be equal to. Because these values are given to us, the values of our function are also given on this. What is the value of the function ? That is fx0, we basically write it as y0. fx1 let's write it as y1. This is how we write. So whatever s xi will be, it will be the same as y i, which means she will pass among these. Ok? So that's why it is interpolating polynomials. Ok? So this which will be true will be true for all the points. 0 1 2 up to n second one second one is that on each sub interval sub interval now we know the sub interval, we write it as xi minus 1 xi and i is 1 2 3 up to n so we get n sub intervals so if this is on each sub interval which is S x is a polynomial of degree n. Let these be of degree n. The third one is S x and it's n minus 1 which are the derivatives. Is it okay? n minus 1 derivative means first derivative, second derivative, third derivative because n degree is a polynomial so if we take its n minus 1 derivatives what will they be? And continue on this I will replace a0 with on open interval x0 to x okay? So this will be ours, if we leave the boundary point, then it can happen on that, meaning we do not have any requirement, but the function which is inside and its derivatives, n minus 1 derivatives, will all be continuous. Ok ? So basically what are we going to do now? Now look, we have to define the spline. Suppose, suppose we have four points. One is this point, one is this, one is this and one is this. There are four points. So what are we doing now ? In this we saw that this was the common nodes, so suppose this is my x0. This is my x1 here. Here's x2. Here's x3. Suppose it is like this. So what we said is that if we look at the linear Spline, that linear Spline, then it's okay. It will be the same. But if we talk about quadratic Spline, then what we will do in quadratic spline is that the function which is coming from the left and the one coming from the right, which is the interpolating polynomial, which is a piece wise polynomial, will give the same value on both here, from left and right, but if seen from the left and the right, their slopes will also be the same. Here also the slope will be the same. What this means is that the derivatives of the polynomials coming in the left sub-interval and the derivatives of the polynomials coming in the right sub-interval will be the same at this point. Same will happen at this point also. Ok? So we'll call it quadratic spline. If it is a cubic spline then its second derivative will also be the same. This means the curvature will also be the same. So what will happen by doing this is that all the polynomials that we have will be connected with each other in such a way that the complete function that will be formed will satisfy all of these properties. What was the interpolating polynomial, which was simple? We had a function like this happening. Ok? But in all those intervals, we saw that in all those intervals, it was a polynomial of second degree to third degree. But the common notes at these points, which were interior points, it could have been smooth or not, so we could not call the total function that it is a function, we could say that the function is continuous but we do not know what its derivative is, so we cannot say that this function is smooth, but now we have got our complete function in the spine, so now we will write these entire functions as s x, so from here we can say that s x is. If it is satisfying all these properties then it will be a smooth spline and what degree will the spline be ? will have the spine of n degree. So this is its definition. So now we have to check how to find this out ? So the first thing we have to do is that we can, means from here we have to understand that the linear spline is the same as piece wise linear interpolation, the piecewise linear spline is the same. Like we have done piece wise in previous lectures. So piecewise is just like linear interpolation. Ok? So whatever we have in it, there is no need to repeat it again. Now the main thing that comes up is what we are going to do in this is the quadratic spline and cubic splines. So now let's start with the Quadratics spline. So in the quadratics plan, which will be an interpolating polynomial, what will happen in it is that the nodes we have are x0 x1 x2 xn so there are n plus 1 nodes. Their values are given to us. f at x0 f at x1 f at xn okay, we consider all these as y0 y1. So all this is given to us. Now what did we do? On this we have defined n sub intervals. Ok? So here we go,x0 x1, x1,x2, and x n minus 1 x n.



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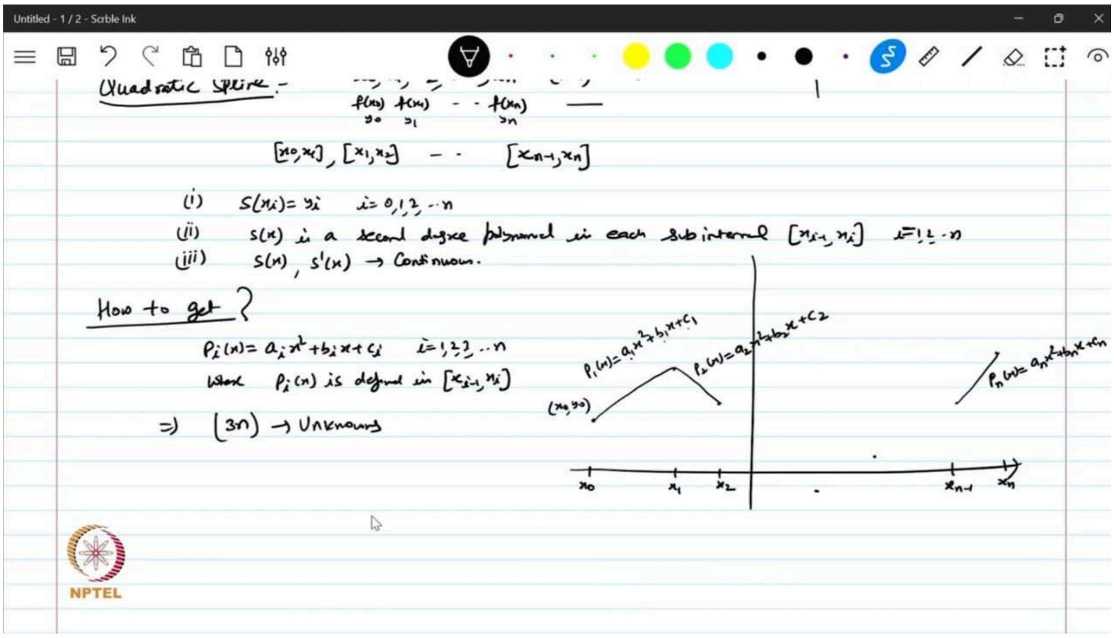
Now see, what should we do next? Now all these conditions should be satisfied. Ok? So if our quadratics spline is s x what should it be? s at xi that should be equal to y i first of all this should happen for everyone. Ok? I should be from 0 to 1 up to N This has to happen. What is the second one in this case what is s x? is a polynomial of second degree. So, this is s x that is a second-degree polynomial in each sub interval, any sub interval like this xi minus 1 to xi i which is 1 2 up to n, so there are n sub intervals, this second-degree polynomial should be there in all of them. Ok? The third one, the third condition that we have is that s x and s dash x, okay, this also should be continuous with us. Continuous means that our polynomials coming from the left and those coming from the right should have the same values at the interior nodal points. Ok? So, s x and s dash x both should be continuous. So, from this we come to know that the derivative of the function is continuous and the function will become smooth. Ok? So now the question is how do we calculate this? Now see how to calculate this. What do we have now? We have a large number of points. I have a point suppose value is this. The data values are this, this, this, this, this. Like this we have the value of last. So, this is our x0. So, at x0 we are given x0 y0, this is given, we got our x1, this is our x2 and at last we got x n, right, we have all these things, so we have given such values. So, what do we do now? One, from here to here, I'm going to define polynomial. Whatever name I give to the polynomial, I will give it that p1x, it is a1x square plus b1x plus c1 because it is quadratic, so quadratic has to be defined. Similarly, I will define it from here to here. I'll give it a name, p2x is equal to a2x square plus b2 x plus c2 and go on like this. And suppose in the end I have xn minus 1 somewhere here, so I had some point like this, so from here I will define a polynomial. Ok? If I name it pnx then what will pnx be? anx square plus bn x plus cn here it is. Now we have defined such a polynomial. Ok? So how to generate or how to get this polynomial. So, what did we do now? We will continue to get such polynomials. Ok? So, what are the polynomials that we have defined in pi x? ai x square plus bi x plus ci correct? Where i what will be? 1 2 3 up to n right and where is this defined where pi x is a is defined in xi minus 1 to xi okay so p1 in the first one p2 in the second one, this is defined like this so if we see, we have three 1 2 3 three unknowns in one sub interval. So, if we have to find out all the values, it means we have 3n, which is three here and three in the next one, so there are a total of 3n unknowns. If we have to find out then we need 3n unknowns. So, if 3n is unknown it means we need 3n conditions so that we can solve a system of equations and after that under these conditions and from that system we get a unique polynomial and we can solve it. So now we have to see how to find out the 3n who are unknowns. So, we need three n conditions.



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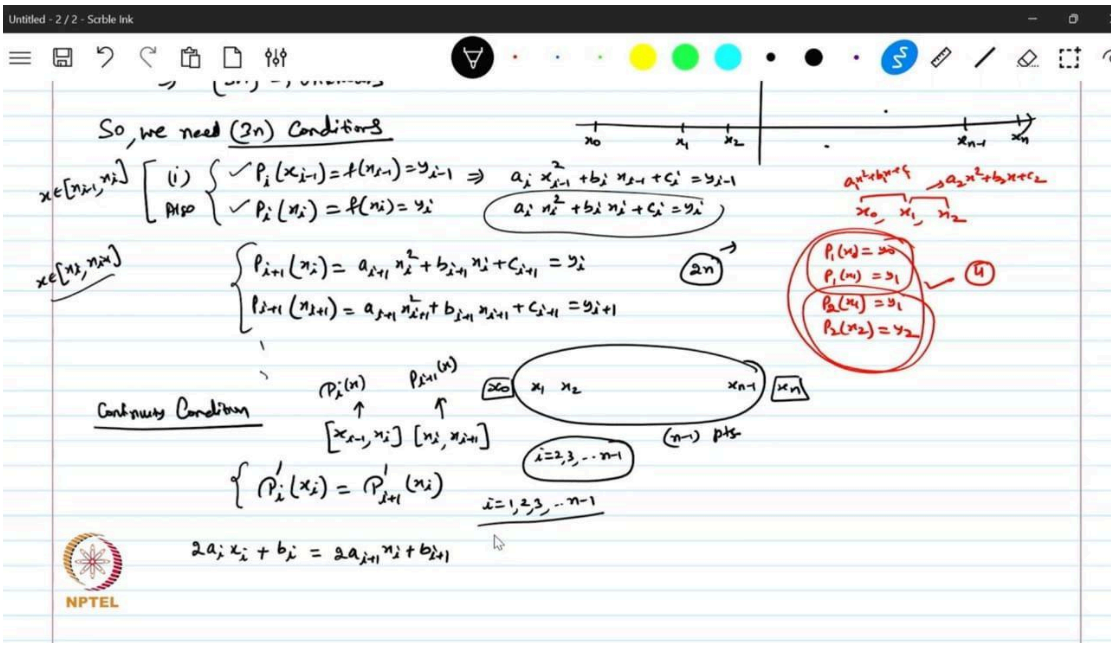
Now let's see how to do this. So now first of all we will satisfy these conditions. So what will happen? Now in this we have to first write that pi x which is being defined in this, so what will be pi x minus 1? This becomes f xi minus 1 and that is equal to y i minus 1 again also pi at xi what is this? This will be at fx i because this is the definition of interpolation, that will be yi so what does this mean? If I look from here, what we get is ai xi minus 1 square plus bi xi minus 1 plus ci is equal to y i minus 1, okay? Similarly, if we take it from here, it will come ai xi square plus bi xi plus ci is equal to yi. Now this will be i minus 1, now if we see, it will go to every point, this is correct, so it means one condition is found here, one condition is found here, so how many polynomials do we have, total there were n polynomials, in n polynomials we have this condition, each one of them is meeting in the interval, it means the conditions that we will get will be 2n. What is the meaning of this? Now if we have two points then we get two values. There are three points, look there are three points, so 1, 2, 3, if we look at it, then I will do a little bit of that to it. So, suppose we have only two points. There are three points. Suppose x0, x1, x2 then what is happening in those three points? We are getting two sub intervals. One is getting this and one is getting this. So, the condition that we get is the first one p1 x, right? So p1 is the first polynomial of second degree in this. So, whatever is p1 at x0, we get y0. P1 at X1 that Y1 was found. P2 at X1 is B Y1. And P2 at X2 is Y2. Now if it's in two then we have three ax square plus bx plus c1 1 we need three unknowns in this and we need three unknowns in this. Ok? So, we need six unknowns. And how many conditions did we get from here? Four conditions were met. So, this means that we can say that this four 2 into 2 four conditions has been found here. So, this is how we got this condition. Ok? So now if we look at the conditions, then the conditions that we have got are two conditions. Ok? How much is there in the second one? Then two conditions have been found. So, if I keep writing it like this, what do I do now? Now look, I keep writing it like this. It is done. Ok? These two conditions arise when our x is between xi minus 1 and xi. Ok? Next our x which belongs will belong between xi xi plus 1. What will that condition be? That would be here. p Now here we have the polynomial. Ok? I will write it as i plus 1, it will be made on xi and if it will be made on xi then what will it be, it will be ai plus 1 xi square plus bi plus 1 xi plus ci plus 1 and that will be y i, okay and pi + 1 will be on xi + 1 which is i plus 1, what will be on this, ai plus 1 xi plus 1 square, okay? b i plus 1 xi plus 1 plus ci plus 1 and this will become y i plus 1 so in the next one also we got two. I got two in this too. So, you will get two in each one. So, we had the interval two, so in this way we got 2n, which is the total condition, from here. Ok? Ok? So, we will continue like this. So, our job was to use the interpolating definition. What do we do now? So, what do we do with the internal nodes that we have? Leave x0 and x n aside. Ok? Because this is a boundary point. So, what is left inside this is x1, x2, xn minus 1. So here we have n - 1 points which have common nodes which will have common nodes with each other. What do we do now? Continuity condition has to be applied here. Is it okay? Now we will apply the continuity condition. Now see what will happen in the continuity condition? Look now, this is what we have. So, we have x i minus 1 xi, this is one interval and the next interval is xi xi plus 1, this is the interval. Ok? And I'm taking whatever i we have. From where to where will it go? 2 3 up to n minus 1 will go like this. Ok? So, we will get that n minus 2, we will get all the intervals. So, it will go like this. Now what is the condition that we have to put in this? Have to look at derivatives. Now we can also take derivatives of the polynomials that we have. Is it okay? So, we took the derivative. The polynomial defined in this was named pi x. The mean of pi in the i th subinterval. So, what should I do? pi dash at xi this, and the polynomial inside it was p dash i plus 1x, which was named. That must be the bean. p at i plus 1 at xi. Ok? So, this condition will be applicable to us and where does a should be true for i equal to? 1 2 3 up to n minus 1 should run here. Ok? n plus 1 is the same and i plus 1 is the same. So, this condition will be a continuity condition. So, the continuity condition will be applied. Ok? Continuity Condition If we look at i here, what will it become? So, this will become 2 ai xi because it was a square, so its derivative plus bi, okay, this came to us because the polynomials that we had, this was in it in the ith and this was in i plus 1th, right, so the polynomial that we had in the ith, it is written above, so if we take its derivative, then we are taking 2ai at xi. This should be equal to 2ai plus 1 xi plus bi plus 1. These conditions will be applied. All i equal Okay, so now what do we have, whatever the conditions are, if these are from 1 to n minus 1, then it means from here we will have n minus 1 condition. Ok. So now in the quadratics spline, we just have to see till here that because we had written in the definition that the derivatives of n minus 1 should be continuous. So, if this is of second degree then it must be continuous up to a single derivative. So, this is where we'll go in the Quadratics splines.



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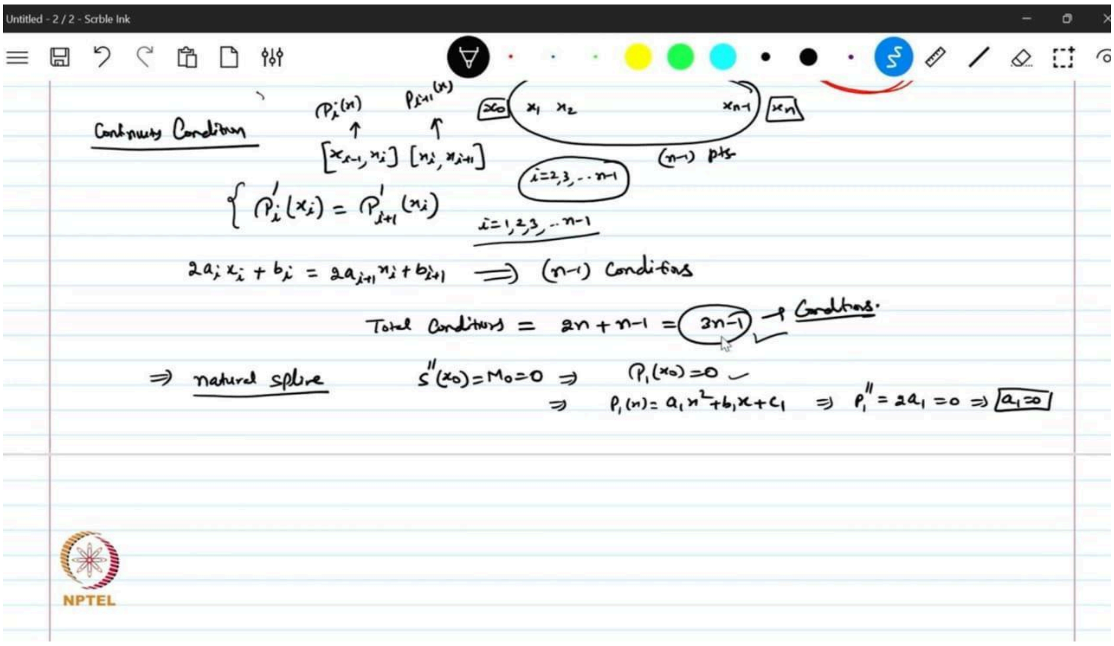
So now look, we have n minus 1 condition. 2n condition came up above. So, what is the total? 2n plus n minus 1 so this will be 3n minus 1 this condition we have got in total. What do we do now? We need one more condition. So, in splines, we take this meaning, a condition and we call it natural spline. Ok? So, what is there in the natural plan is that we are assuming that the boundaries point is like any spline, as we had defined, whatever spline will be there, what will happen on it is that a curve will be formed like it is here. So, if you look here, then the curve is a straight line. Is it okay? Meaning it has a slope. There is continuity but there is no curve, no curvature inside it. So, what natural spline said is that we will assume that the function which is our spline is s double dash at x0, we will write it as m0. That would be zero. Ok? I will erase this. This double dash second derivative is going to be zero. What does it mean? What should be the function that we will have on the first boundary? The polynomial need not be quadratic. It should be a linear one. What does it mean? However we make the supply, you will see carefully that I will walk from here to here like this. Ok? So, the condition that should be applied on these boundaries is that in our first sub-interval, our approximation, which will be the interpolating polynomials, will be linear. We have imposed this condition. Ok? So, if we impose this condition. We can apply the same condition either in the starting one or in the last one. Can be installed on boundaries. So, we imposed this condition. So, what we got to know by applying this condition is that from here we got to know that my P1 at x 0 will be zero. Ok? So, if we look from here, what was P1? Ours was a1 x square a1x square which is b1 was a1 x square plus b1x plus c1, so from here we put the condition that this at x0 will be zero, its second derivative. So, from here if I take the two derivatives of p1, what will it be? If you see from here then it will come 2a1 only this will be left and I have taken this 0 so from here our a1 will automatically become zero. So, our number of variables automatically reduced. One variable became zero. Ok? So now we can say that we have 3n minus 1 condition here. One condition will come from us here. So total 3n conditions will be formed and we have only 3n variables. So, we will solve it like this and we will get a unique solution. Ok? So, we can calculate it with its help.



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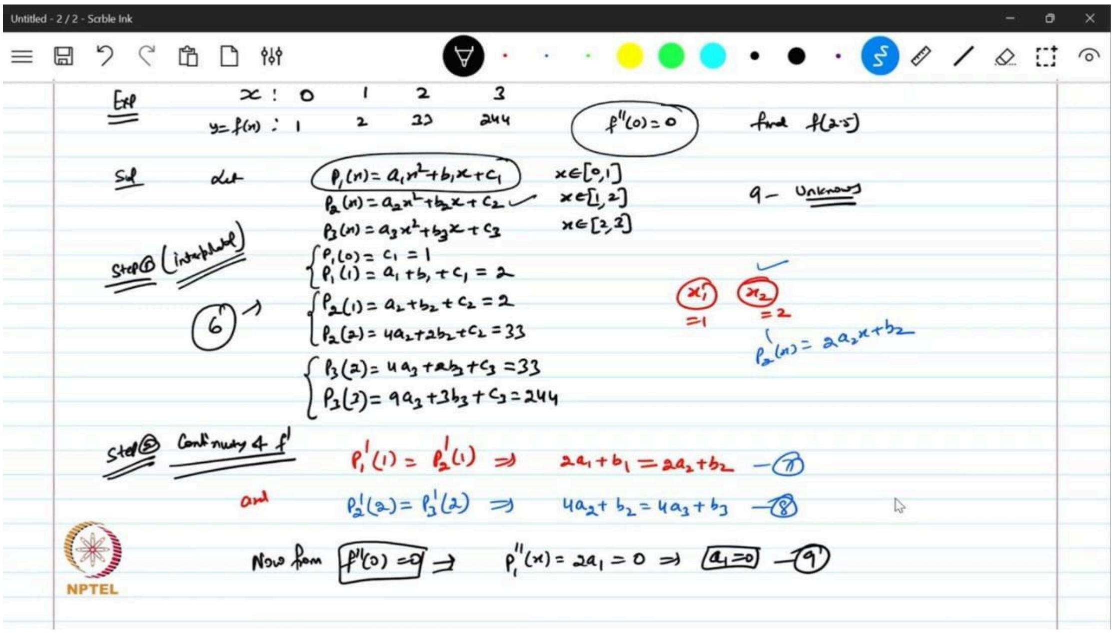
So this is the work, you mean if we take an example, suppose I take any example, then on the basis of that example you will come to know that now suppose I have taken an example, our example is the values that I have x, we had taken them earlier also 0 1 2 3 and the function f x which is y equal f x is 1 2 33 244 this value is okay and we are given the condition that f double dash at 0 is 0 this condition is given and we have to calculate find f 2.5 so what will we do in this now see, this is our x0, this is our x1, this is our x2, this is our x3, so in this we will have three polynomials, okay, so now if we see in this, let p1x be, I have taken a1x square plus b1x plus c1 x belongs to 0 1 p2, then ours is a2x square plus b2x plus c2 x belongs to 1 2. Okay, and the third one will be a3x square plus b3x plus c3 x belongs to 2 to 3. So, these three give us the quadratic polynomial, which we have defined. Now what will we do with its help? We will make splines. So, what will be the spline? We will create quadratic splines. So, what's the first thing to do? First of all, we have to impose conditions on it. Ok? So how to solve this? So first we have to apply the interpolation condition. Ok? So, our first step will be P1 at 0, let us see this now on common nodes. So P1 at 1 is okay? What would that be? a1 plus b1 plus c1 is what should be equal to because y1 is here. Ok? Now P1 is here. Let's take P0 first as well. P1 at 0 what should that be? If we take zero then we get C1 okay? What and how much should it be? 1. So, we took P1 on both points. Now P2 has arrived. What should be the P2 on one? A2 plus B2 plus C2 what should that be? Two P2 at 2 now this common one has come. p2 at 2, So how much should it be? 4a2 + 2b2 + c2 okay and this should be 33 just in this if we put two in place of x then 4a2 plus 2a2 plus c2 it will become 33 so we got two conditions from here, we got two conditions from here now from the next one we will get p3 so what will be p3 at 2 4A3 plus 2B3 plus C3 and that is 33 okay? What will happen in P3 at 3, the last one? If you three putt you will get a 9a3 plus 3b3 plus c3 gives 244. So, you see, here all the three were intervals, so we have the condition of six and how many unknowns do we have? There are three here. Three here, nine is our unknown, so now it has come here. Next thing that came was the continuity condition. Ok? This was an interpolating condition which is interpolating the meaning and passing it through. Ok? The second one that came, we got continuity. Continuity of f dash means continuity of the derivative here, of the function. Ok? So that he can pass through it. So, what to do now in this case? Now look at the values we have, the nodes are x1 and x2, which are the common nodes. We just have to look at this. So, what should we do? P1 dash I'll take it at one. X1 is one and X2 is two. Is same as P2 dash at 1 so from here it will come to us, I have taken its derivative. So, this comes 2 A1 plus B1 which is on the left side should be equal to 2 a2 + b2. So here we go, we have a condition, okay, and on x2, okay, so we have to find out on x2, now we have to find out on this, so where is p2 coming and p3 is coming on this, so p2 dash, if I see at 2, what happens, it should be p3 at 2, so from here, let us see, what will come to p2? 2A2 and 2 will putt again. 4 will come. Then 4a2 + b2 is correct, right? If we take its derivative, what will we get if we take the derivative of t2x? 2 a2x plus b2 So I put two in place of x and we got 4 plus b2 this is the same 4 a3 + b3 okay which is b3 so from here we had six from this and this at this condition came okay now we need one more condition so we will take one condition from here that we are given that this is our function, the values which we will get of the spline function its second derivative at zero is zero. So now from f dash 0 is 0 which is the natural spline, what do we get? Our P1 is the first polynomial defined in the first sub-interval. So, from here we will get that what will be the second derivative of P1? 2a1 That should be our zero. So, from here the condition comes a1 should be zero. So, this is our ninth condition. So now we have nine equations and we will solve these nine equations and all the values of unknowns that we have, a1 b1 c1 a2 b2 c2 a3 b3 c3, will be found out. Ok? So, we will calculate this.



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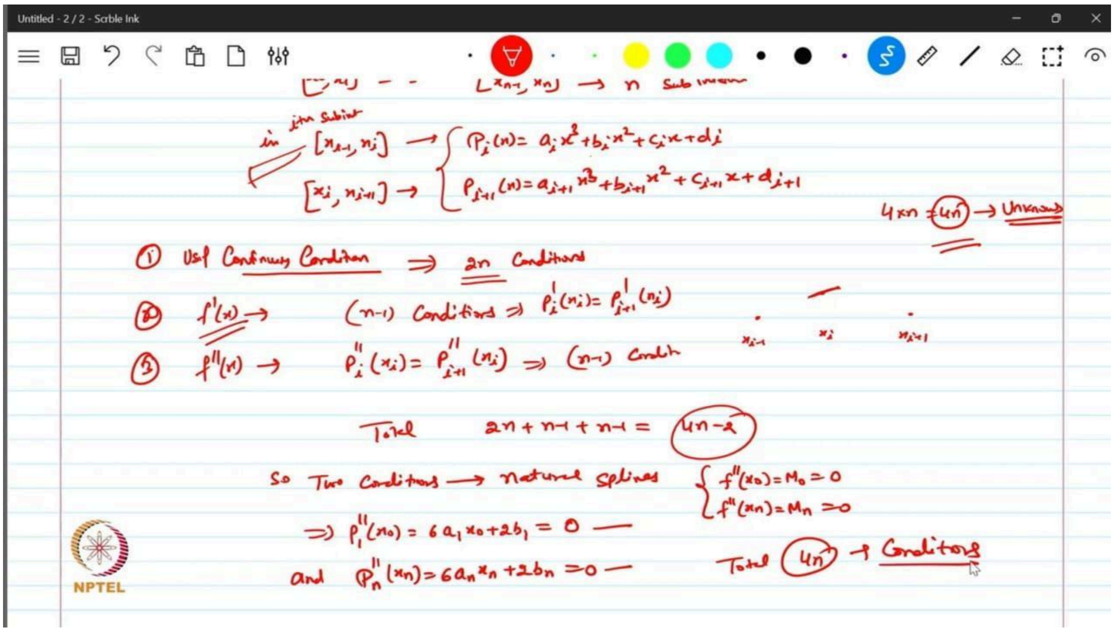
So, we will do some calculation regarding this thing. So, we will see what will come according to the program which we have in Python. So now that we have done this part, it is the quadratic spline. So, what are we doing now? Now let's talk about cubic splines. So, basically the problem with the quadratic spline is that the quadratic spline swings a lot. Ok? And what is there in this is that the straight lines which are there in the beginning or in the last ones are straight lines because we have put such conditions. Ok? And what happens in this is that this fluctuation occurs due to this. So, the quadratic spine is mostly not used. But cubic spline is used a lot. So, what we will discuss now is cubic spline. So, we have similar points for the Cubic spline. Ok? It is certain that we have to write like this in everything. x0 x1 x2 xn These are n plus 1 point. In this way we will get all the intervals. x n minus 1 x So these are n sub intervals. It is all the same. Just what do we have to do now? We have to define the polynomial in each subinterval. So, we will define that these xi minus 1 xi any sub interval rectangle sub interval, which is first see one is coming here. In the second, so rectangle sub interval in i th sub interval, our polynomial will be pi x and what will it be, ai x will become bi x square ci x plus di in it, why? Because now Cubic has arrived. So, we got four unknowns for one. Ok? So, we will define this and I have written i. Now what will happen in the next one? What will be i and xi plus 1 in this? Why? Because we need it because how do we define the common node xi? What would that be? pi + 1xi is i plus 1x plus bi plus 1x square plus ci plus 1x plus di this becomes di plus 1, right? So, in this little subscript, sometimes there may be some mistake while writing. But now we understand how to write the polynomials we need. So now we know how to apply conditions? So, we know that we will get two of these intervals. Is it okay? So, the two conditions that were coming in it, we came to know that using continuity conditions, continuity condition which is the first one, means from the interpreting, now look in this case, the condition that will come on 1 2 3 4 will be 2n only, 2n end conditions. Now we have to check that we have the total, if we see, 1, 2, 3, 4, four one is there in it, so we have 4n which is unknown. Ok? So how do we remove the Four Unknowns now? So 2n unknowns will come to us from the first continuity condition. sec What do we do now? The derivative of the function f dash will come to us from those conditions. Ok? So, what do we do? The continuity of its derivative, the derivatives which we are taking should be continuous. At the interior common sense there. So, from here you see, in the previous we calculated and we got 2n minus 1, sorry not 2n minus 1, but n minus 1 like it came here. We are doing the same work. These are n minus 1 condition. So, we will get n minus 1 condition like this. n minus 1 condition. Ok? The third one. Now what is there in the third one? There must also be a second derivative. So, this means it must have curvature also. What does it mean? Which is a common node, like we have one and one, this is a point. Suppose this is it. Ok? So, this is our xi. These are the values at xi minus 1 and these are the values at xi plus 1. So, in that we have said that whatever point is there here, its slope should be the same and the curvature should also be the same. If it is taking a curve here then it should have the same curvature on the left and right. So, what does that mean? That f double dash or whatever our polynomial is, we write it as s x. So, what should happen? s double dash at xi is it okay? We are taking it in. So, if either I make it well right like this that pi dash at xi should be the same as p dash i plus 1 at x i okay? This condition was pi at xi should be same as p dash i plus 1x so this applies to n minus 1 condition. From here also we will get n minus 1 condition. So how much do we have in total? 2n plus n minus 1 plus n minus 1 that is 4n minus 2. Here we go. We need 4n so we're still short of two. Ok? So, the two conditions will come from natural spline. Ok? These two conditions will come from the natural spline. So, what is called natural spline? In which the derivative of the function, the derivative of our function, the second derivative at x0, we will write it as m0. That will be zero and xn will also be zero. Ok? So, if we apply the condition from here, then what do we have from here? If we look at a condition, p1 is the double derivative at x0, then if we take the double derivative of it, what will be the result? 6 a1 because three will come. Then take another derivative to so 6 a1 okay? x0 plus 2b1 is all that remains. Ok? Because we are taking two derivatives and that will come to zero and pn at xn what will that be in the last one? 6 an xn plus 2 bn 0, so one condition becomes this and one becomes this, so if we see, now we have a total of 4n conditions. Ok? So, we got 4n conditions. There were 4n unknowns. So, with its help we will solve this entire system. Ok? So, we will solve this entire system.



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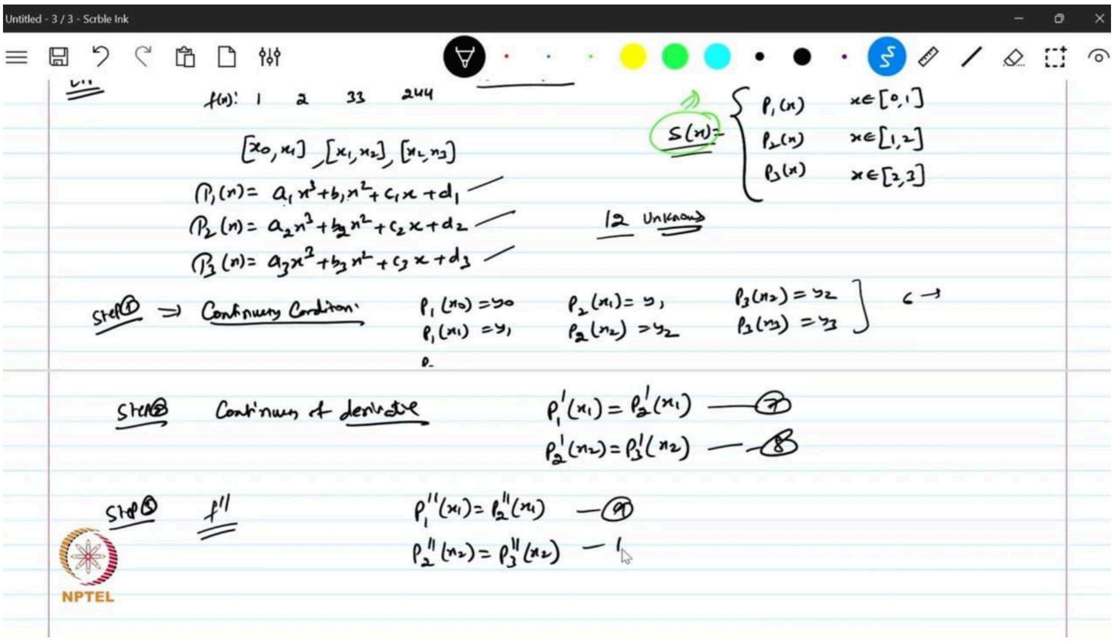
Then if we solve it, we will get a unique solution and our splines will work. Is it okay? So now this thing, we have seen that the cubic spline have been defined in many books at many places in an alternative way also. But our method is direct, which is the basic method. Is it okay? So, we can take example in this case also. And I can take the same example. The same example that we have just defined for 0 1 2 3, we can take the same example. Ok? So, in that case x we have 0 1 2 3 suppose I took this. Ok? And we took fx 1, 2, 33 and 244, so what is our purpose now? What do we want to define? We want to define Cubic spline. So, if we look at the Cubic spline, what will be the Cubic spline? will be s x and what will be sx? p1x p2x if seen in this, there are three then p3x like this spline will be defined. x belongs to 0 1 x belongs to 1 2 x belongs to 2 3 So what will be our spline? There will be cubic spline. Is it okay? So, as we have found out above. So here if we are asked what is the value of our spline? So, if we see, we can rewrite it like this, in this case which is quadratic, what will be s x in it, in quadratic it will be our p1x x belongs to 0 1 p2x x belongs to 1 2 p3x x belongs to 23 so in this way we will get the spline that we have. So, if we look at it, what is s x? Now s x becomes a smooth curve. How is a smooth curve formed that if we take its derivative, we have seen that the derivative will be the same. They are bound to come in between. But there are common nodes, same will be there also. If I take its second derivative, it will also come out the same. So, we can say that the polynomial that we got, the s x that we got, we got a smooth curve. Is it okay? So, we will do it with its help. So now what do we have to do in this case? So, as we moved backward, we got three. So, three came. x0 to x1 x1 to x2 x2 to x3 okay? So, this is our x0, x1, x2, x3. So now we have defined polynomials. So, I'll define p1x. I will define a1x plus b1x square plus c1x plus d1 p2x. a2x plus b c2x plus d2 and p3x will become a3x plus b3x square plus c3x plus d3. So now we have these three polynomials. I defined it in the first interval, this in the second and this in the third. So now we will see that we have nine, there are 12, sorry 1, 2, 3, 4, so in each of those intervals there are four unknowns, so we have a total of 12 unknowns. Ok? So, what did we get by meeting 12 unknowns? Now we will apply the continuity condition for this. So, in step one, we did the same thing, step one continuity condition, right? What is a continuity condition? That is, P1 is Y0 on X0, and P1 at X1 is Y1. P2 is P2 at X1 on Y1. So, this and this both will be the same values. And P2 is X2 which is Y2. Ok? Similarly, P3 at X2 is Y2 and P3 at X3 is Y3. So, the six conditions which are 1 2 3, these six came to us from here. Ok? So now what will happen with such step one, step two derivative continuity of derivative? The derivative of P1 dash x1 is same as the first derivative of P2 dash at x1 and the derivative of P2 dash at x2 is same as the derivative of P3 dash x2 so we have these two, one and two come from here, so six comes from here, 8 comes from here, okay now step three comes, we have the continuity of the second derivative, what will be the second derivative continuity, okay, what will that be p1 double dash at x1 same as p2 double dash derivative at x1 p2 double dash at x2 p3 double dash x2, so we will take it on x2. Ok? So, this and these 10, 10 conditions have come to us.



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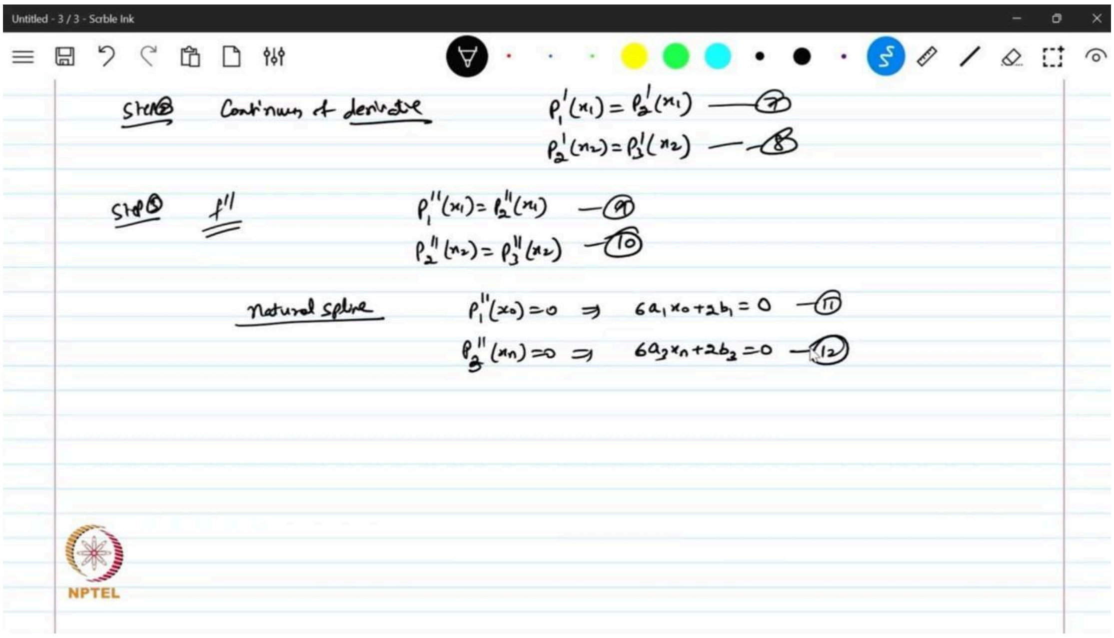
Just what to do now? The conditions for the derivatives which are natural splines have to be applied. So, what we have to do in natural spline is take the second derivative of p1 at x0 which is 0. Ok? It is given to us that we have to take it at the first value. So, what do we do? We took its two times derivative, and we got 6 a1 x0 plus 2 b1, which should be zero. Ok? Because we have imposed this condition from here. This is okay on cubic and the second one has come, so this becomes our 11th and P2 sorry P3 will come at XN is zero so this comes 6 A3 at XN plus 2 b30 this comes 12. So, we have 12 equations. Only 12 are unknown. So, we will compile all the equations together and we will get a system of equations. We will solve that system of equations. So let us see how we can do this in code now. So now this system that has been formed, we will solve it. So, we have created these codes. Ok? So first we will check the quadratic spline, what is it, and what will we find out? So, we used quadratic spline. Ok? So, in that we need three unknowns. So, we know that we will meet three unknowns. So, we have defined a matrix of the first 3 unknowns. So, its dimension will be 3n into 3n and the right-hand side vector B will be of 3n dimensions. Ok? So, we have defined it. What will we do after that? You look at this. So now we will start substituting values. So, from here, the initial equations will come from at each nodal point. Ok? Continuity conditions will come from here. All the conditions which are continuity. i from 1 to n was the whole thing. We have told you i from 1 to n minus 1. Ok? Whatever its conditions, they will come. And start from the first row, row zero, row one, row two like this. Ok? Now the last one was the natural spline condition. So, it depends on the natural spline condition as to what it will be. So, the condition that we had taken in that was this. Here we go, 2a1 is equal to 0, so this means that the value we have to put here in the matrix, we have to put it. Is it okay? So, we will do two put and it will come to us from there. So, look, we have put value in the matrix that we are creating. Ok? So, the two values have arrived. So, we had a system. The more unknown there are, the more the system becomes. So, we have the n cross n system. We solved it using linear algebra. Input the matrix a and b., Ok? And after that we wrote the spline. And so now, we ran it for the data that we had done this for, 0 1 2 1 2 3, and we also put its value at 2.5. I found it out. So, you see, we have polynomials. So, you see s0 so what I said was that it would be linear. So, it became linear. Look, its quotient of x2 is 0 but if you look at the quotient of x1, it is coming out to be one. So 1x plus 1 becomes ours in the first one, in the second one it becomes 30x square minus 59x plus 31 and s2 becomes 150x square minus 539 plus 511. So, this is our third polynomial, the two quadratic polynomials have come and this polynomial is defined in these intervals and the value that came at 2.5 is 1001 and we plotted it, so see, this is the plotting. So, in the first interval we have linear. After that it became quadratic. This became quadratic. So, you will see that the node which is an internal node, the function there is continuous but its derivative is also continuous, smoothness has come, we did not have the piece wise interpolating polynomial in the function because there was no smoothness there, but here, smoothness has come, right? So, we have defined it well in this manner, similarly we can check it on other data, now we can increase the data that we have, we can increase the data points because once we write a code, then after that you can increase its data points. So we increased the data points by four and we got a polynomial. Now if you see in this, then s0 is there, then once again we have a linear polynomial. Ok? The second one has a quadratic. The third one also has quadratic. The fourth one also has quadratic. But in the first one we know that it will come linear only. So, look, it has come to us. So, look in the first one, there is a line. Then after that it became a quadratic curve. The curve was formed like this and like this. So, look, the common nodes here, 2, 3, 4, the function there has become smooth. So, this is our Quadratic spline.



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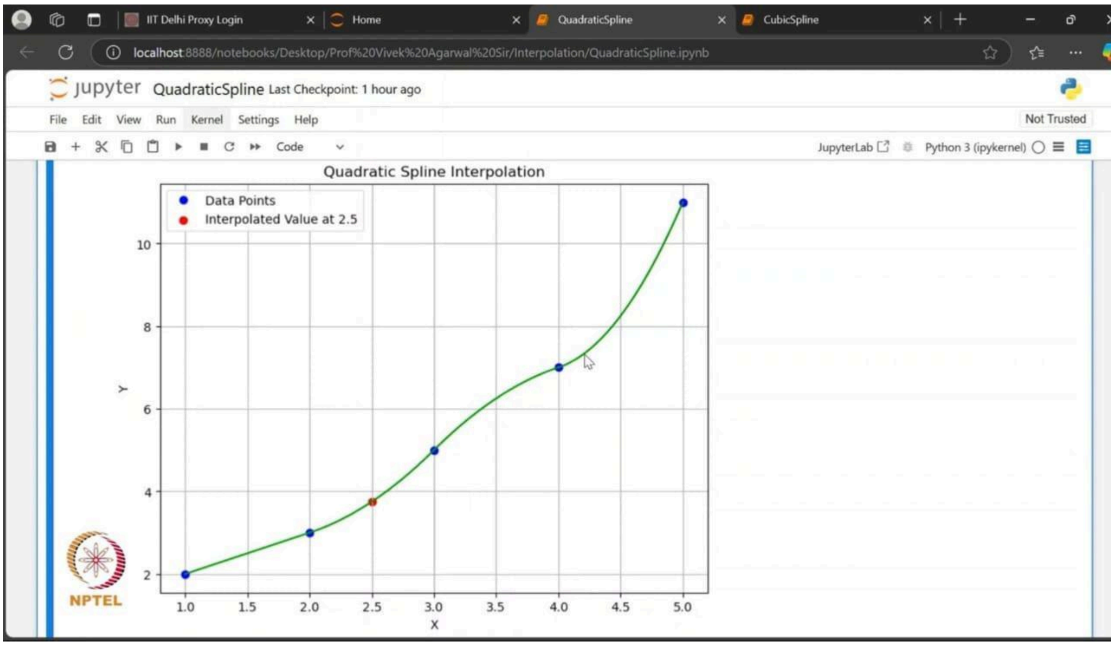
Similarly, we can define cubic spline. So cubic spline is the same but we will need a system of 4n into 4n. So, in that, first the continuity, then the continuity of the derivative and the second derivative continuity, the continuity of the curvature, we will define that. So, the matrix that we have, now what will be the initial conditions, the initial boundary values? So, we will have to multiply by six and by two because two was coming. In the second one and in the last one also six and two. Is it okay? So, after that we will have this system. So, this natural spline condition has to be imposed. And then we will write it to the supplies. And look, these are the same data points. But what should we do about it now? Supplies were installed. So, you see what is there in S0? We got Cubic spline. Ok? Cubic spline is also available in S1. Cubic spline was also found in S2. Ok? So, we calculated this at 1.5 and the value came out to be 1.75. So, see what is here? Even in the first one, the function was linear in the previous one but it is not linear in this one. If you see, there is cubic in this also. The second one also has cubic. The third one also has cubic. So, this is the function we have that has been smoothed out. In fact, the curvature you will see on common nodes is also the same.



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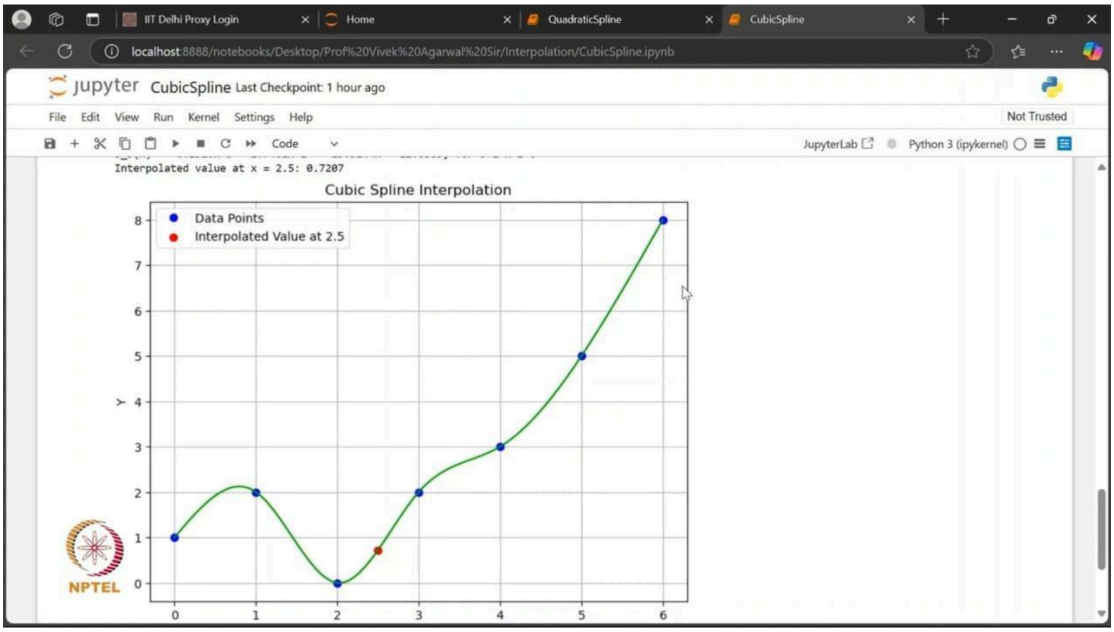
So here is this cubic spline. So, we found the p value to be 1.5. We can see this at its value which is 2.5. Ok? So, 2.5 comes to 121. Right? So, we have to check the values in this. So, you can find out what its value is. So we took this bean and when we calculated it, it came out to be 121. And our polynomials have come, this is it, the cubic polynomials have come, this is it. Is it okay? So now, once we have written the functions and written the code, what can we do now? We can solve this by using a large number of examples. Ok? So, what to do in this? Now we have s0, s1, s2, s3. Now we will get four. And these are cubic polynomials. So, you see how much is there in it. Here's a function that's also cubic. Then this cubic, next then this cubic and then this cubic. So, if someone is asked then they will feel that if they see this graph then we will say that it is some interpolating polynomial. But yes, this is definitely an interpolating polynomial. But this is a Cubic spline. Because if it was an interpolating polynomial, then its degree would be 1 2 3 4 5, there are five data points, right? So, its degree is a polynomial of 4 degree, but what is there in this, it is cubic, right, so if we look at the interpolating polynomial which we have defined in Lagrange's and Newton divided by it, then if we look at its base, then we have the number of points, now in this the number of points is five. Similarly, if the number of points we have is 100, then the interpolating polynomial will be of 99 degrees. But if we define cubic spline then we will get the cubic spline like this. Is it okay? So, this is the Cubic polynomial, if I now suppose I put five in it, I should make it six here. Like this I increased the value 5 and 8, it was like I increased two more values. I ran this and now look; we got five or six polynomials. This became ours. Is it okay? 1 2 3 4 5 6 7 So we have seven points. So, if we created the interpolating polynomial, we would get 6 degrees. But it is not like this in this. This is what we got, 3 degrees. Ok? So, our even we will have 100 points. So, each one we have to get is cubic of degree 3 and it has also become smooth. See how smooth it became. So, what will happen with smoothness is that our functions will give us better accuracy for what we need to use.



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So, if we look at it, like we discussed cubic spline or quadratic spline, then what we got in this is that it is piece wise and it is also smooth. Which we are not able to find in the piecewise interpolating polynomial that we had earlier. So, due to its smoothness, this supply is used extensively in various areas. So, I hope you have understood this lecture and thank you for watching this lecture.