SCIENTIFIC COMPUTING USING PYTHON

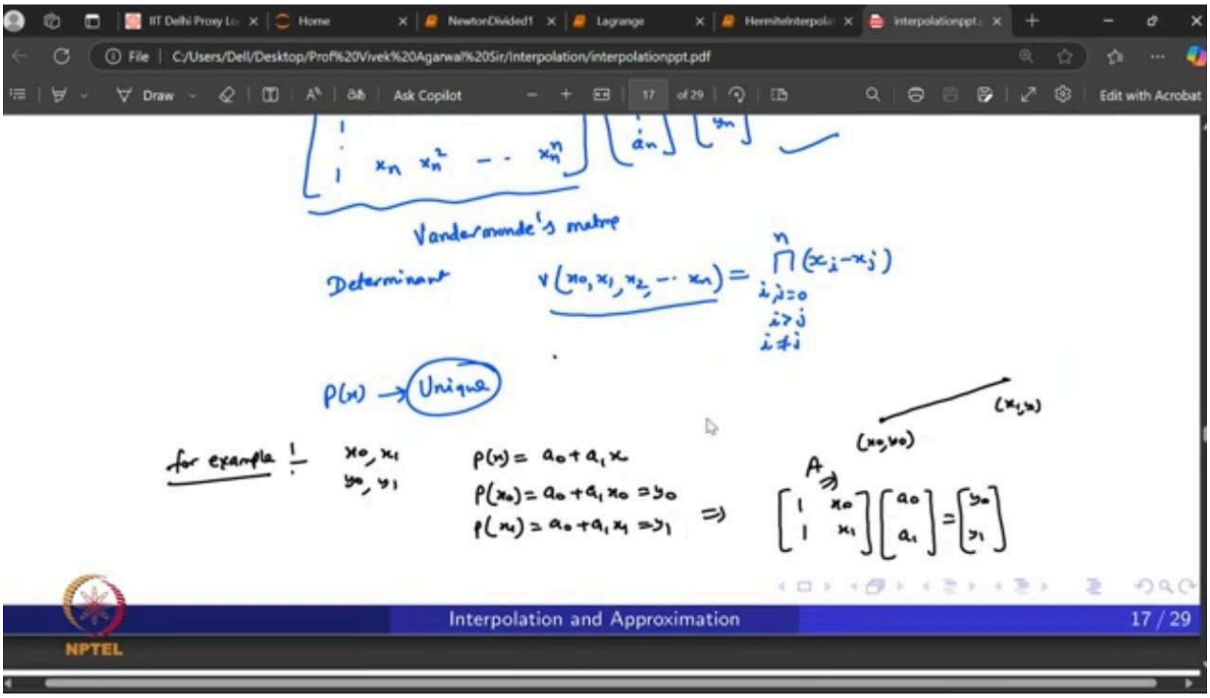
Professor Vivek Kumar Aggarwal & Professor Mani Mehra

Department of Mathematics

Indian Institute of Technology, Delhi

Lecture No. 22

Welcome to Scientific Computing Using Python. So, as we started in the last lecture, how can we find out interpolating polynomials. So today we will continue with that. Let's start. In the last lecture, we saw that if we have a set of n plus 1 point on which we have given data and those points can be equally spaced or unequal spaced, then how can we find out a polynomial of degree n which the interpolating will do and all the elements, all the points. So, what we did was, first we defined a polynomial of degree n and after that we saw that the interpolating polynomial will be p at xi will be equal to y i and then from there we got a matrix vandermonde. So, we saw that if all the elements are different, then the determinant of the vandermonde matrix will be non-zero and we get a unique solution from it. So, we started this example that let's see how to find out a linear polynomial. So, if we have two points and we have a linear interpolating polynomial, from there we got these values.



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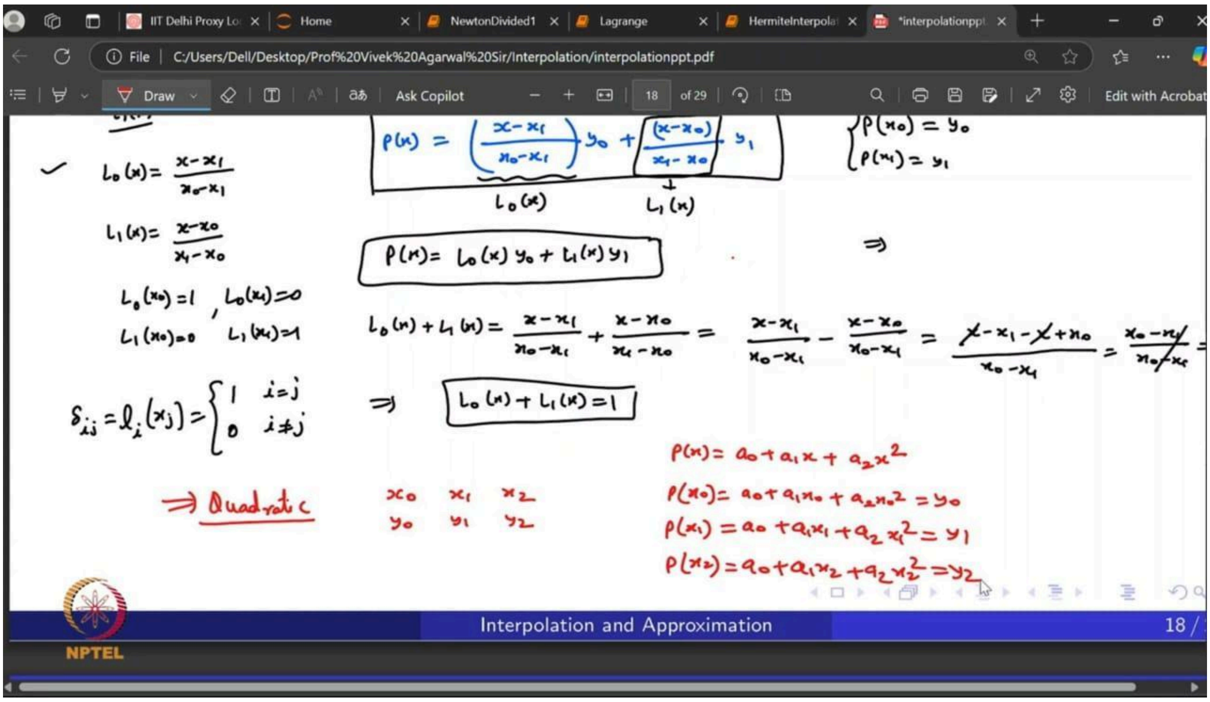
So let us see further how we can find it out. So, we have a polynomial. So, we have to calculate this polynomial. So let us see, it is the same polynomial. I can write this polynomial like this. So, we will get this polynomial p x. Now let me rearrange it a little bit. What I will do is, I will separate the term of y0 and I will separate the x zero and y1. So, if I will separate the term of y0, then it will come x1 by x1 minus x0. I took it from here one y0 will come to us from here. Okay, minus x1 over x1 minus x0. So, this is our y0 plus now we will take the term of y1, so what is in y1, it will come here. So, I will write x in place of 1. It will come x over x1 minus x0 and y1 is also coming from here minus x0 x1 minus x0 y1. We will write it like this now. What do I do with this because the denominator of both is coming out to be the same as x0 minus x1. So, I would write x minus x1 x0 minus x1 y0 and here we have written x minus x0 and x1 minus x0 y1. This polynomial will become ours p x. Now we have written it in this form. So now if we want to see, if this polynomial is an interpolating polynomial or not? or I look in this polynomial and if we put p x not, then instead of x, I will put x0. This will become the same factor. It will be cancelled. Here x0 minus x0 will become 0. So, if you see, it will become y0 0. Similarly, if I want to see p x1, then it will become zero. And this x1 minus x0 will cancel out. It will become y1. So, it means that this is the interpolating x1 and x0 x1. So, this polynomial will become we have this linear polynomial. So, now let us give some name to this polynomial. So, what did we do? This is a polynomial. I named it L0x. So l zero which is x means we are doing it with l, so this polynomial we will write it as l1 x, so this l0 x is l1 x, so this is a Lagrange Polynomial, so we call it Lagrange Polynomials or Lagrange fundamental polynomials, we call it fundamental polynomials, similarly it is, so what is it in this case, it is linear, so now if we see, we have formed px as L0x y0 L1 x y1, so if we see, we have taken L0 x and L1 x which are linear polynomials, so we consider it to be the basis and on these two polynomials we have defined an interpolating polynomial p x is defined and taken their linear combination in terms of y0 and y1, so y0 y1 which are our coordinates, which are our values, and L not x and L1 x become fundamental polynomials, so we have written this It is written like this. Now if we see, if we see in this, it satisfies some properties. So, we saw that L0 can be written like this. L0 means it is defined at x0, so we will leave x0 and write x minus x1 and see, x0 minus x1 is written here. Okay, and if we want to write L1 x, then what will we do? We will leave x1 and write x minus x0 after that I will put x1 minus x0 in it. This comes to us. Now if we see, what is L0 x0, L0 x0, if we see is one, what is L0 x1 is zero, what is L1 x0 is zero and what is L1 x1 is one. Okay and after that we saw that L0 x plus L1 x, we summed these two, L0, so the condition we have, we have to see what will be the sum, L0 x plus L1x, let us see, so what did we do in this, x minus x1 over x0 minus x1 plus x minus x0 x1 minus x0 did here, so I can write here from here. I am x minus x1 x0 minus x1 minus x minus x0 and it will become x0 minus x1. So, if we look at it, we have x minus x1 minus x plus x0 divided by x0 minus x1. This is what we have. Okay, this gets cancelled. Now if we look at it, we have x0 - x1 divided by x0 - x1 gets cancelled becomes 1. So, from here we will have another condition that L0 x plus L1 x is one. I took the sum of them, so that is one. So, from here we can write some properties. So, we can define a delta function, delta ij, so what will be delta ij, L i at x j. So, this will be equal to one when i,j is equal to one and zero when i,j is not equal. So, these properties, which we have, are fundamental polynomial and satisfy it. So, we have defined this polynomial for linear. So similarly, if I had two points here, now let us see what would happen if we take three points, if it was linear, then it means that if it is linear, then it can also be quadratic. So what is there in quadratic, we have three points, x0, x1, x2, y0, y1, y2 so what would happen in this, so what would we do in this, let's assume that our polynomial is a0 plus a1 x plus a2 x square, and after that if we pass it through these points, it will come out a0 a1 x0 plus a2 x0 square and that is equal to y0, just like that, p of x1, we took a0 a1 x1 plus a2 x1 square becomes y1 and p of x2 becomes a0 a1 x2 plus a2 x2 square is equal to y2.



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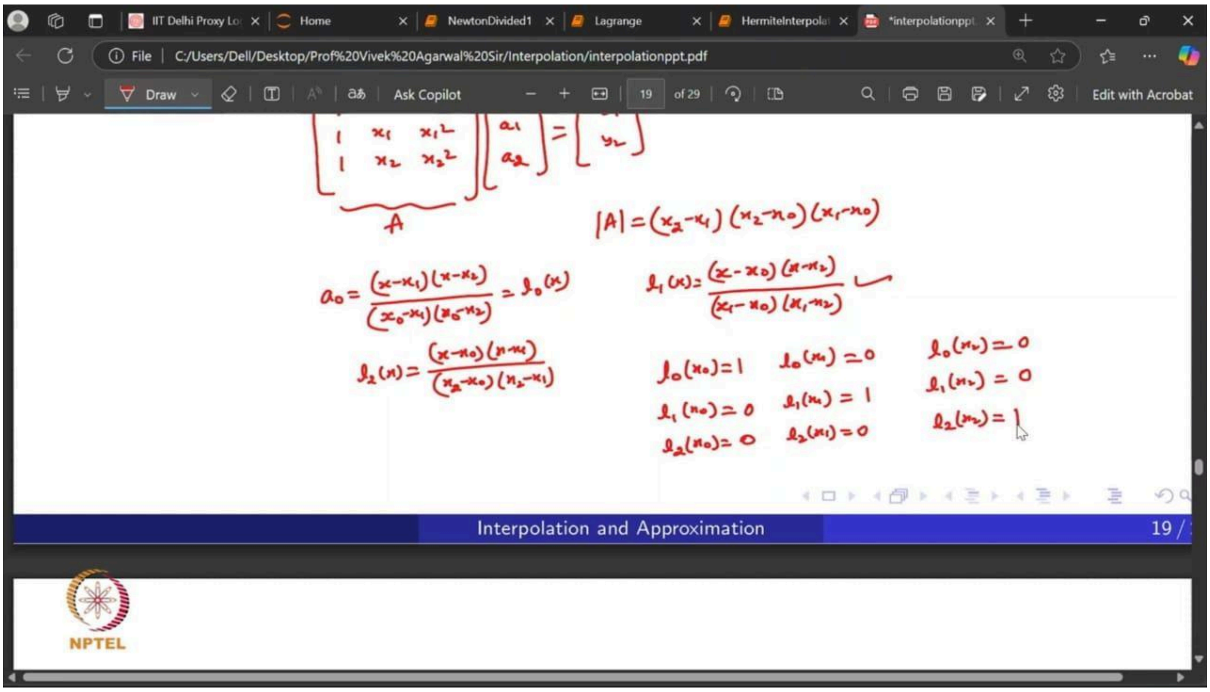
So, if we write it in its form, then we will get this. So, what we have is, then we will have this system of equations, 1, x0, x0 square, 1, x1, x1 square, 1, x2, x2 square and here a0, a1, a2 are unknowns and these y0, y1, y2 are knowns. So, if we solve this, then this is our matrix, suppose I name it A. So, the determinant of A will be x2 minus x1 x2 minus x0 x1 minus x0, which we had defined earlier, we can write it on its basis, so this becomes our determinant and all the points x0 x1 x2 are all different distinct points, so they are non-zero. So if we do this, we will know that our a0 or a1 or a2, so if we calculate, then the a0 that comes out will come out like this x minus x1 x minus x2, which we will write coefficients, okay and x0 minus x1 x0 minus x2, okay, so if we see, this comes to us, we will have L0 x, similarly we will have L1 x and we will get x minus x0 x minus x2 and this comes to us as x1 minus x0 x1 minus x2 okay, in the same way we will get L2 x because it will go up to two, here what will it be x minus x0 x minus x1 divided by x2 my x0 and x2 minus x1, this is it now if we see, we have this polynomial, this fundamental Polynomial of the second degree, this is it, so if we see, it is satisfying, now if I see L0 x0, what will come, x0 x0 will come one, if I see L0 x1, what will come, we will put x1, okay so now I put L0 x0 to this, now I put x1, now I put x1 in it, it got zero, I put L0 x2, this also came out to be zero, okay now let's look at L0 x1, so if we put on x1, it got zero, now let's see, this is it, let's see from L1, now we have to see what is coming, L1 is this one, if I put x0 here, it became zero, now I put L1 x1 If we put x1 then it will get cancelled, it will get one and if we put L1 x2 it will become zero, so yes, let's see what is L2 x0, it has become zero, L2 x1 also becomes zero and L2 x2 will become 1. So, in this way we have these values. So, the condition of the delta function that we had got is satisfied. So, if we write on its basis, then what can I do now, I can write my interpolating polynomial, which is a Lagrange interpolating polynomial, then in this case it will come p x is equal to L0 x L1 x y1 L2 x y2. So, this is our second order Lagrange interpolating polynomial, and it is satisfying all the properties of our fundamental polynomial, it is satisfying all the properties, so we can define it in this way.



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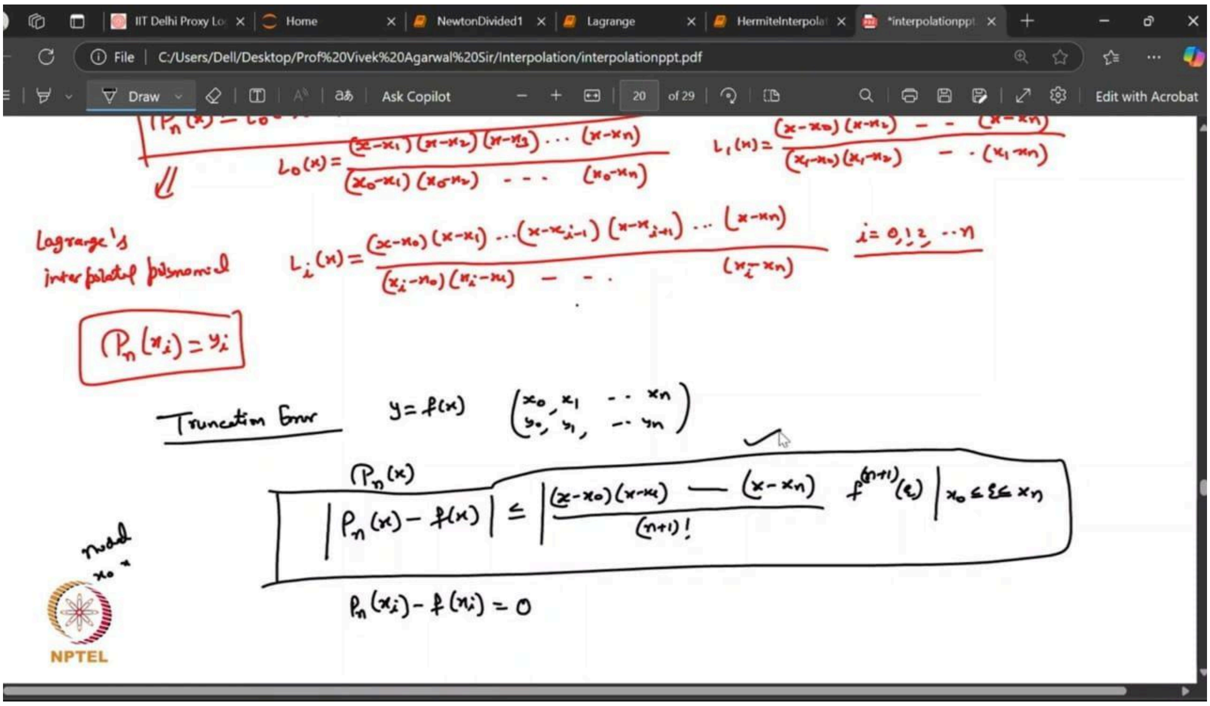
So, now we have this, I am doing this, so what can I do now, now we can write it in generalized form. So if we have x0, x1, x n values given y0, y1, y n like this we have data given so if I have to write the value of n degree then I can write it as p 2 ok but now I have to write p n p n means n th degree polynomial so what will be the polynomial of degree n L0 x y0 plus L1 x y1 plus and at the last we will have L n x y n x ok so we have taken linear combination of these now it depends on what L0 x now see what will be the polynomial of n degree will be the polynomial of n degree and the value of degree is now the polynomial of n degree so its meaning and in L0 we also saw above that when we write L0 like we wrote L0 then a x 0 is not appearing anywhere above in it then we replaced x by x0 in denominator so we will do this work in this also so now see if I write this then you can directly write x minus x1 x minus x2 x minus x3 x minus x n I wrote this and I wrote this here I will write x0 in place of x so x0 minus x1 x0 minus x2 x0 minus x n so this factor is here so now see if it is up to n then if we multiply them then we will get a polynomial of n degree and a constant value is there in the denominator so if we write this then how can we write L1 x, we will write that x minus x0 x minus x2 and the product of all of them is okay divided by x1 minus x0 x1 minus x2 x1 minus x n so we will keep writing like this so if we keep this in mind then we can write any factor now so let me write L i I will write a general and what will be x minus x0 x minus x1 x minus x i minus 1 x minus x i plus 1 okay and in the end x minus x n all these will be multiplied now in place of x x i will come or it will come x i minus x0 x i minus x1 and in the end we have x i minus x n okay and this is true that from 0, 1, up to n, okay so, in this way we will keep writing the total which we have is L0, L1 and L n so this is a n plus 1 polynomials and each polynomial is of n th dimension n th degree, so if we write it in a general form, then this form will be, this polynomial which we have, we will call it Lagrange interpolating polynomial, this we have come to know clearly that it is an interpolating polynomial, because if I find p n x i anywhere, so if i take x i, then we have seen that x i x i if we substitute it in it will get cancelled, so it will be equal to y i, so that is why the condition of interpolating that it should pass in it is getting satisfied, so if it is getting satisfied, then it means we can do it like this, now if we want to see its error, whether it is a polynomial of n degree or a polynomial of second degree, so if we want to see its truncation error, that is, in what type of truncation error can we write it, so like the error that we have above was of second degree, so we will do the same in this also, the error in is there a truncation error or approximation error, suppose we have a function given f x, okay, and suppose we have values of its x0, x1, x n, then y0, y1, y n , suppose we have a function y is equal to f x is given and values on it is given and if we approximate this with the n degree polynomial and that is Lagrange’s interpolating polynomial, if we see this, then p n x, okay, minus f x, the magnitude of this will always be less than equal to x minus x0, okay, x minus x1, x minus x n divided by n factorial, and here what we get will come this, if one is n is so n plus one will come n plus factorial, and this n plus 1 th degree derivative of xi and this xi coming, okay, and this xi which belongs to x zero to x n okay, so ,if this is positive let's take this, so here we get this bound that the maximum error for any x will be this, it is obvious that if I look at p n at x i, minus f x i, then i will come somewhere here, if we substitute it from in the middle then it will become zero, right? So what does it mean that if we look at the nodal points, our nodal points x0 x1, then when the polynomial passes, then obvious that error will be zero, otherwise if we approximate the error on any x, then the maximum error we can take we can calculate it from here, right, so with the help of this we can calculate it, so this was our truncation error or we can do it with the example of approximations map.



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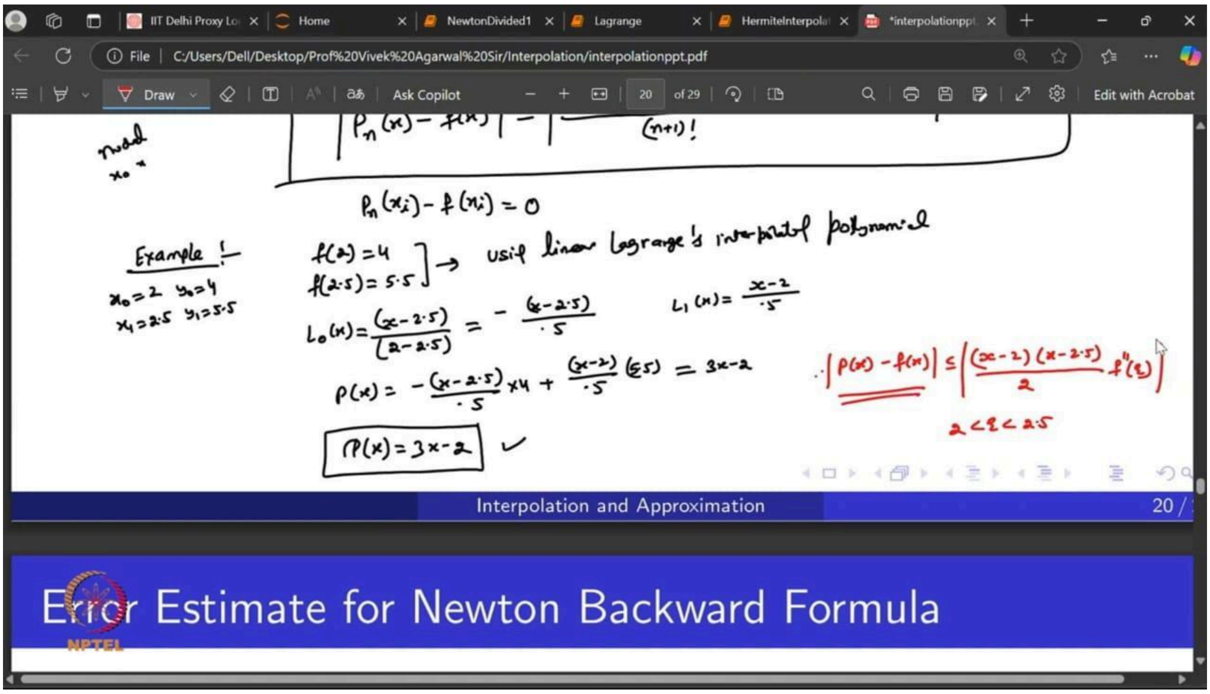
So, let's do it, now see what we have done here, so we have done the Lagrange’s, now suppose we have an example, so I took any value and we have, let's see an example, so suppose we have the value of a function given on 2 and that is suppose 4 and f at 2.5 which is given and is 5.5. So, if we want to do this, then I can approximate it using Linear Lagrange’s Interpolating Polynomial, so this comes to us. If we want to calculate this, then the first thing we will do is our L zero. So, see, what will be L zero in this. Now in x0, there is 2 in it, y0 4, x1 is 2.5 and y1 is 5.5. Now see, the gap between the two, which is h with that f is L0 x, what is x minus 2.5s. Now if we leave x0, then x1 will come. How much is x1 2.5. So, this will become 2.5 divided by x0 so 2 minus 2.5. So, this will be the value. So, if we see its value, then we will have it in minus minus x minus 2.5 divided by .5, this comes to L1 one x, that will come to x minus 2 divided by x1 minus, so this will become .5, it will become positive. So we will substitute this, so our p x becomes, what does it become minus x minus 2.5 divided by .5 into y0 4 is here plus x minus 2 divided by .5 and y1 is here .5 so whatever value we have put for it we will calculate it so after calculating we found that its value is 3 x minus 2 it means our interpolating polynomial that was linear polynomial that is 3 x minus 2 comes. Now see if I put x as 2 then it will get 4 if I put 2.5 in x then it will get 5.5 so this is our approximation so if we want to make it second order if the number of points increases then it will become second order okay so if we want its accuracy or error then the error which we will get in this case I can write it as that the error will be p x minus f x because we have not given it yet if is given then we can do it very easily so it will come to x minus 2 x minus 2.5 divided by 2 factorial and the f second derivative xi and the xi is laying from where to where it is taking it is from 2 to 2.5. So, if we get the function, we will find its second derivative, we will put its maximum values in this interval and after that the error between them will always be less than this, so we can say that this is the bound, it gives the maximum bound, so with the help of this we will do it.



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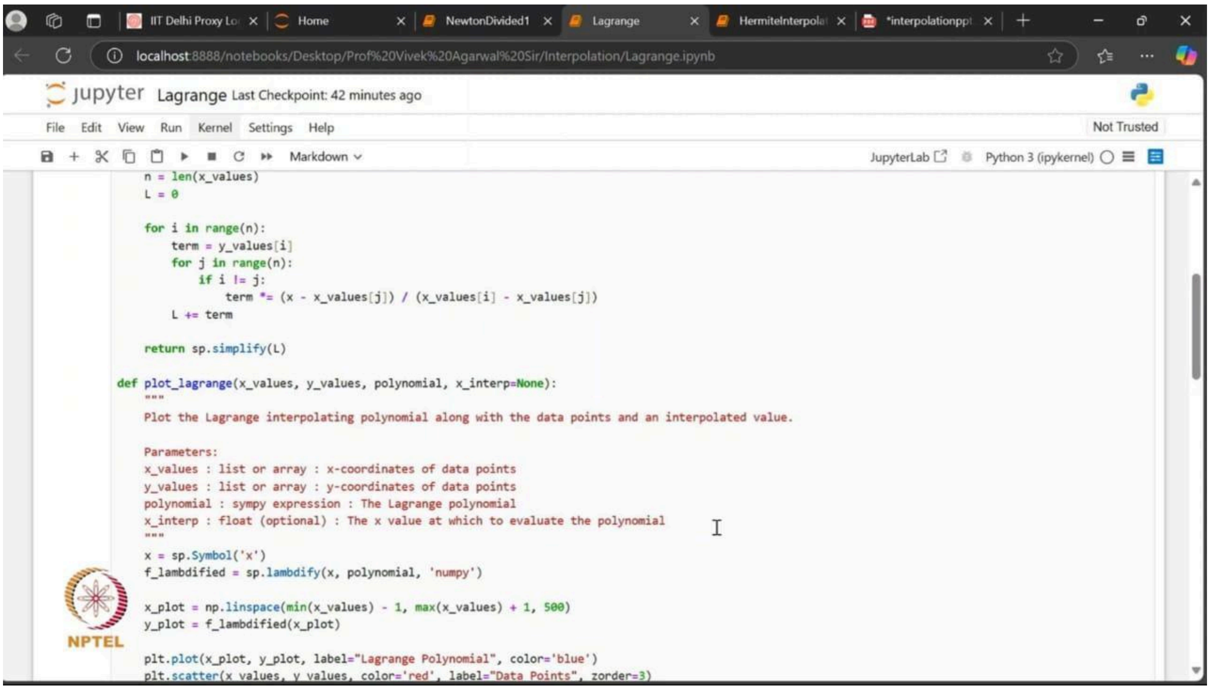
So, we will increase the number of points like this and as we increase the number of points, we will see that the calculation will become a bit difficult for us to calculate, if it becomes difficult to calculate it, then it means that we will have to take the help of the computer, so we will figure out how to use the program in this case, so if we take this polynomial or any other polynomial, then we check it, so suppose now this is my code and the code of Lagrange's, we try it on different values or different questions, so what did we do in this, we wrote its name Lagrange is ok, this is Lagrange's interpolation, so in this as we have been doing earlier, we will use NumPy, uses NumPy, SymPy, Matplot ok, so we have defined a function Lagrange's interpolation in which we will input the x value and y value and what will we get in return, we will get a polynomial and the value on which we have interpolate, ok, so the interpolation that we have to do is any value that we need in between, we will do it there, so like this, we have defined x in symbolic, defined its length, started from L equal to 0, so this value is x minus x values which is divided by x value i minus x value j, so what will we do, the Lagrange's which was the fundamental polynomial, we will keep creating it, ok and return it and after that we have defined a plot Lagrange's, function x value, y value and polynomial and interpolation on where we have to do interpolation, ok, x value, y value. Now we will get the polynomial and this x value of x at which to evaluate the polynomial. So, we have defined it in such a way that how to plot it and what we will do after that is after doing this, we will see the data for different types of data.



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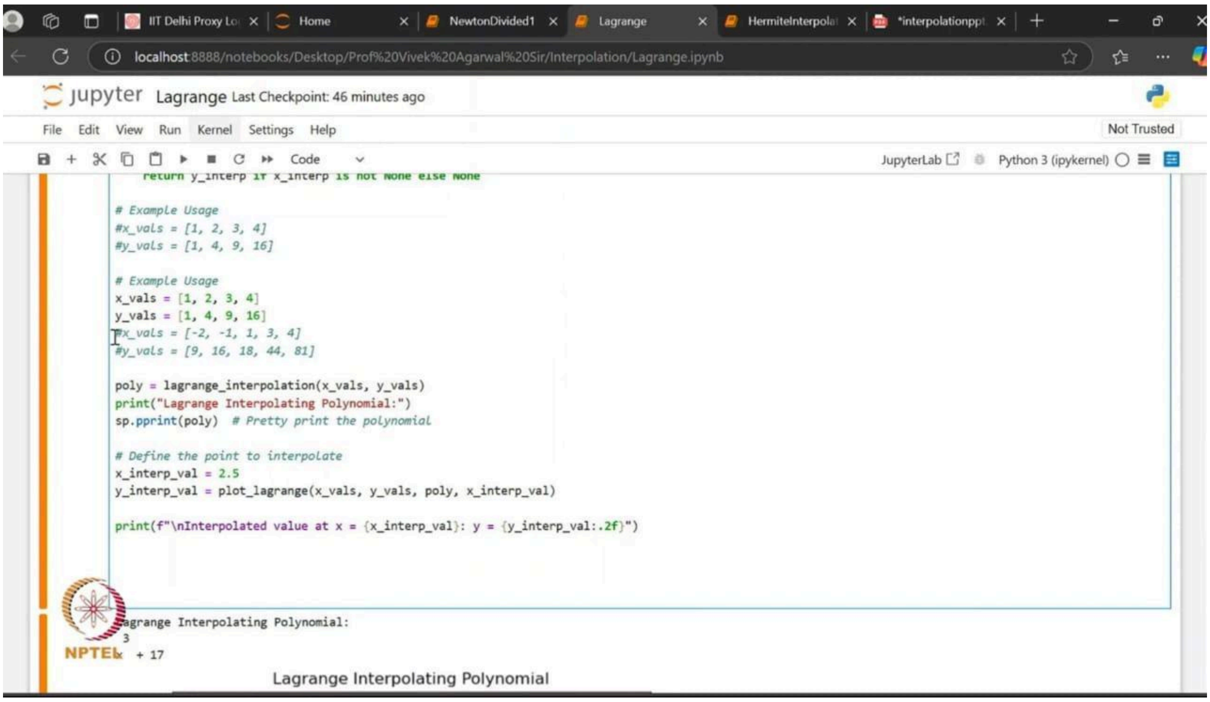
So, we have defined this data for the time being, now what do I do, we have defined the x values and the y values -2 -1 0 1 3 4 so you see this is unequally spaced, there is a difference of two between one and three. Now what do I do in this, I have defined this and from here I have called a function from Lagrange's interpolation, I have transferred the x value and the y value from here and input it. From there what do I do, we will get a polynomial, we have printed that polynomial and then we have defined a value where I have to check it, so suppose I take 2.5 between 1 and 3 and what did I do, I called the y interpolated value there and I will get the value of y and after that I got it printed. So, if we do this, I ran it and as soon as we ran it, we saw that the polynomial we got was x cube plus 17. And then we plotted it and saw that there were 1, 2, 3, 4, 5 six number of points and I saw on 2.5 so the value on 2.5 that is coming out is 32.6 our interpolating value came. So now see, 2.5 was somewhere between 18 and 44. So here we got it approximated and we got 32.62 somewhere here. So, the values we got from interpolating polynomial. Now I can change the data a little, so suppose I removed the zero from this and reduced the number of points. So, like I reduced the number of points like before it was 6 now 5. So, if we see, we will have to do all the calculations again. The polynomial is the same x cube plus 17 and the polynomial is same and the value came 32.62. But we have to keep in mind that all the calculations in this will be done again. We will have to take some more values. Let's take this value, 1, 2, 3, 4. So we have taken a value which is Equi spaced, it will work for unequal space and equal space as well, the value of f is 1, 4, 9, 16. Okay, I plotted it and ran it.



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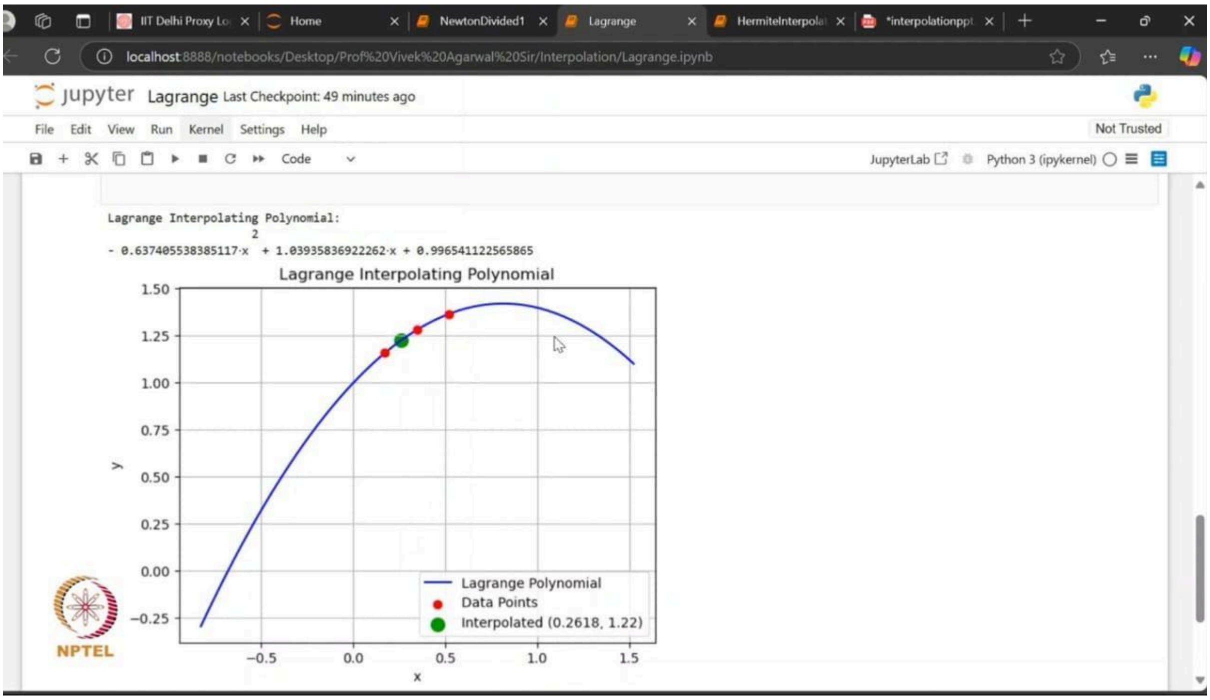
So, you can see that x square came because the value that we had defined was done in this way that I am taking x square, so x square came and the value of 2.5 is 6.25 came. So, from here we took out the value, so our exact value came out. In this case, now I change it a little bit. So, what do I do, I take 2 on one and on this I take 5. On this I took suppose 10 and in this I took 7. So, what did I do? I wrote a x square plus one and let's see what will come out. So, see, x square plus 1 came and its value came out to be this. Okay, so what happened in this case, for our polynomial of second degree, we got the Lagrange's interpolation polynomial, we got this and on its base we took Once we have calculated this, what we have done with the help of this is that the values are a little easy, suppose we get a little complicated value, suppose I change this value a little, so suppose I have the value of x, if we have 0.1745 1745, the value of x1 is 0.3491 3491, okay and the value of x2 will come to us 0.5236 5236, suppose we have given values on this, 1.1585, this comes to 1.2817 after that 1.366 or comes, so this is given to us, okay, so if we have to calculate any value, then suppose I calculate the value on which 0.2618 2618 this value, on this I have approximate values, then i ran it the polynomial of degree 2 comes because three number values three points, which are data points, three. If it is there then you can see that the coefficients that have come in it are in fractions so if we do it with pen and paper So if you are not able to do the calculation then the same code will work in this as well, so when we calculated it, we got this value 1.22, so we calculated it with the value of this, so look, this was what it was, so let's say we have it very easily, we can do it.



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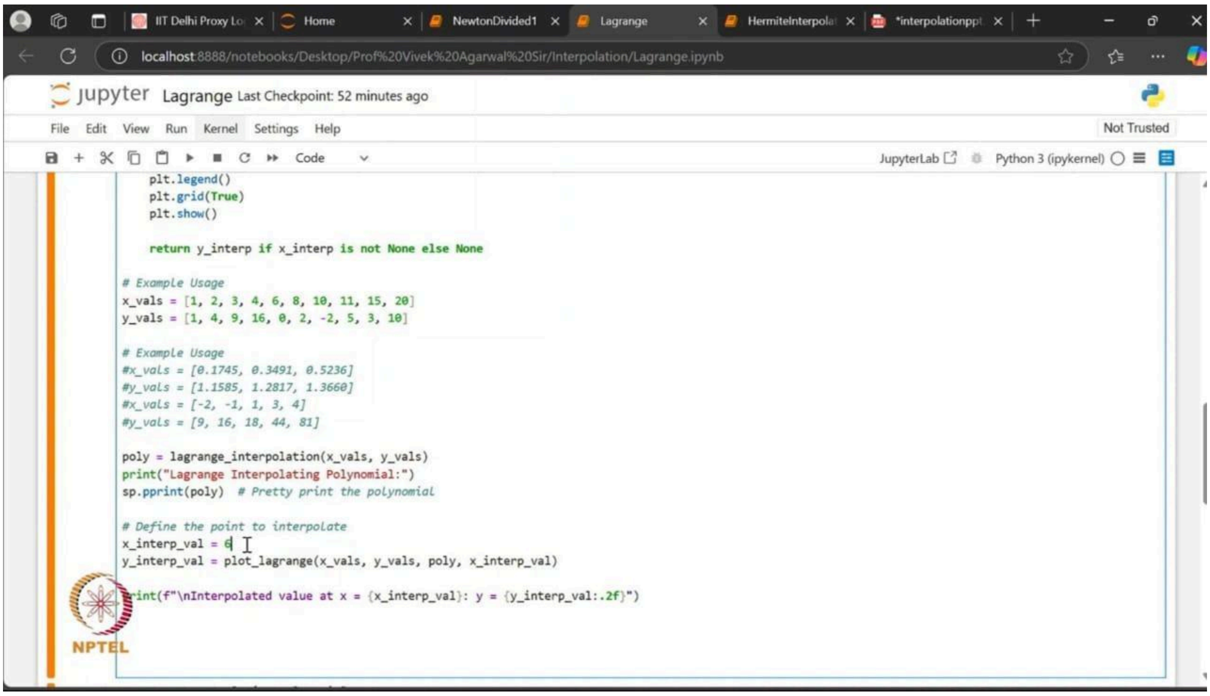
Now I will make it a little bigger and then let's see what happens, so suppose I took this value 1, 2, 3, 4, after that I took six, then 8, then 10, then 11 and then 15, I took this, okay 1, 2, 3, 4, 5, 6, 7, 8, 9 and I will take one more 20, have taken 10 values 10, okay, after that suppose I took the value zero, okay after that I took 2, after that I took minus 2, 5, 3 and 10, I just took such a value because we checked that the function was giving good value, okay so we are just taking a random value and what am I doing in this, I am fluctuating. The value after 16 came directly to zero, then it went to two, then to minus 2. Now what I do in this is that I approximate a value which I do not have. So, I do not have the value of seven, right? So, the value of seven is not defined here, so I see what the value would be after that. So, I ran this, now let's see what happens. Now see, we had 10 number of points, so the maximum polynomial that we can get is the polynomial of n th degree. So, we got a polynomial of n degree. So, whatever we want to represent it, we can represent it like this. Very random numbers are coming, very denominator. See how big the values are coming. So, if you see, we have interpolated it from here, this polynomial is of this type. It fluctuated here and then it went back up. So, we calculated from here, the value of 7 is minus 2.72 here. So, these values are the polynomial that would be Lagrange interpolating polynomial will be defined in this way, so we can see that the same code works for 10 data points as well, right?



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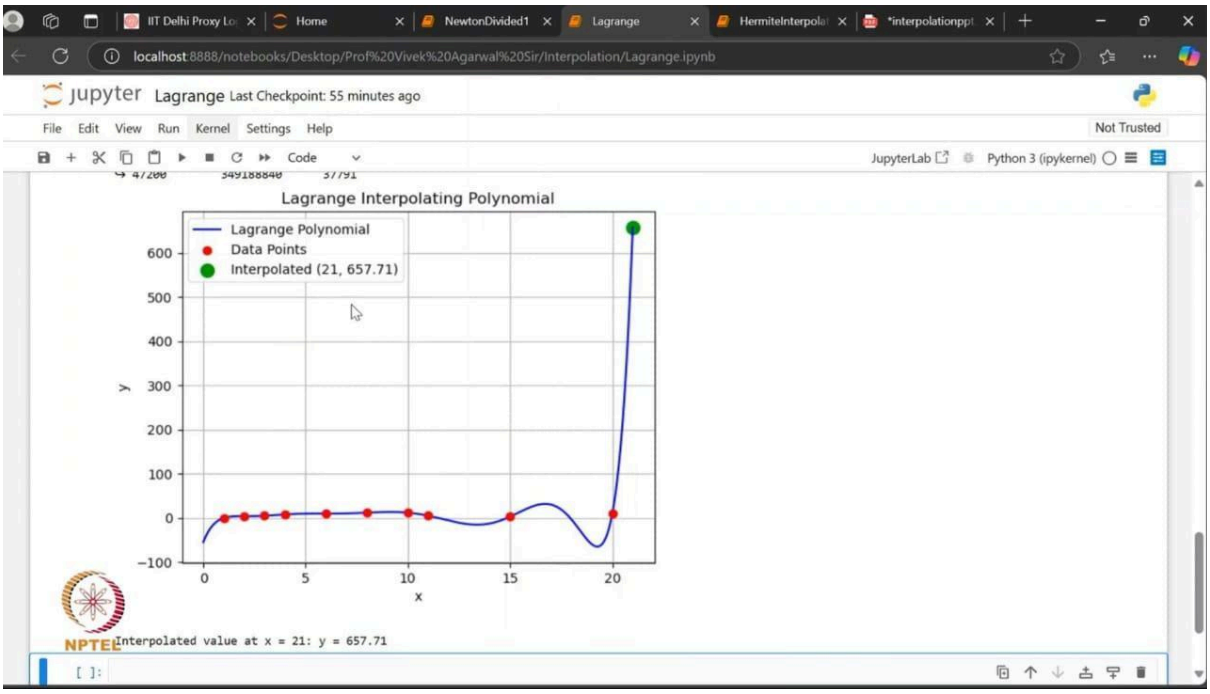
So if I define this at suppose six, I want to see what happens at six, the exact value will be zero, so what will happen in this, the error will be zero at this point, so here we can see that at six, the value of zero will come, so in this way we can input big data here, if you have data and this data is anything, suppose it is the oil price every day in the last one year or it is the dollar price or it is the price of a stock of a company, so the x can be taken out corresponding if days are chosen then days and value of y will be its prices, in this way we can see the its prices and can find the interpolating polynomials, but there will be a problem in this that as the number of points increases, the polynomial will go here and as the degree of the polynomial increases, the calculation will become hard, even the computer will take a lot of time and the computer will take a lot of time and we know that If we have a polynomial of degree nine, then its nine roots will also be brought, right? So, we also know that its roots, the nine roots, will also be brought. So, as we keep increasing the degree of the polynomial, it is not necessary that our approximations get better, maybe it doesn't get better and it is possible that the computer may have to work hard to calculate it. Like this, now I will change it a little more, okay, instead of four, I made it 5 or 8, then I made it 10, made it 12, okay, I made it 12. Now I did this polynomial, see how it works. Here also there is a fluctuation, so where the values are zero, this polynomial is passing. So, these are the data points and these values, which are green and exact values, with the help of this we can measure any approximation and I can also do extrapolation. In this, I want to calculate it on suppose 21. Okay, if we calculate on 21, then see, 657 came. The value on 20 was 10. The value on 21, 657, is extrapolation. Now, it is possible that the value is correct or wrong. It will depend on the fact that if we had a function that defines all these values, then we could have calculated from there what the error is. Now, we can tell the maximum error, we can tell the bound. So, the value that we saw on 21 is 657.



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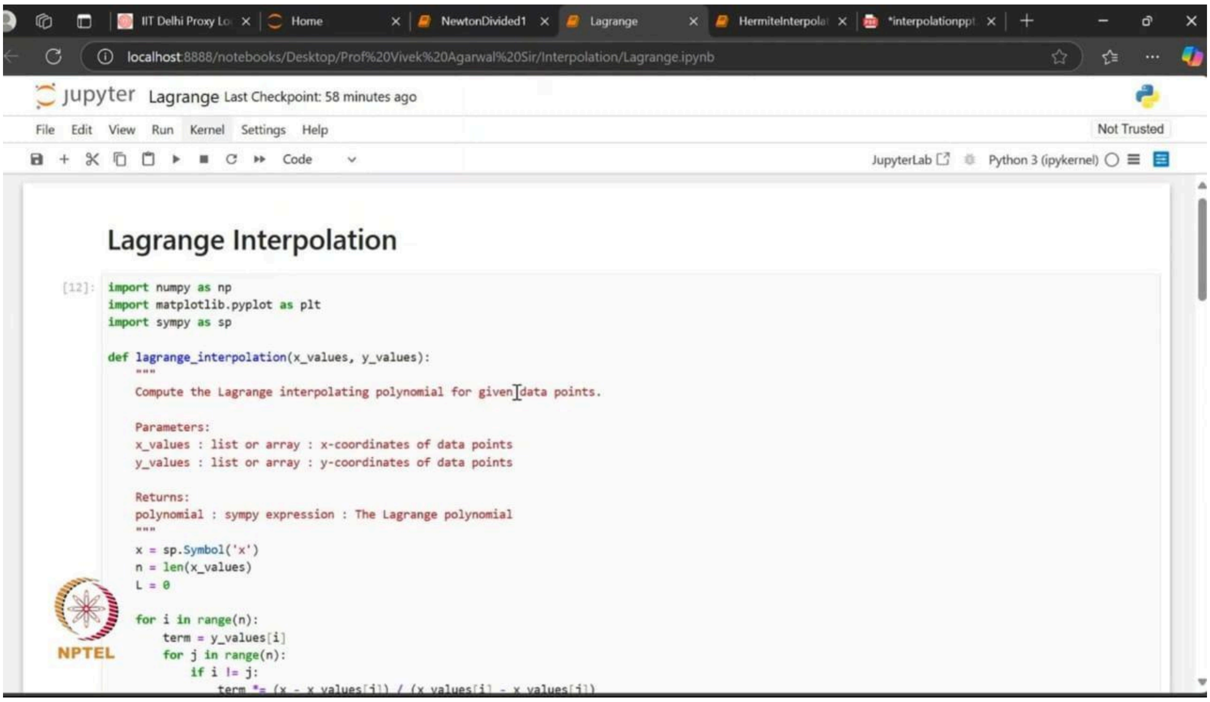
And let us see what value is coming on 20.5. So, the value that we saw on 20.5 is 214. Okay, why is the value changing so much? It is possible that the error has increased because it is doing extrapolation, it is going outside 20. If I go inside 20, then suppose I make this work 19. See, as soon as we do 19, the values came under reach of minus 59 here how much is there 20 over 10 so this value which was 20 over 10 so it has to go down because it has to go down Polynomial has to go down because the Polynomial was at 15 was at 3, right and right, it will fluctuate like this and this value will be calculated, so according to the way we are doing it with the interpolator, so according to this we can play a lot with our code and the values which we want to interpolate can substitute, so if I take extrapolation, then if I see what value is coming on minus minus 1, like if I saw on minus 1, then minus 326 comes because this was polynomial here, okay, so extrapolation is a prediction and depending on how we are doing it, it is based on that, but interpolation polynomial, interpolating polynomial is good for the interpolation, so we have used this code which We have run the Lagrange interpolating polynomial and seen that it is giving the same values for different data.



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So, if we check it for different data values and if it is working then we say that this is our method and the code that we have created is robust. So, I hope that you have understood how the Lagrange interpolating polynomial was generated and how we can create a code for it. And, thank you for watching this lecture.