SCIENTIFIC COMPUTING USING PYTHON

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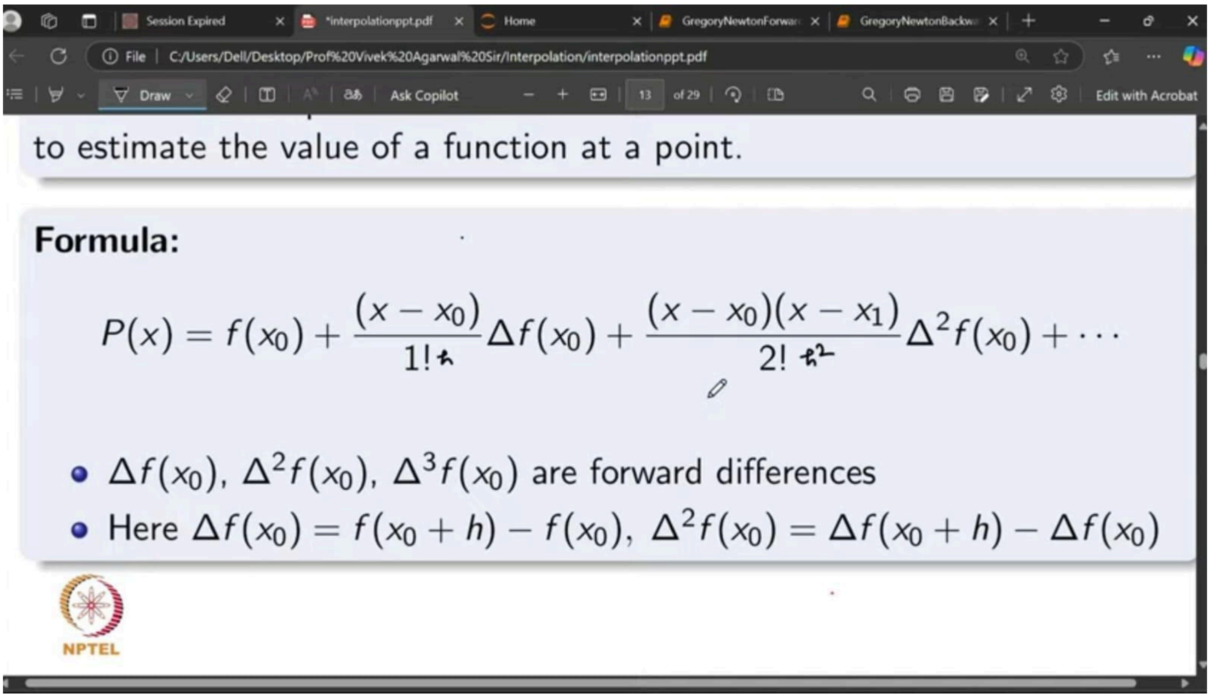
Lecture No. 21

Welcome to Scientific Computing Python. So, in our last lecture, we used the Newton forward difference formula. We saw how we can find out a value by using interpolation. So today we will discuss the Newton backward difference formula and create a Python code related to it. Let's get started. So, we did the same thing in our last lecture. We used this formula. We saw through the Python code how we can find out the interpolating value.



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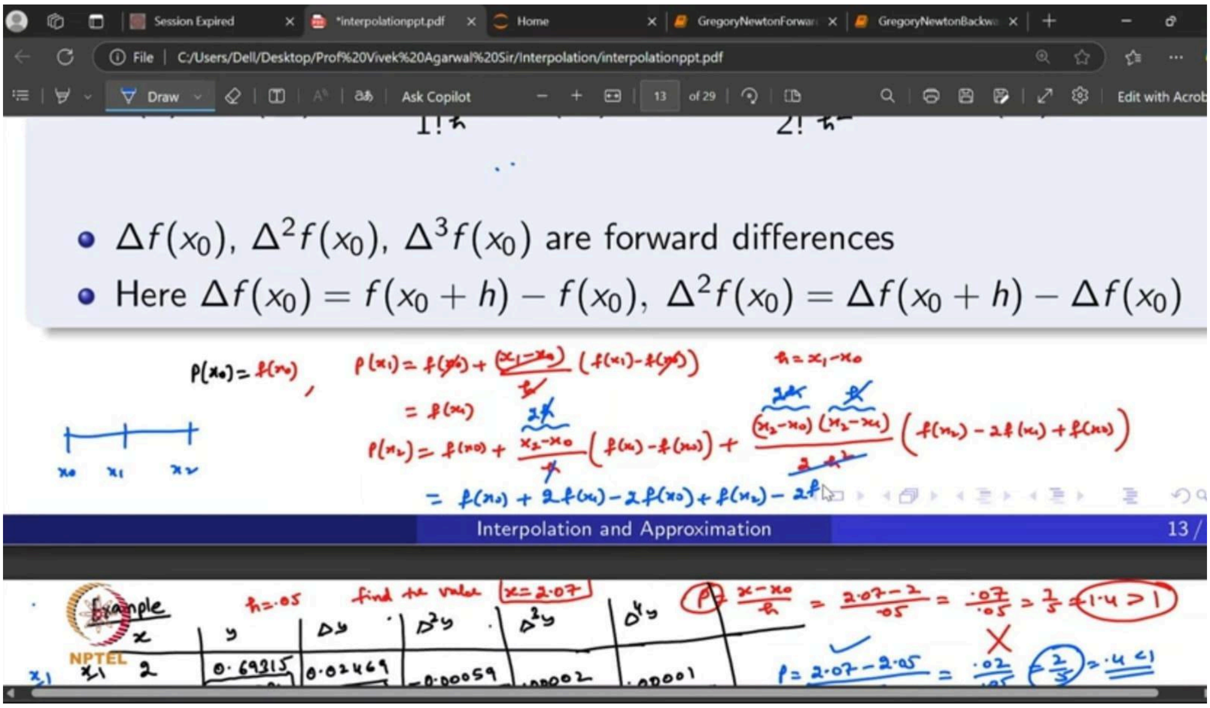
So, this Polynomial that has come, if someone tells us that we need to check this. Suppose the Polynomial, these values are coming out to be x not or x one, if we have to check this, then see, if I am putting this Polynomial forward, then if I have to calculate it as p x not. So if I have to calculate this from the Polynomial that we have found out from Newton forward, then I will put x not In all these, zero will come in all the places, so this will become f x not, if I calculate it at p x one, then what will happen, it will become f at x not plus x one minus x not by h, okay and this is f x not, so we know what is f x not, the forward one, so it is f x one minus f x not, okay, so we will calculate from here and the values beyond this will all become zero, now let us see what is h, a, in this case, it is equally spaced, so it is x one minus x not , so I will cancel it from here, it will get cancelled from here, so here comes our f x one, now we need f x one, so what will happen we need p x 2, the value will go till here, so now what will we do with this a x not plus x two minus x not by h f x one minus f x not, here comes our x two minus x not x two minus x one by two h square and these values will come we had seen f x two minus f x one plus f This will come, okay so if we calculate it, then we will see x two minus x not which is h, so this quantity which is h, what is it, this is 2h, what is it, this is 2h because two values are coming in between, this is x zero, this is x one, so from here to here it will become two h, okay from here to here 2h, so if we see, this will get cancelled by this, this will get cancelled by this, the value left from a will be okay, ours is cut, so if we see, what will be left, see, here t comes, so now minus 2 fx1 f x two f x , so how much value came, it became a x one plus two times because two will be left on f x one minus 2 times f two this is left, so from here everything gets cancelled, so from here we will have fx2 minus 2f x one plus f x zero, okay so now let us see how it happens, this gets cancelled, okay f x not f x not Minus 2 f x not gets cancelled so if we see, we are left with f x two, okay.



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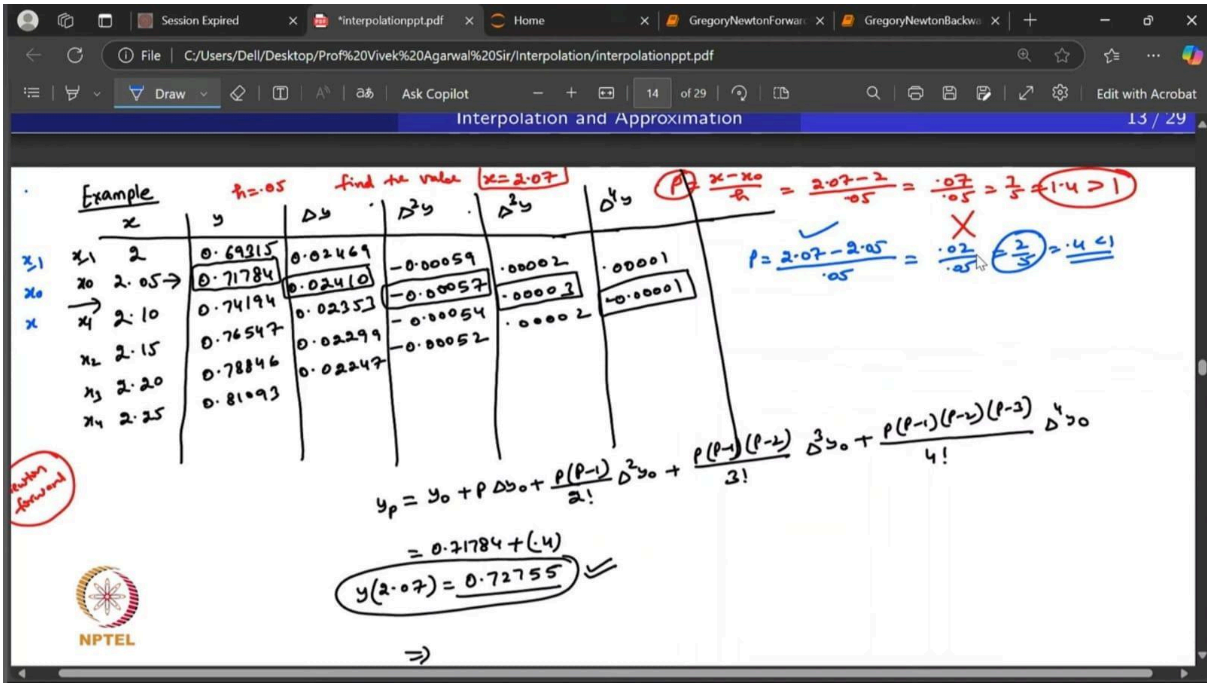
So if all these qualities are happening, then from here we are satisfying all these values here, so from here you will see that we can say that this polynomial has come, the Polynomial we had, that Polynomial, if this has happened then we will say that the Newton that we had, we have passed the interpolating to the Polynomial, okay, this is coming, so we can tell it that this is the interpolating Polynomial, if this condition is cleared only then we will say okay, so we have come to know that the Newton forward difference formula gives the interpolating Polynomial and its degree depends on how many data points we have, now we did the same work for forward and backward, okay, so in backward also it will happen that its values will match, the values on the nodal points will match, now if I do the same work in this Let me do the question and I just want to calculate its value, now what do I do, find value at x is equal to 2.21, I want to calculate on this 2.21 so now see 2.21 which will come somewhere here, so what do we do, if 2.21 is coming here, then what do I do, I will assume it to be x not, so if I assume it to be x not, the value that will come out will be y not, if we see the difference between the two, then .04 divided by .05 comes out to be less than one, so what does it mean, the that the p that have come out we have is how much, 2.21 minus 2.25 divided by h .5 this comes minus .04 by .05, so the less than one ok, the magnitude of this is less than one or if we see, the p that we have, if p is between minus one and zero, then what do we do, we apply backward, okay, and if p is between zero to one, then we apply forward. So, if we need the value here, we will use this. Now we can use this table only for what we need to calculate. So, we have the calculation already done. So, we will assume it to be y not. So, this will be used. This will be used. This will be used. This will be used. And this will be this value. And one more value will come in the end. Okay, this minus this. I will write it as minus 0.00002. Okay, so this is y in the end. So, with the help of this, I can calculate the values. So, we have calculated it. Now what will we do?



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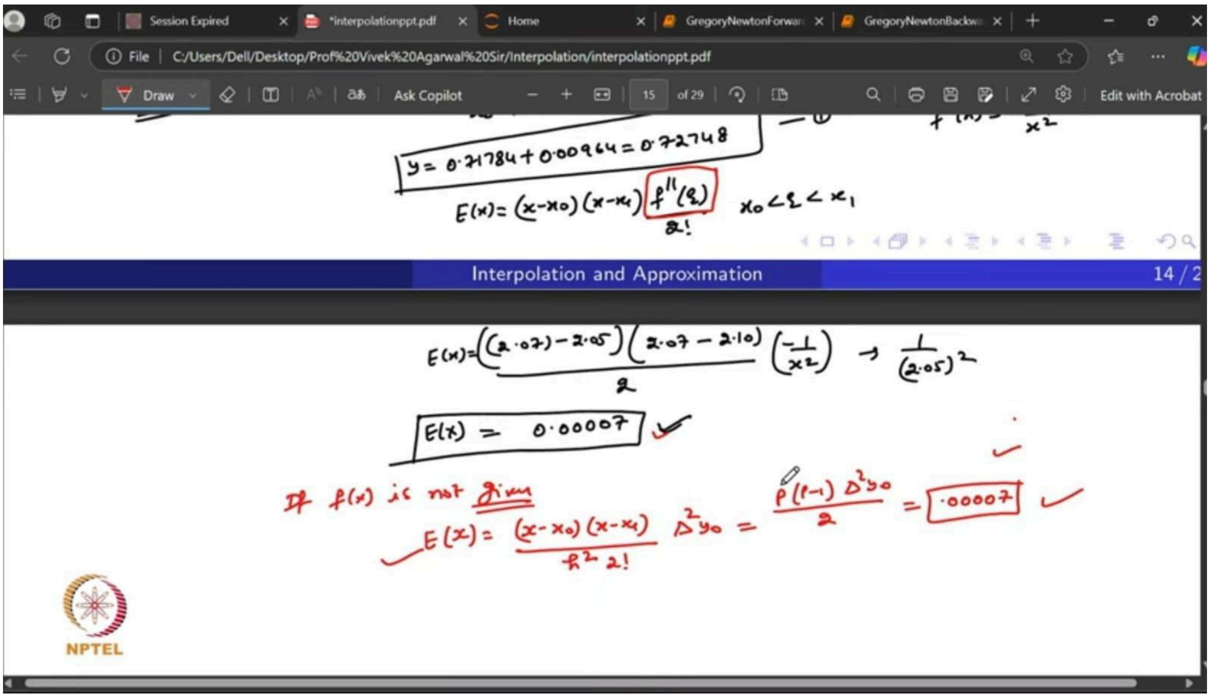
I will substitute it in Newton backward. Okay, so if I substitute this in Newton backward, then what should our values be y not, which is ours, so I have assumed it to be y not. Okay, so y not plus p plus p p plus one by how much will come 2 factorials, we will go on like this and we will keep substituting the values inside it where the value of p is this, so if we see, in this case p is close to minus 0.8. So, we will substitute it in it and we will get the values, so when we calculated it, I saw that at x is equal to 2.21. The values that we got after calculating this are 0.79299. This value is okay, so we will also see this with the help of the python program, whether it is coming out correct or not after calculating, so we have done this work, now what will we do, now see, if the values that we have put are approximate, then there should also be some error in it. So, what do we do, let's try to find the error, so let's see, I am taking the same example, okay, so in this we will see how to find the error, so we were told that the values that were there here, okay, were calculated by this function, and that function was f x equal to ln x ln x means base, so they used this. We substituted and found the value from here. We did not know it earlier but now we know that the function is ln x, we calculated the above one. So, what did we do in this? Now we want to see how much error came. So, the question is that we should have used only linear interpolation. Linear interpolation means because we have to find out the error, so what are we doing? I am using only two values. So, what did I do for this? Now let's see using two values. So, we have to find out the linear interpolation by using this and what do I have to do? I have to find out the approximate value. x is equal to 2.07. Okay, we found the value at 2.07 and we have to use linear interpolation. So now what did we do? We will calculate this. So, the x not that we have is 2.05, we used and our p was 0.4. So, what did we do? We have to use these two values. We have to find out the value from this. What does linear interpolation mean? We will go only till here. Till now just so if we calculated from here then the value that we have is 0.71784 plus 0.00964 if we calculate it then the value will be 0.72748 this we calculated it with the forward formula okay so this value ah let's give it a name, I have given it some name it is approximate value right so let's give it one now we know that we have the function also given okay so now we need the error so if we see the error then what will be the error in this case x minus x 0 x minus x1 because I said that it will be multiplied into now the function is given so we can use the second derivative of that function by 2 factorial okay and divided by h square so this will come we get this error okay so if we calculate this so if we are using the function then this h square will not come only this value. If we get an error, we will get p x is equal to by two factorial and the xi that is going to go, from where will it come from x zero to x one. So, we have calculated it. We have substituted the value of substitute, so what do we get, E x we get 2.07 minus 2.05. This is the first value and the second value is 2.07 minus 2.10. So, I have divided it by two. Now we have to find the second derivative, so the function that we have is ln x, so what will be the f dash x, 1 by x and f double dash x, which will become minus one x square so what do we do, in this there is minus one by x square, so this will come, we will take the absolute value, one by x square, so this is one by x square, so we know that this is a reciprocal, so this maximum value will be 2.05. So, what will we do in this, find the maximum, alright, so I will write it down, one by 2.05. whole square so we calculated this, after calculation we will get the value 0.00007 this comes this we have error ok we have taken the absolute value and if we see the value from this then it will come negative here so if I convert the negative with this as minus then the value will be eliminated so the value let me take it as let me write it down as it is instead of magnitude so this will come minus 1 by x square ok values that we will get will be this a so this error we got in approximating these values using the interpolation now this error that we found out with the help of function see this was a function this was given now what would happen if we assume the function was not given so if f x is not given then what would happen so our error will become x minus x not x minus x one by x square now this here is by two factorial ok or I can do it like this p p minus one by 2 factorial this will come now look we have the function The value of p was not needed, we just used its value, so if I substitute it, we will get to know the value of p. Okay, point four, I will substitute this value, so we saw that this value is also .00007, so we saw that the error is when we did not know the function, even then this much error occurred and when we know the function, even then this much error occurred, so from this we get to know that the Newton forward difference, if the values of p are between zero and one, then the error will be very small, okay, so here and the lesser the value of p, the better it is, that means the better accuracy will be obtained, so what happened in this case, we got the solution, so we found out the error with the help of both the formulas, and the error formulas are Newton or this, which are ...fourier..Taylor series used to calculate I can do it, okay, so we calculated with the help of this so we can take any other example as well and we can calculate in that as well.



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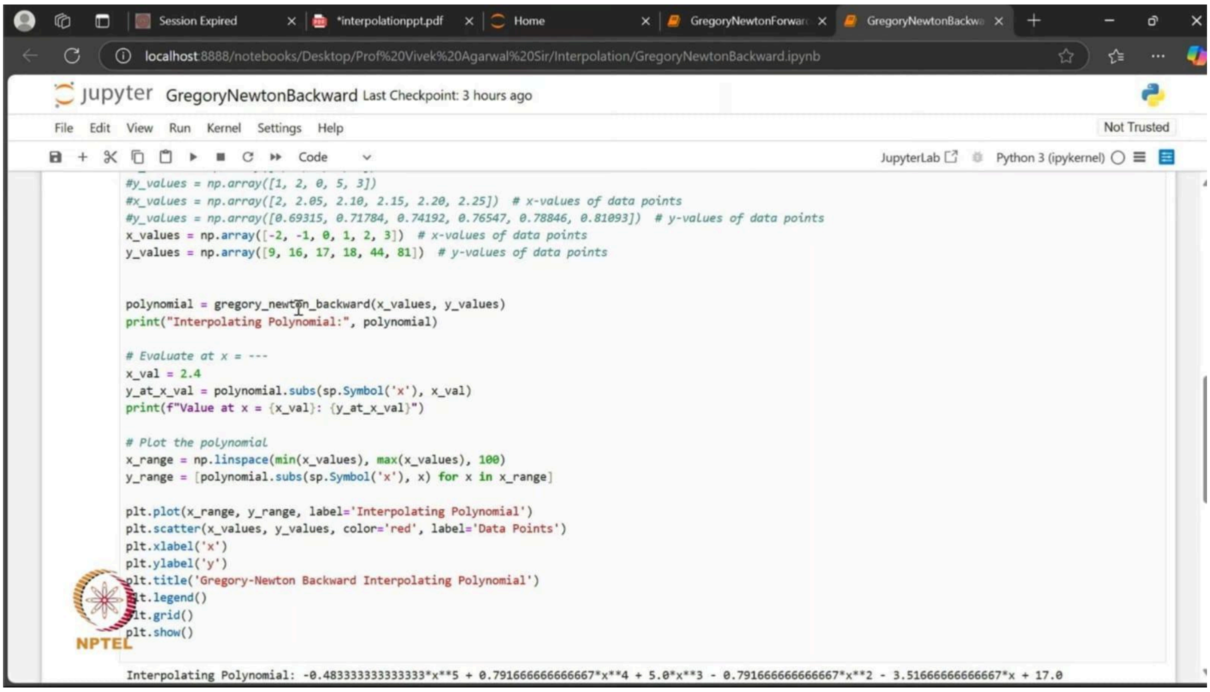
So, suppose we have the values of x, suppose we have given 1 1.2 1.4 1.6 1.8 and 2 is given and the value of y's is given, we have 2.7183 3.3201 4.0552 okay, 4.9530 and 6.046996 6.496 and is 7.3891 so if we see like this, what is the exponential value, so we have some exponential value, so given and if someone tells us to find the value of e raise to the power 1.12, now we know at 1.1 and 1.2 and if someone asks us to find it, then we cannot find it with the help of this, in this way we can give an idea, a rough idea and what is the rough idea that in between this The value will come somewhere but we don't know what the actual value will be. So, what do we have to do here? We will have to find out the interpolation that we have given by Newton forward. Or if someone tells us to find out the value of e 1.68, then 1.68 will come somewhere here. So, we will have to calculate it here. So, depending upon whether we have to calculate the value here or calculate the value, we will have to do all this according to that. So, if we have to calculate it here, then we know that I will consider it as x zero and I will consider it as y zero. Why Because the value of p will be 1.12 minus one and divide by .2 and if the difference is .2 and divide by point 2 point 2, then the point 1 by point 2 comes out to be 1.2 by 2 is point 6. So, less than one is, so in that case we will consider it as x zero y zero or we will consider it as x one. So, we have three six match points which are nodal points. We will get a polynomial of 5 degrees. Okay now. We need 1.68 here, so we have to see whether we consider it as y zero or y zero, so we will consider it as y zero, so it is possible that the value more than P may be less than minus one, so we may have to consider it as y zero, so depending upon where we have to see, which values we have to use, we can do this with the help of that, so if I use this and I see this backward, let's see what is coming on doing backward, okay, so we calculated backward and the backward value, now the Gregory Newton f backward, we found out, so in this we know that backward difference table is created, okay, we have called this function, that means we have defined the function, okay, so after that we can do this for different data, so like I did this last time, I used this and I am calculating its values, suppose I can do it on 3 2.8, so I have done it on 2 point Suppose if I did 4 then 2.4 is coming in the last interval.



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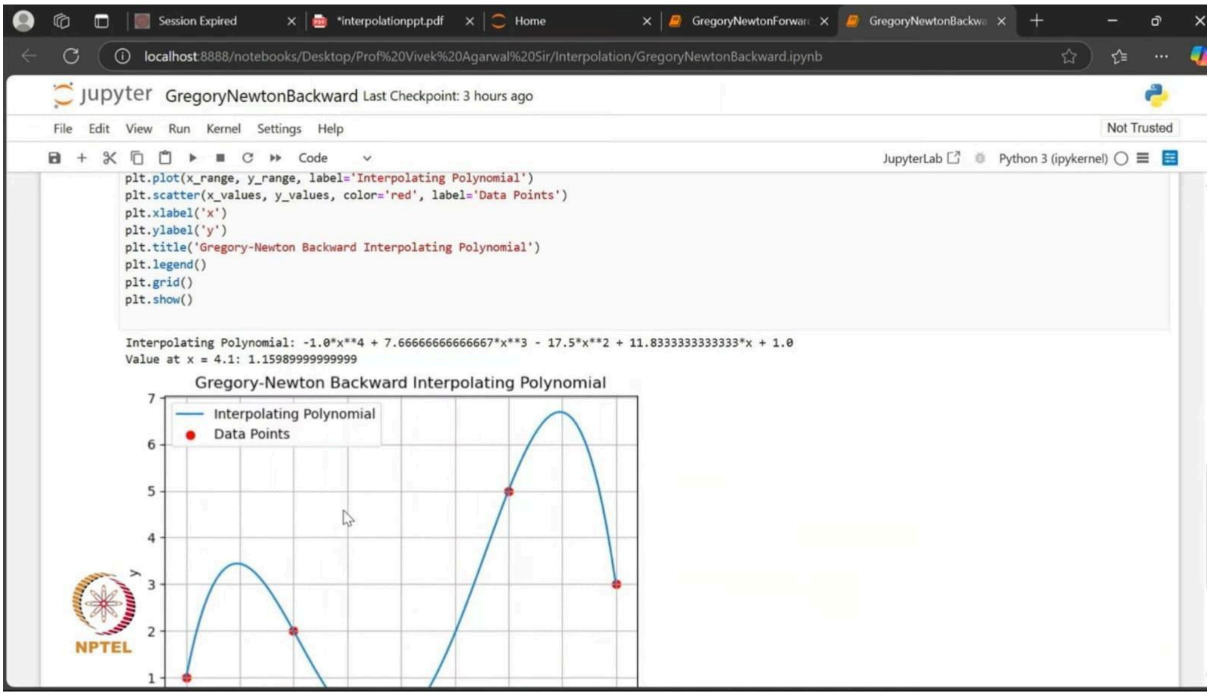
So, I will try to calculate it and I saw that there were six data points, so we got a polynomial of 5 degrees and the value of 2.4, 60, came here and this is our corresponding interpolating polynomial. So, I can calculate it in the same way. Now, suppose if someone asks me to tell the value of zero, then the value of 17 on zero is known to us. Here, the value is given. And we have also seen that the interpolating polynomial gives the exact value, so here we have the exact value. So, if we have to find the value of 1.9 from backward, suppose we have to find 1.9, then what will we do with 1.9, the value of 40 came here. So, with the help of this, with the help of the polynomial that I created, I can calculate it. I can see that 1.9 is coming somewhere here, so the values will come near it. The value of 80 will come somewhere near it. So, with the help of this, I can approximate any value. I can take this, after that I can take another data. I have taken this. So, in this, we have seen that I have to find out the values, suppose around 30, so let me try to calculate 3.1. So, I calculated it. Look, this is our interpolating polynomial and I have checked it at 3.1. The values of 3.1 have come somewhere here, 5.53 or 5.55 has come. So, in this way we can calculate its value. Once we know the polynomial, all these things are interpolation even if someone asks me to check this and tell what the prediction will be, then suppose I check it at 4.1. We do not know the data. But as we got this polynomial, we put 4.1 in x and see, it came out to be 1.15. Now see it has come out 1.15, here it was 3, after that it came directly to 1.15.



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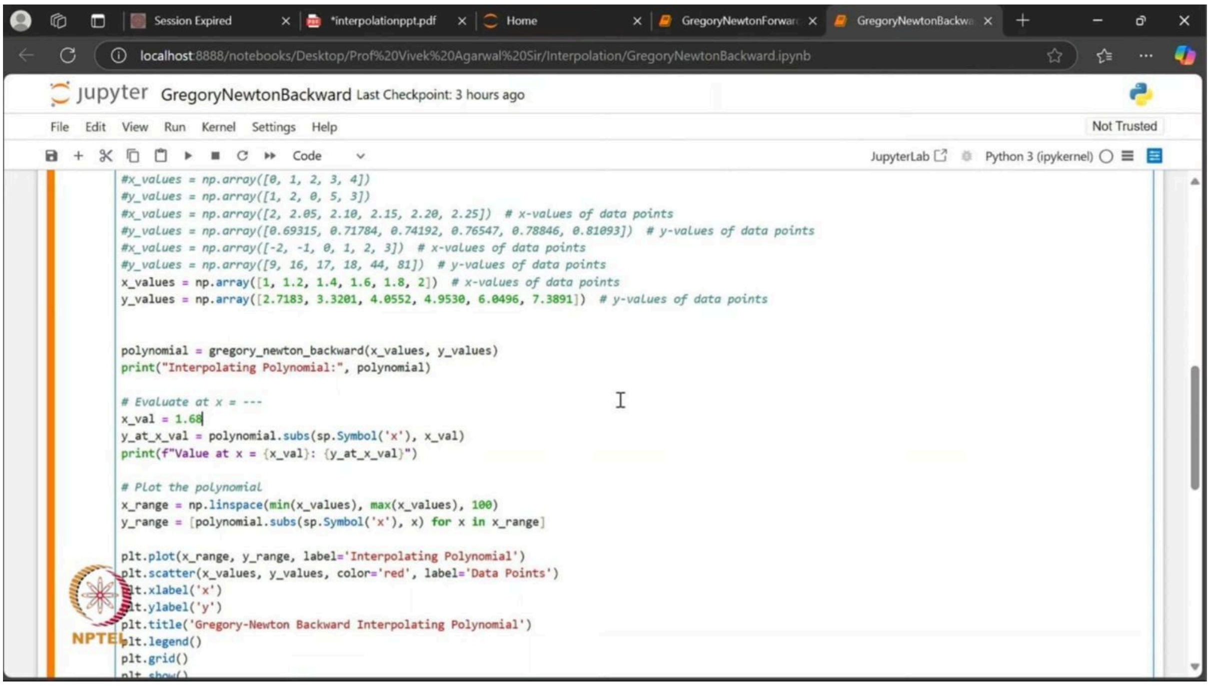
Now if I look at it at four, you see what is coming, the value is coming at four, right? So, the value on 4 is exactly three. As soon as I made it 4.1 here, this value is there and now what happened, it went out of the range of our x. As I calculated, see how big the error was, 1.15. So, it is possible that this is the correct value because if you see, the graph is going down completely here, so as soon as it comes to 4.1, there is a very high slope here, the slope is also very high, so its value will come out. So, we do not know whether this value is correct or how accurate it is, how much error is there in it, that depends on our function, we find out, we know how to put the error. So, we can play it like this and however big our data is, we can find out its values from here by using it. So let me calculate the values that we have one more time. We have just taken an example, Control C, okay. And I have just defined an example of exponential, so I can write it here. So, the values of x here are suppose one, 1.2 1.4 1.6 1.8 and 2 these values are given to us. Let's see what its value is 2.7183 3.3201 this value is given. 4.0552 this value is given. After that 4.9530 this value is given 6.0496 this value is given and 7.3891 this value is given. So now what we have to do is that we have to find out 1.16 1.68. So, what did we do, we used this.



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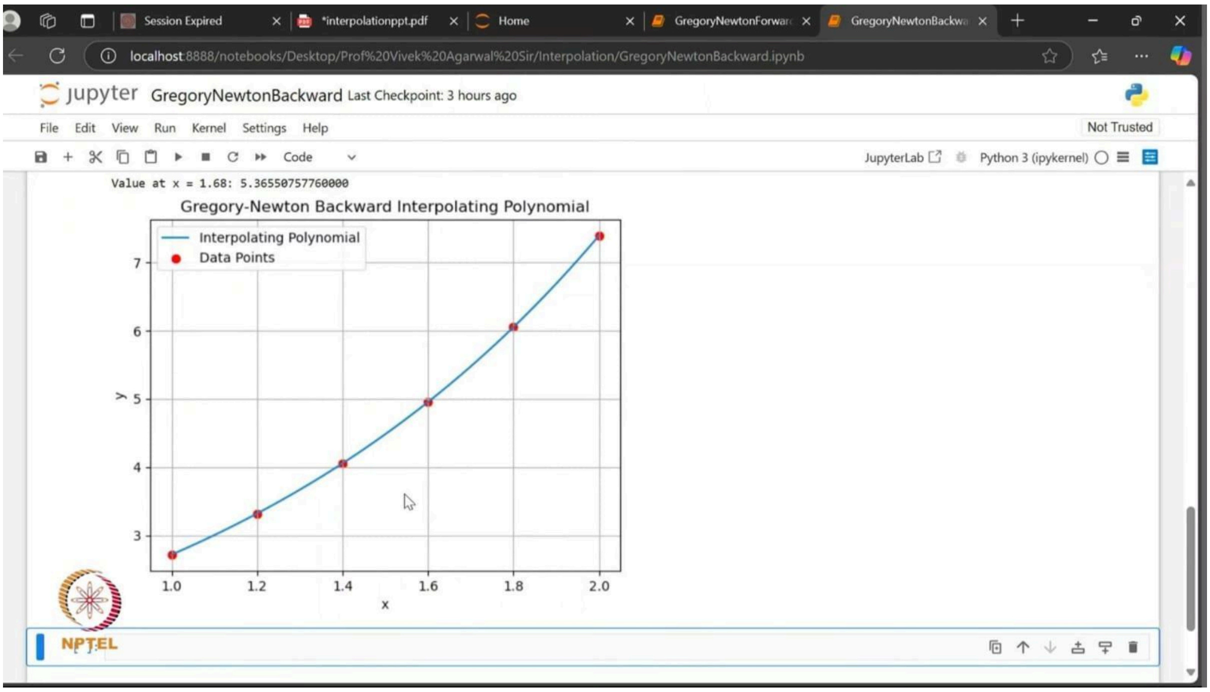
Now look, the polynomial is the interpolating polynomial, so we got it to the degree of five and the approximate value that we got from the 1.68 is 5.36 okay 5.36550. So, this value is and this is our interpolating polynomial, so we can find it using this. Now suppose there was 1.12 in it, so if 1.12 comes somewhere here, it will be around 3, okay. So, when we calculated it, its value came out to be 3.064, so it came around 3, okay.



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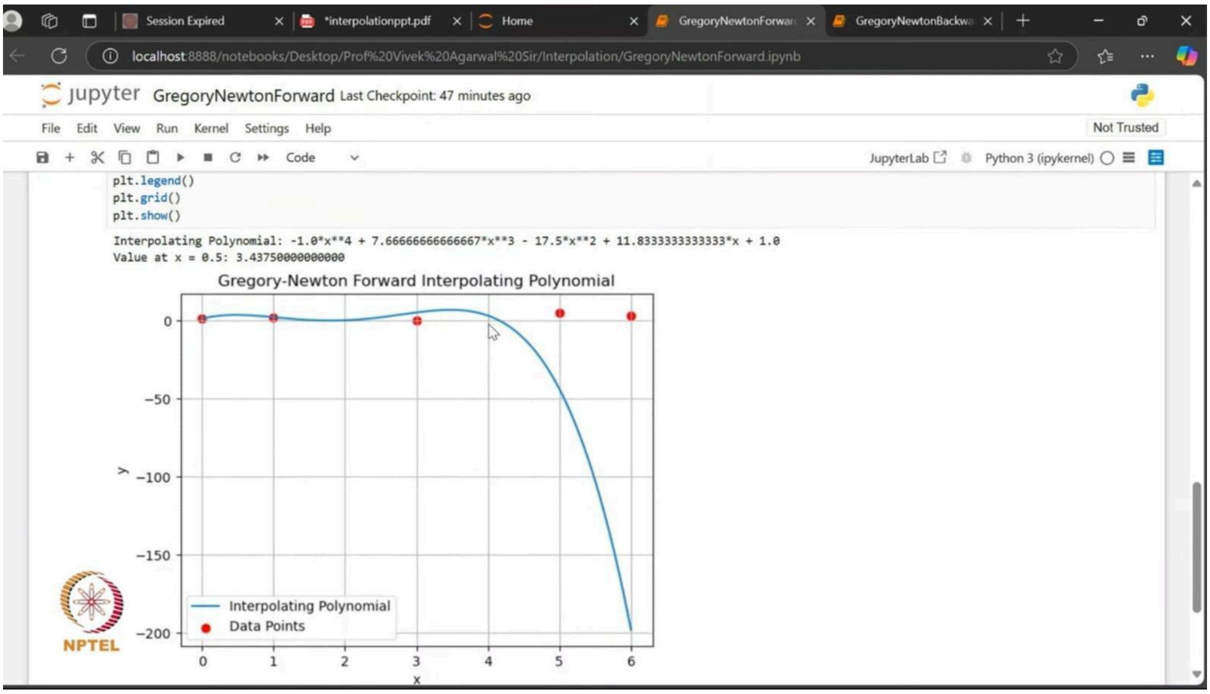
But we know that depending upon the p, which values do we have to use, which operator, which difference formula do we have to use, okay. So now we have done this, now the next case that comes to us is what will happen if we have to find out this, suppose we want to calculate e to the power of 1.5, the e to power of 1.5, which is 1.5, will come somewhere here, it will come in the center. So, if it comes to the center, then neither can it be applied forward nor backward because if we forward it from here, then if I consider it as x zero, then we will be left with only four data. These two data will become useless. If I consider it as y zero in backward, then we will be left with only these four data and the last one will become useless. But our main purpose is that all the data should be used because if we use all of them then only, we will be able to get better approximation, so in these types of problems we have to apply central difference formulas. So, in this way, whether it is forward, backward or central, then everything depends on the value of p. On the basis of that, we have to derive the formulas that we have. So, we have to keep in mind that what is the value of p. So, what will happen if we become independent of p. So, we do not have any problem. Where is the value of p coming from? Where is x coming from? It is coming from the middle. It is being done in the beginning, it is being done in the last, okay, so now that we have done forward and backward, now we will do some such formulas which will work for all of them, okay, so for all of them to work, what does it mean that now I and Newton have worked out forward and backward, in that we also kept in mind that the spacing was equal, so now if we have data given, so we have data, like this, given one, given one, then on three , then on four, then on five, then on 8 ,then on 10, this can also happen, okay, so now the values on this are the values of given x, so this is our x zero, this is x one, x two, x three, x four, x five. Now what will happen in this case, if in this case, if I apply Newton, then it will give an error, why will it give an error because the spacing has become wrong, what is the spacing in it, x one minus x not, the spacing became two, x two minus x one, that spacing became one and x5 minus x4 spacing became two x4 minus x three spacing became 3 so the spacing is different. So if we put different spacing like this and we did not pay attention and what we did was that we did it with Newton forward or backward, so see, if I apply this formula that I have applied, in which the spacing is different, then from zero to one, so after that I took 3, after that I took suppose 5 and after five I took six, okay, and I did not know, I tried to calculate it, so look what happened to our Polynomial, the Polynomial is passing only beyond the first two points, and it is not passing through the third point, and then it is going from a very far distance, so what does this mean, our interpolating Polynomial does not exist right now, okay, so the Newton Gregory forward interpolating Polynomial or backward is there only as long as the spacing is the same. If the spacing is wrong, if our spacing is non-uniform or unequally spaced, and if we apply this method, then the results that we get will be wrong. So, to avoid this, why don't we apply such methods in which whatever values are there, whether the spacing is equal spaced or unequal spaced, it will work for all.



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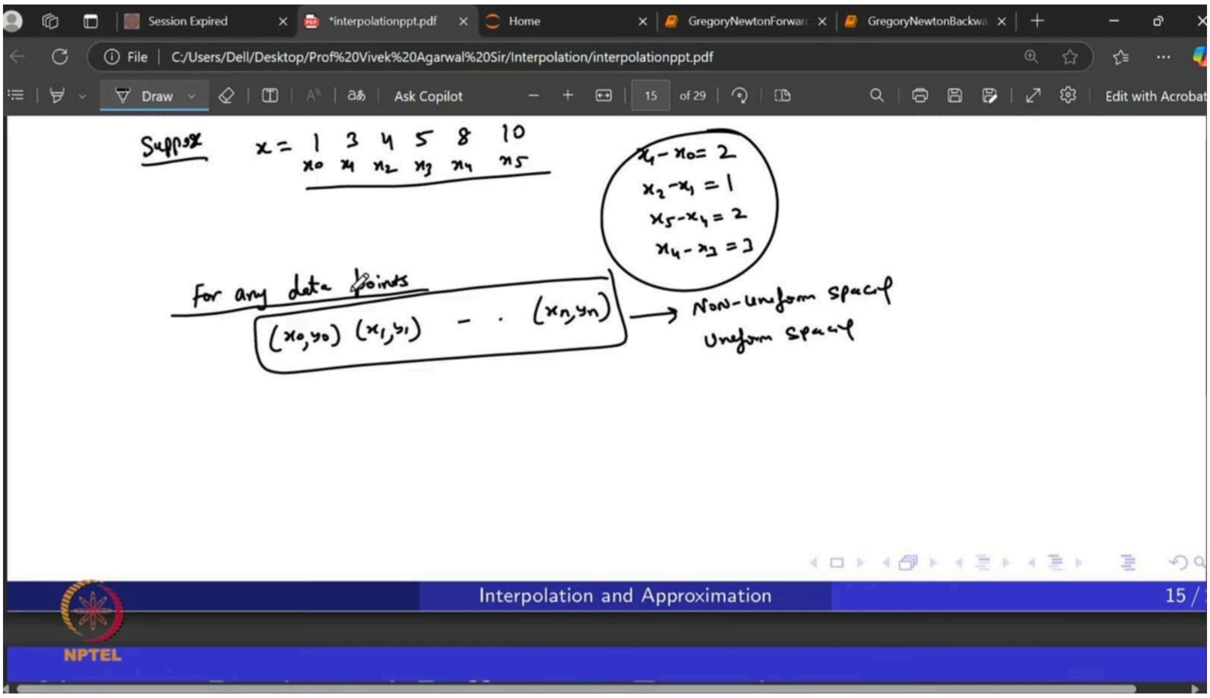
So our next target is that for any data points, for any data points, our work in it is that we should have x zero, y zero, x one y one should be x n y n, okay, so this can be non-uniform spacing, it can be uniform as well, okay, which will work for non-uniform and it will also work for , so our main purpose is to calculate it, so what are we going to do now, which we will use, now we will define how we can find out an interpolating polynomial.



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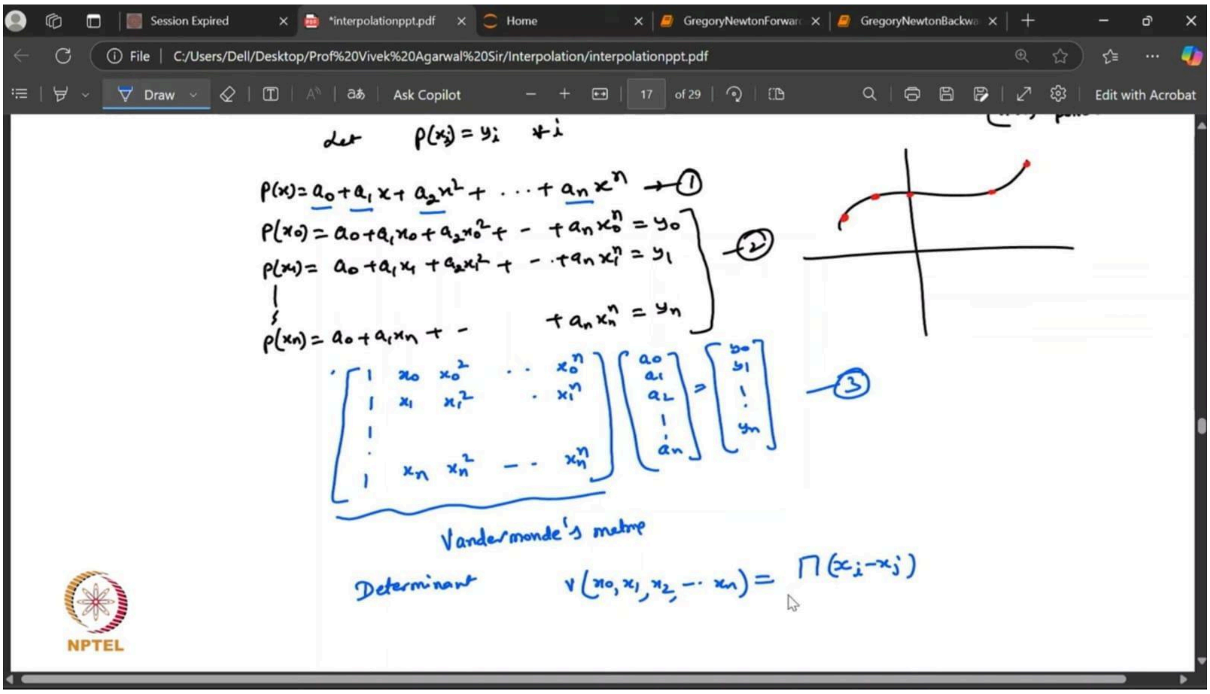
So, now our main target is how to find this interpolating polynomial, okay, so let's see this How can we calculate this because in this we can have any number of points. So, the spacing is equal to them. The spacing is not equal. It doesn't matter to us. It works for everyone. So now we will go to the basics. So, what will be the basics of this? Let's see now we have points. Let's say we have these points. Let's say we have these points. x zero y zero x one y one x n y n are n plus one points. Now I know that I want to calculate an interpolating polynomial from these. So, let's say the definition of an interpolating polynomial is such that P x is an interpolating polynomial that P x is equal to Y i for all i. What does it mean that if we consider a polynomial, it will pass through these points because we have already seen that these points are interpolating polynomial. So, what will happen? The points that we will lay down here can be anything. They can be unequal or equal. So, what do we do now? What should I do to calculate this? So what do I do, so let's assume that p x is of this type, zero a1 x a2 x square a n x to the power n because n plus one points, so this is the polynomial maximum polynomial which we can construct, of maximum degree and this is what we can do, so we have assumed, okay let p x is the polynomial, so let p x is the polynomial interpolating polynomial and I have given it a number one, now this is an interpolating polynomial, so what will happen, what will be p at x zero, so what will happen, it will become a0 plus a1 x not a2 x not square a n x to the power of zero to the power n and what should this be equal to or should it be y zero, now it will pass it through, now the next element will come x one, what will it be a0 plus a1 x1 plus a2 x one square a n x 1 to the power n, y 1 it comes like this, then it should be passed through all such elements, so it means the last nodal value will also be passed by us and we will We will substitute, okay, so we have put a condition, I will give it number two. This is the condition of interpolation. If we have to make a polynomial, then it should be like this. We used to call it interpolating polynomial. All the points that are given points should be passed through them. So, this is the definition that we had written for interpolating. So, we have used it. Now what are we doing? Let me try to calculate it. Now we have defined our polynomial, but we do not know the coefficients, so how will we get the coefficients? We need a0, we need a1, we need a2, we need a n. We should know all these coefficients, only then we can use it. So now see, what do I write for equation number two? Let me write it like this. One, x zero, x zero square, x zero to the power of n, and this will come, a0, a1, a2 a n, will come on the right side, y0, y1, yn and this will come, x one, x one square, x one to the power of n, and 1, x n x n square, x to the power of n. A this will come so this system 2 which was there I have written it in this matrix form, okay so this is our matrix and the coefficients which were unknown have come here and this is known value so if we have this matrix, it is a nonsingular matrix and we find its inverse, okay so we can find the value a0 a1 a2 up to a n so this matrix which is formed we call this matrix as vendormond's matrix, okay and this is vendormond's matrix and its inverse means its determinant, basically we need determinant, so if we calculate the determinant of vendormonde then you see it depends on x zero, x one, x two up to x n on all these values, so this determinant of this matrix which we have got, so the determinant of this matrix we represent this matrix like this V x zero, x one, x n is depending on these, okay so if we have it we can write it like this So the determinant of this matrix is called vendormonde and basically I remove its determinant, its determinant is written like this and its determinant is the product of all x i minus x j, okay, for all i,j starting from zero to n such that i is greater than j, so all these products come and what is happening in this is that all the values x zero x one we know that all the values are distinct values, if they are distinct, then this value xi minus x j cannot be always the same, so these values will not be zero, then we will take their product and we will get the determinant of this vandermonde's matrix and after getting the determinant we will come to know that this system exists its solution and we will solve it, so this matrix will be formed with the help of this one, so this This will be a matrix. This matrix which is p x will in fact be unique. If this happens, then the polynomial that we get will be the unique polynomial.



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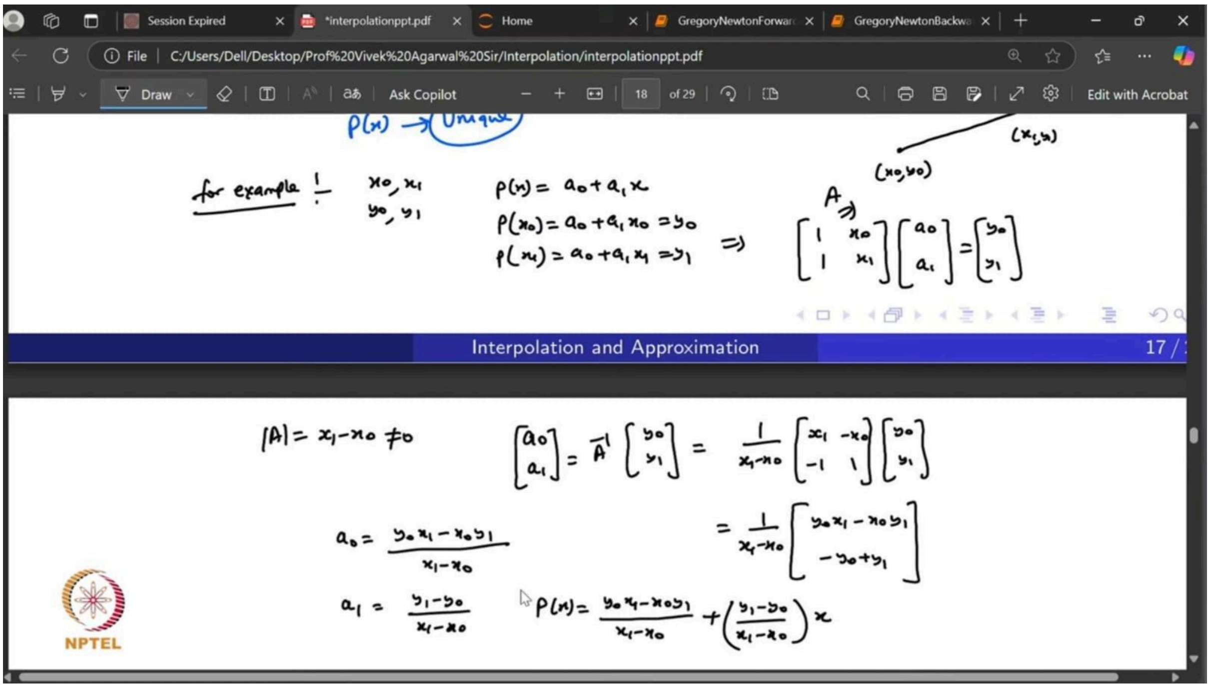
So, we can either solve this with the help of a computer or suppose I want to check this as well. Suppose what do I do, for example, I take two points, x zero and x one. So here the value y 0 and y 1 are given. So, we have two values. So, we know that there are two points, one is this x zero and the other is y zero, x one, y one. So, if we can construct a line from y, then let p x be a polynomial a0 plus and a1 x is obtained. Now if the polynomial is p of x zero becomes a0 plus a1 x0 and it should be equal to y0, then p of x1 will become a0 plus and a1 x1 and should be equal to y1, because it has to be because it is an interpolating polynomial. If we see from there, it will become 1 x0 1 x1 a0 a1 and this y0, y1 is the system we have created. Now we will solve this system. So, this matrix, this vendormonde matrix, I will name it suppose if A we see the determinant of A, then it will be x1 minus x0. Okay, x1 minus x0 will come and if both of these are different values, then it will be non-zero because if we take distinct values, we do not take the same values, we do not repeat them, so our data should not be repeated. The value of x should not be repeated because we are assuming that these values might have come from some function. So, I will calculate this. Now we need a0, a1. So, if we want to calculate a0. a1, what will come out to be A inverse y0 y1 will become and what is A inverse, what is its determinant, so see, what will come out to be x1 minus x0 and we know that it is a 2 by 2 matrix, so what is its inverse, diagonal element changes and here the negative sign comes and this will become a0 sorry y0 y1 so this comes. Now I will solve this okay so if we solve this then what will come 1 over x one minus x zero here so this is what we have y0 x 1 minus x zero y1 here and here comes minus y0 plus y1 this comes okay so if we calculate from here then how much will a0 come here a0 has come our y0 x1 minus x0 y1 divided by x1 minus x0 and a1 comes a1 how much will come y1 minus y0 x one minus x0 this value so if we substitute this then our polynomial will become p x so this becomes this is a not x1 minus x not plus y1 minus y not x1 minus x not into x now this comes so the polynomial we have is this okay so I can name it and I can call it p1 p1 means linear polynomial so we get linear polynomial which is From here we get x1 minus x not which will always be non-zero, so we can define it very clearly. So, what can we do now? We can collect all the things in terms of x not, y not. So, if we collect this, we will get polynomial. So, this polynomial that we have got, this polynomial is a linear polynomial, so in this way we can find out second degree polynomial, we can calculate even eighth degree polynomial.



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So, we will do this part till here in this lecture and we will do its extension in the next lecture. So, I hope you understood this and thanks for watching. Hello