

# SCIENTIFIC COMPUTING USING PYTHON

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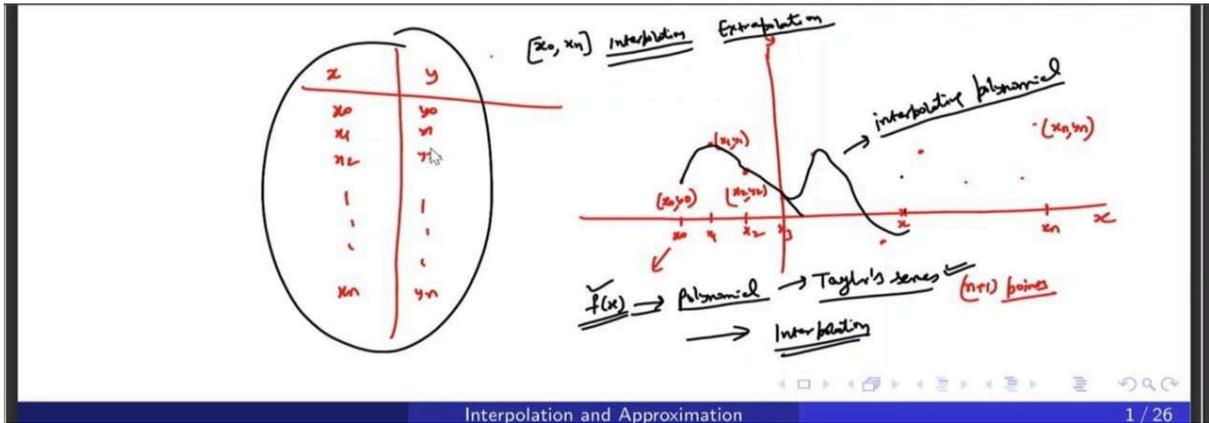
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**Lecture No. 19**

Welcome all of you to Scientific Computing Using Python. So today we are going to start a new topic, Interpolation and Approximation. So let's start. So today's new topic is Interpolation and Approximations. If we have to do something or make a prediction, then how can we find it out with the help of interpolation approximation. Is there any function, we want to write it in terms of series or in terms of Polynomial. So how can we write it? So all these things, these are interpolation and approximations, so first of all we define operators. So first of all we will discuss what are difference operators. Let's first see what is difference operator and how can we define it and what does it mean. So see, difference operator, as the name shows, there is a difference. So in that we have defined forward difference operator. So what is happening in forward difference? Now see what we are going to do, like our We have interpolation, see what we have to do now is, if we have some data and in the data we have given a value, or we have given a value here, or given a value here. Okay, like this it is given here, then it is given here, so we have  $x_0, y_0$ , a point here, the  $x$ - $y$  coordinates, I am just taking in  $x$   $y$  in the plane,  $x_1, y_1$ , like this  $x_2, y_2$ , so we have a lot of points, so suppose in the last one I took  $x_n, y_n$ , so here we have  $n+1$  points, so these points which are defined here, this is defined at  $x_0$ , this is defined on  $x_1$ , this is defined on  $x_2$ , okay I took this  $x_3$  and this last one which is  $x_n$ , so this is these points, we have given the values of the function, so we have written the coordinates of all of them, now the values which we have, these values which we have, we have only the value given, we have the data and someone gave it to us, we have  $x$  And  $y$  this is  $x_0, x_1, x_2, \dots, x_n, y_0, y_1, y_2, \dots, y_n$  we have values and nothing else, we know we only have this data, so what do we have to do now using this data we have to tell that suppose we have got some values here somewhere, these values which are  $x$ , there is some value  $x$ , for the values of this  $x$  we need to find here what will happen for this  $x$ , so what do we have to do, this is called interpolation, if we within the range what is our range  $x_0$  from  $x_n$ , if we have any value approximations, then we will call it interpolation, okay, if we want to take it out from this then we will name it extrapolation, that extrapolation will come, so there is one interpolation and one interpolation, so we have to discuss interpolation in it, so what will we do now, we will find out with the help of these data points, this is all we know and we do not know from where these  $y$  have come, from which function they have come, okay, these are just values given to us, so on the basis of this we can do something If we create methods then we will see that we will try to find out some polynomials that pass all the points through it. We will create something like this. So this polynomial that comes will be called an interpolating polynomials. So what we have is that if we have a function, we can write it in the form of a polynomial, we can express it in the terms of a polynomial. We know that it has been told to us with the help of Taylor series and there are some conditions that the value of the function at a point and the value of its derivative should be known. So we can

approximate that function with the help of Taylor series with a polynomial of degree 'n'. So in this, we have a function, so we can talk about its derivative. Now the value of the function is not a function, but instead of the function, we have given some values. Those values, if we approximate it with respect to a polynomial, then interpolation is done, so now we have to see how we can find the derivatives. Now this is the set that we have. If someone tells me to find the derivative of this, how do we find the derivative of this, because we have a set of values, discrete values, so we will do that with the help of operators, and the operators will be difference operators.



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so what are we doing in this, we are applying the difference operator, so first of all we define the forward difference operator, okay, so I have written the forward difference delta in the form of a triangle, it is called delta, okay, so we have written it in its form, so what does it mean, now see the values that we have, we are always given like this,  $x$  takes zero, then  $x$  zero plus  $h$ , so our  $x$  one,  $x$  zero plus two  $h$  becomes our  $x$  two, we will keep writing like this, so these values are called equispaced nodes, these are nodes, okay, so equispaced nodes and the width between them is  $h$ , so we can write this as equispaced second, unequispaced, suppose there is a value in it, here  $x$  zero, there is a value here, here  $x$  one, here  $x$  two, then here  $x$  three, then here  $x$  four, there is not such equispaced If it is unequal spaced then we will see how to calculate it. So now the work we are doing is this one for equispaced. So what we have done is we have to see the forward difference. So at some point we have forwarded. So here is the  $x_i$ 's, there we have given values like  $y_0$   $y_1$ . So we have a point somewhere a  $f(x)$  and I can write it as  $\Delta y$  at some point. Okay, so if I put forward then what will it do in forward, it will go one step forward like here there is  $y_0$  and this is the next value  $y_1$ . So what will it do,  $f(x+h) - f(x)$  comes here and it will calculate and tell us. So what does it mean, this difference has become the forward difference. Why are we saying forward, we are going one step forward and after that we are using it. So this is our forward difference. Similarly, if I do backward, then we will write this triangle upside down. Okay, so what is happening in it now, what do I have to do, I have to go to a backward point. So here I was standing somewhere and after this If we want to go backward, I will write here  $f(x) - f(x-h)$  so at this point if we want to find the value, then we will have to go one step backward so this will become backward difference, okay so a forward difference, backward difference backward like this we have some different differences, we represent central

difference with small delta, I represented it with small data, so this is ours, so what will we do in this, x plus h by two went half forward and minus x minus h by 2 went half backward, okay so we have only h, but what happened in h, half forward, h went ahead and h went behind, so we have calculated it, so this is the central difference, the average difference of f, so what are we doing in this, we will take the average of these two, the front half, half backward and their mean, okay, so whatever these values are or whatever will be its mean, then we will say average difference, like this we have shift operator, what do we do, we represent it by 'E', we will apply it somewhere f x so next The value will shift it to the next value. So this is our shift operator. All these values are used. We will use them when we have to deal with derivatives. All these difference operators are used.

**Difference Operators**

- **Forward Difference ( $\Delta$ ):**  $\Delta f(x) = f(x+h) - f(x)$
- **Backward Difference ( $\nabla$ ):**  $\nabla f(x) = f(x) - f(x-h)$
- **Central Difference ( $\delta$ ):**  $\delta f(x) = f(x+h/2) - f(x-h/2)$
- **Average Difference ( $\mu$ ):**  $\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$
- **Shift Operator ( $E$ ):**  $E f(x) = f(x+h)$

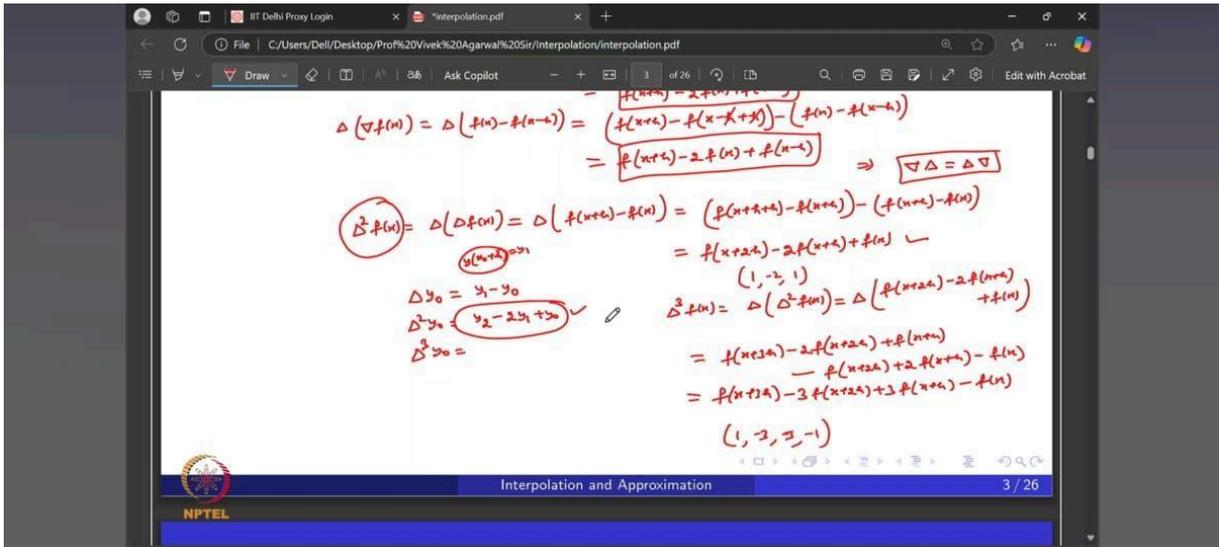
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So we have defined them like this. Now we have to see that we have defined the forward difference operator. This is called first forward difference. Now we have to see what it means. So what does it mean? We have to calculate it in this as well. So what will we do for this? Now see, I am writing it like this. Delta Delta of f(x). I am writing it twice. So before that, let us write one thing. I will define it here. These difference operators that we have defined above are linear and they will be linear and commutative. So, it means that if I apply a linear operator on any function,  $L a f$  one x plus  $b f$  two x then we can write it here as  $a L f$  one x plus  $b L f$  two x. And this will be linear. So, the second one is done. If we have  $L$  one of  $L$  two f x equal  $L$  two  $L$  one f x, then we will have a linear operator. On this we applied the function so first we did  $L$  two then we did  $L$  one this is same as  $L$  one then  $L$  two of f x, so what does it mean that  $L$  one  $L$  two same as  $L$  two  $L$  one. These linear operators are commutating each other then they get commutated so these difference operators that we have defined above all of them are linear and commutative so if they are commutative then what does it mean now suppose I want to check it so look I applied this here so for this suppose I do this first what do I do first I apply forward on any function F and then I apply backward

let's see what happens now see if I wrote it like this so I wrote and what does forward mean this will become  $f(x+h) - f(x)$  now it has to go backward okay but we found out so this we have off now this will cancel so it has come  $f(x+h) - f(x) - f(x) + f(x)$  minus  $h$  minus  $f(x) - h$

so it has become  $f(x+h) - 2f(x)$  and here it will become plus so this has come  $f(x) - h$  okay so this we have quantity Now what do I do, first I apply backward and then forward, so let's see what happens in this, forward is outside, if I apply backward, then this comes to us,  $f(x) - f(x) - h$ , now I have to take it forward, so what do I have to do in forward, if I have to go forward before  $x$ , then  $f(x+h) - f(x) - h + h - f(x) - h$  the same gets cancelled, so if we see here, then this will come  $f(x+h) - 2f(x)$  and this will become plus, so  $f(x) - h$ , so you will see this quantity and this quantity is the same, so we can say from here itself that first we did backward, then forward the same as forward and backward, so both are commute with each other, okay, right? So now what did we do, with the help of this we used first forward difference operator or first backward difference operator, center difference operator, we have found out all this, now we have to see what is the meaning of this second, what is happening in second, second means this like We take the second derivative, right? First second derivative. So we are doing this work with respect to that. So now let us see what does it mean?  $\Delta^2 f(x)$ , like I did here. Here it is happening  $\Delta \Delta$ , okay, so what does it mean? It means  $\Delta(f(x+h) - f(x))$ . So it has been forwarded. Now if we apply it on this again, then it will become  $f(x+h) + h - f(x)$ , this has been forwarded minus the same function  $f(x+h) - f(x)$ , what does it mean?  $f(x+2h) - 2f(x+h) + f(x)$ , this will go forward two steps, the value of this is minus, this is also minus, so this two times of  $f(x+h)$  came here, this was one step plus  $f(x)$ , so this is the second forward difference operator, okay, so what did it do? It took it two steps forward minus two times one step is plus  $f(x)$ . So here we have the value of the function, okay, so it means telling the difference. So if we see that our difference is, I want to take what will this be? So now the value of  $y$  not will become  $y_1 - y_0$ , what does it mean  $y_1 + h$  and what is  $y_1 + h$ , if we see the value of plus  $h$ , then write it like this, it means  $y$  at  $x_1 + h$ , what is that  $y_1$ , has come minus  $y_0$ , so here we have the first difference ratio, like this, if we take out the second, then what will happen, I can look from here and write  $y_2 - 2y_1 + y_0$ , it will become three values, okay, so if the three values are there, then we can take it like this, okay, so what will happen next, I can also write it like this, so I had  $f(x)$ , so I wrote three  $f(x)$ , let me see, I can write it like this, see, this square  $f(x)$ , so we have already taken out the second, so we can put this on it like this, so this becomes this and value  $f(x+2h) - 2f(x+h) + f(x)$  this value, if we see this, what will happen here, then one more step will be taken, so it will become  $f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$  will come, I have to add  $a$ , I will have to add it in it, so this will come  $2f(x+2h) - 3f(x+h) + f(x)$  will come plus  $f(x+h)$  will come minus of the same quantity, so the same quantity will become minus plus two times  $f(x+h) + f(x)$  this is the minus sign, right, so minus will come here minus off now if we see, these quantities have become  $f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$ , this is plus two  $h$  and minus minus so three comes, so minus three  $f(x+2h) + 2f(x+h) - f(x)$ , okay, right, but the coefficients that come in the form  $1 - 3 + 3 - 1$  this is what we have, okay, so our coefficients will keep coming in this way and the different difference operators that we have, first order, second order, third order, we will keep finding it out from here.



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It is backward, so what do we want to do backward is to go backward, if this value is Suppose we have  $y_n$ , then the value behind this will be  $y_{n-1}$ , the values behind this will become  $n-2$ , so now we will have to go backwards to calculate this one, so suppose I have this and I apply  $y_n$ , then what will this be,  $y_n$  minus  $y_{n-1}$ , so it went backwards, now we apply it one more time, what will happen to this, I applied this one, I applied this one, so what do I have, so this comes out to be  $y_n$  minus  $y_{n-1}$  and  $y$  is linear, so we can write it like this, now look backward, what will be  $y_n$  minus  $y_{n-1}$  what will happen to this  $y_{n-1}$  minus  $y_{n-2}$ , so what do we have from, look  $y_n$  comes out to be minus two times and plus  $y_{n-1}$  minus two comes out, so what did we do, I went backwards from there  $y_n$  was  $y_{n-1}$ , and from  $y_{n-1}$  minus two, I went two steps backwards, if we want the second one, if we want the third one, then I will go back one more step, so this means how many differences will we have. We can take operators of how many orders, it depends on how many points we have, if we have only two points, just one point and this one, then we can only apply first order operator on it, right? First order operator means we can apply only one of the two, forward or backward, other than this we cannot do anything, so we applied the forward difference operator like this, we applied the second one, we applied the third one, so in this way we can calculate all its values, now what do we have to do, now we have to see a little bit about their relation with each other and how can we define it, now like we have defined it, if someone tells us what will be this,  $\Delta$  of  $y_1$ , what will be this, what does  $\Delta$  of  $y_1$  mean, to center, so  $y_1$  was this, somewhere my  $y_1$  was coming,  $x$  zero, somewhere there is  $y_1$  and this is somewhere  $y$ , if I am taking from here then  $y_0$  and here is  $y_2$ , so okay so what do we have to do in  $y_1$ , we have to go to half, half, so when we The value is also defined on its half, so what will we do, we will calculate it, so what will it come out to be  $f(x+h)$  by two minus  $f(x-h)$  by 2 between. Okay, so for this we can apply it only when we know the values in between as well, so what will we do, we will find out the values, so half from  $y_1$  to half will go here and go to half here and we will take the difference of these two, that will give us the central difference, similarly we will get the second central difference, so we can do this, we can apply shift, now if I apply shift on  $y_1$ , what will we get, we will get  $y_2$ , if I apply shift on  $y_4$ , I got  $y_5$ , I do shift like this, suppose I take the inverse like this, then what will happen, the value of  $y_0$  will go backwards in  $x$  zero minus  $h$ , okay, like if I apply inverse on  $y_1$ , then here we get  $y_0$

zero, there was plus h here minus h. Okay, apply minus two y one, then this is what will happen, y will reach y minus one, it will go two backwards. It will become minus two h, so this is the shift operator and this is its inverse shift operator. If we want to see the relation between these two, then from here we will get to know what is the relation between them. So like this, what did I do, I applied forward, okay, so forward of f x, so it came f x plus h minus of f x, I applied this, I can take it like this, see E of f x minus f x and how much did it come, e of minus one of f x, it came, so from this we came to know that the forward difference is equal to e minus one, so from here we can take this which is e that is one plus delta, okay, so this is the relation between these two, that we have defined the relation between them, okay, so in the same way I can define something else and how will we define it.

The screenshot shows a presentation slide titled "Backward Difference Operator". Under the heading "Definition:", there are two bullet points:
 

- First backward difference:  $\nabla f(x)$
- Higher-order differences:  $\nabla^2 f(x), \nabla^3 f(x), \dots$

 Below the text, there are handwritten mathematical derivations in red ink:
 
$$\nabla y_n = y_n - y_{n-1}$$

$$\nabla^2 y_n = \nabla(\nabla y_n) = \nabla(y_n - y_{n-1})$$

$$= \nabla y_n - \nabla y_{n-1}$$

$$= y_n - y_{n-1} - (y_{n-1} - y_{n-2})$$

$$= y_n - 2y_{n-1} + y_{n-2}$$
 To the right of the equations, there are two diagrams. The first diagram shows three points labeled  $y_{n-2}$ ,  $y_{n-1}$ , and  $y_n$  on a horizontal line. A red arrow points from  $y_n$  to  $y_{n-1}$ , and another red arrow points from  $y_{n-1}$  to  $y_{n-2}$ . The second diagram shows a single point labeled  $y_n$  with a red arrow pointing to the left, representing the backward difference operator  $\nabla$ .

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Now, like this, it is backward, so how will I define backward, so what is happening in backward, of f x I am writing f x minus h, so I will write in it f x E inverse f x. Then i will get one minus E inverse f x. So from here we got to know that the backward is equal to inverse. So from here, the one minus backward is E inverse will come and from here we will get e is one minus backward E inverse. So, this value will come now. Next, we have seen what we have. Now if we go to delta, I can write delta of f x, f x plus h by two minus f of x minus h by two. So, what will come here, e half I will take f x minus of e minus half take f x. So, from here, what will come e of half minus e of minus half of f x. So, from here we have this delta. Okay, this central finite difference operator and this shift operator have a relation between them. Similarly, average is mu. So, from this, we can find e half plus e minus half by two f x. So, this mu is basically the average of these two. So, we have found it like this here. So, we can do this. Now, what can I do? We have another operator which we use a lot. That is the difference operator D. So there is another operator which we write as D of f x. So basically what is a derivative. So now see how I use it. Now see, we have the f that we have, suppose I am writing it as x plus h and I am expanding it with the help of Taylor

series. So it will come to us as  $f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$

Now what are we doing, I can write it like this  $f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$ . I have put it like this. Now what do I do, if I take this  $f(x)$  as common, so from here I take it, it will become  $1 + hD + \frac{h^2}{2!} D^2 + \dots$  and so on  $f(x)$ . I can write like this and from here I can write of  $E$  of  $f(x)$  on the left. So if we look at it, what quantity will it become? We have  $e^{hD}$ . It becomes a series of exponential and this one came and it came  $f(x)$  to us, so from here we got the shift operator, this is the same as  $e^{hD}$ , so we can apply this operator, whether we apply this operator or we can apply this operator, both sides have the same value, so we have some relations and we can define them like this, we can define more relations, it is not the case that we have, but it depends on how many points we have, how many polynomials there are, like we have this, like I have this function,  $x^2$ , I have defined it, okay, what do I do, if I take  $\Delta$  of  $x^2$ , then what will come out,  $(x+h)^2 - x^2$ , this is what will come out because  $f$  is this, if I expand it, then it will come out,  $x^2 + 2xh + h^2 - x^2$ , this gets cancelled, so we have the quantity left,  $2xh + h^2$  is left, now I have applied it one more time, now let's see what happens, now we have this quantity left, so what have we done in this. What we have to do is instead of a square plus two  $x$  plus  $h$  in place of  $x$ , this will come out as  $h$  into  $h$  minus the same quantity. We have to apply the resultant once on it. Now if we see, this  $h^2$  gets cancelled by this.  $2xh$  will get cancelled by this. If we see, here we will be left with  $2h$ .

$$\Rightarrow \delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2}) = \frac{1}{2} E^{1/2} f(x) - \frac{1}{2} E^{-1/2} f(x) = \frac{E^{1/2} - E^{-1/2}}{2} f(x)$$

$$\Delta f(x) = \frac{E^{1/2} - E^{-1/2}}{2} f(x)$$

$$\boxed{D f(x) = \frac{d}{dx} f(x)}$$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \dots$$

$$E f(x) = \left( 1 + hD + \frac{h^2 D^2}{2!} + \dots \right) f(x)$$

$$E f(x) = e^{hD} f(x) \Rightarrow \boxed{E = e^{hD}}$$

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So if we carry forward the same thing, if we want to take its cube, it will come out to be zero. Because if we shift over the constant, it will come out to be zero. Same value constant. So if

it comes out to be zero, it means if there is a polynomial of second degree, if we apply the difference operator twice, then we will have this quantity. If we apply it more times, it will come out to be zero. So if we want to univariate on its base, we can write a theorem. Here, what is the theorem, that the  $n$ th forward difference operator, the  $n$ th forward difference of  $f(x)$  is equal to  $x$  the power of  $n$ , okay? What will be the  $n$ th forward difference of  $x$ , which will be  $n$  factorial in  $h$  to the power  $n$  this means this is equal to  $n$  factorial into  $h$  to the power  $n$  so as we have seen above, but what was coming to us, if we see then this was coming so I can write it like this  $2$  factorial into  $h$  square three factorial into  $h$  cube and if we go beyond that it will become zero so this is the theorem we can just write it. We will remember how we can use it in the future. So now we have applied forward and backward. Now we have to see how we can use it. So what we do for this is we have a difference table. We will calculate the forward difference table and backward difference table. What we are going to do now is the forward difference table. Now see what is there in the forward difference table. We have some  $x$  values given. So the value of this is suppose  $x$  zero. This is  $x$  one. This is  $x$  two. This is  $x$  three, this is  $x$  four. This value is  $y$  zero. Because  $y$  equal  $f(x)$  is taken. This is  $y$  one. This is  $y$  two. This is  $y$  three. And this is  $y$  four. So we have five values. Now what we have to do is to see the first forward difference. So what will be the first forward difference? So we know what will be the first forward difference. So this is  $y$  one minus  $y$  not will be our first forward? What will be the forward?. this  $y$  one minus, this  $y$  one minus  $y$  not. So what is this? What is value one? So this one which will be there will be  $\Delta y$  not,  $4$  minus  $2$  this will become  $\Delta y$  one so this is  $8$  minus  $4$   $4$  so it will come  $\Delta y$  two  $10$  minus  $8$   $2$  so it had come three and there is no beyond this so we cannot calculate this value. So if we want to calculate this here  $\Delta y$  four, then it is needed beyond, so we do not have any beyond so we will stop here. So what happened now earlier we had five items, we took their first difference, so four remained, so one got reduced. So similarly we have in the function, if we take the derivative of  $n$ , then its polynomial becomes  $n$  minus one, it means we are going to that side, now what did we do, I put this first difference now we have to see the second, so what do we have to do in the second, we have to see  $\Delta^2 y$  not, to see  $\Delta^2 y$  not we have to calculate  $\Delta$  of  $\Delta y$  not what will this be, what is  $\Delta y$  not  $y$  one minus  $y$  not basically what has it become, so it means what we calculated earlier If we take the difference of the same once more then it will become the second time, it means if we take  $2$  minus one then one comes, so this is what our  $\Delta^2 y$  not becomes, okay  $4$  minus  $2$ , this is  $\Delta^2 y$  one two minus  $4$  minus  $2$ , this is our  $\Delta^2 y$  two. Now here, if we want to take it further, the value is not with us, so if we leave it here, then the value will be one less, okay now we will go to the next, so what is our  $\Delta^3 y$  not, and if we see, what is happening,  $\Delta^2 y$  not and what is  $\Delta^2 y$ , this is what is  $\Delta^2 y$  not, what does it mean,  $\Delta^2 y$  one minus  $\Delta^2 y$  not, so if we take the second derivative difference of the same, if we take the difference once more, then our third will come, its value is this minus this two minus one, so this is what we have,  $y$  not this minus this minus two minus two minus four, so this is what we have  $y$  one, we can go only till here and in the last if we see,  $\Delta^4 y$  not no we will get this and the quantity will end so in the end we will get a value and if we go next then nothing will happen so if we have a maximum of five data points then we can go upto the fourth difference maximum is fine we cannot go beyond that it means that if we have a polynomial of degree five then we can take five derivatives they become constants later on right so we are working related to it like this but you have to keep in mind that we first got  $y$  zero then next we got  $\Delta y$  not then next we got square we got cube we got four so we

always get y zero this is what we call forward what is happening in forward is that we are going forward and to calculate its value we went forward to apply the difference operator on it then one more forward so like this we keep going forward and in the end we will have a quantity of one that will become with us delta four y not so this is ours in this way It is going ok so this is the forward difference.

The screenshot shows a PDF document with the following handwritten content:

$$\Rightarrow \delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2}) = E^{\frac{1}{2}h} f(x) - E^{-\frac{1}{2}h} f(x) = \frac{E^{\frac{1}{2}h} - E^{-\frac{1}{2}h}}{h} f(x)$$

$$\delta f(x) = \frac{E^{\frac{1}{2}h} - E^{-\frac{1}{2}h}}{h} f(x)$$

$$D f(x) = \frac{d}{dx} f(x)$$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \dots$$

$$E f(x) = \left(1 + h D + \frac{h^2 D^2}{2!} + \dots\right) f(x)$$

$$E f(x) = e^{hD} f(x) \Rightarrow \boxed{E = e^{hD}}$$

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we can do it backwards in the same way what to do in backwards is to go in reverse ok so what is happening in backwards now this quantity is my x zero this is x one this is x two this x three it we have just added four values this y zero y one y two y three data point is just like this so what do we have to do is apply it backwards so to apply it backwards see what do we have to do apply y not so its value will be y not minus y minus one we have to go backwards so if we don't have this value then it means we can't calculate it so instead of calculating it what will I do I will try to calculate y one backwards if I calculate it then what will it become y one minus y not f x minus f x minus h y two what will it become y two minus y one y three what will it become y three minus y two see if we have to calculate it then delta y three becomes eight minus four left four so this is our y three two out of four two so this is our backward y two and this is our one from two one so ours has come here now see in this y zero is not coming now we have to go to next now if we want next we need square of y three then we will do it in the same way as we have been doing till now right so what was in del y three this right so we will calculate it and take it above it so what did we do 4 minus two so what did it become 2 minus 1 so this came one so y three will always remain and y two will change and in the last if we see then look what was happening in the previous one y zero was always coming now ours y three is always coming so first it was y three then delta three came then delta square y three came then delta cube so if we have some quantities such that we have to always use the last quantity then we will have to use it backward if we have some quantities such that our initial starting quantity has to be always used then we will have to

apply forward so this has become backward and forward difference table right so this is like this According to this, the difference table that we have is quite a lot of difference tables that we can apply. Similarly, we can apply a central difference table.

**Backward Difference Table**

$x$	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
$x_0 = 0$	$1 = y_0$			
$x_1 = 1$	$2 = y_1$	$1 = \nabla y_1$		
$x_2 = 2$	$4 = y_2$	$2 = \nabla y_2$	$1 = \nabla^2 y_2$	
$x_3 = 3$	$8 = y_3$	$4 = \nabla y_3$	$2 = \nabla^2 y_3$	$1 = \nabla^3 y_3$

~~$\nabla y_0 = y_0 - y_{-1}$~~   
 $\nabla y_1 = y_1 - y_0$   
 $\nabla y_2 = y_2 - y_1$   
 $\nabla y_3 = y_3 - y_2$   
 $\nabla^2 y_3 = \nabla(\nabla y_3 - \nabla y_2)$

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So, if we want to do something after calculating it, then how will the central one that we have here be, so see how will we do the central one, suppose I take any value in the central one, then how do we write it? To write it, we have to be a little careful, like this is  $x$  i's, so  $x$  zero,  $x$  one,  $x$  two,  $x$  three,  $x$  four so on value and this we have  $y$  i's  $y$  zero,  $y$  one,  $y$  two,  $y$  three,  $y$  four, so now what do we have to do, first take out the central one, so what do we have to do in the first central one, now see how will we write it, I will take out  $y$  half, so we will write  $\Delta y$  half in between them, now tell me what will be  $\Delta y$  half, if we have to go half ahead, then it will become like we have  $\Delta f(x)$ , what was that  $f(x + h)$  by two minus  $f(x - h)$  over 2, we have  $y$  half if I go  $h$  by 2 forward, then it will come  $y$  one its half will go behind, then it will come  $y$  not then  $y$  one minus  $y$  not, that is what we have. It will come so that's why I have written it in between it. So if we want to write it then we will write it like this. Next I will write  $\Delta$  in between these two. So  $\Delta y$  3/2, I will write that in between them. In between them I will write  $\Delta$  of  $y$  5/2 and in between this  $\Delta$  of  $y$  7/2. So here we have the first central difference, the value of which we have come, we have to see the second one. So what will we do in the second one. Now we will use both of them. If we have to use both of them, then I will write in between it,  $\Delta^2$ . Now see what comes between half and 3 by two, so I will write  $\Delta^2 y$  one, what will I write here,  $\Delta^2 y$  two and I will write here  $\Delta^2 y$  three. Okay, so the value of these is in between and we will take next,  $\Delta^3$ . Now if we want to take  $\Delta^3$ , then now we will have to use both of them, so I will write in between them, so  $\Delta^3 y$  one and two means 1.5 so 3 by two in between this  $\Delta^3 y$  five by two, so it has come in between these two. Now we will take the fourth one, fourth. How will we write this? Between these

two, we want the power of delta four 3 by two and 5 two y two. So it means that y two have come. Look, y two was used here and then this is the central difference table in which we have five quantities. Similarly, we do the forward in the same way. In its backward, we keep taking it like this. In forward, we keep taking it from the top. And in center, we go from the middle. So, we call it the central difference table. So, in this way, we can make many tables. Forward, backward, central.

The screenshot shows a presentation slide with the title "Central- difference table". It contains a handwritten table and some formulas.

$x_i$	$y_i$	$\delta$	$\delta^2$	$\delta^3$	$\delta^4$
$x_0$	$y_0$				
$x_1$	$y_1$	$\delta y_{1/2}$			
$x_2$	$y_2$	$\delta^2 y_1$			
$x_3$	$y_3$	$\delta^2 y_2$	$\delta^3 y_{3/2}$		
$x_4$	$y_4$	$\delta^2 y_3$	$\delta^3 y_{3/2}$	$\delta^4 y_2$	

Formulas shown on the slide:

$$s f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$$

$$s y_{1/2} = y_1 - y_0$$

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Okay. Now, we have one more thing that we have to see is how the error is propagated. If we have an error in any quantity, then how will we propagate it? How will we consider that the error is coming in the difference table. Now, for example, we have a quantity. For example, if I take this value, we have  $x$  i's. Okay, I have taken  $x$  i's.  $x$  zero,  $x$  one,  $x$  two,  $x$  three,  $x$  four,  $x$  five,  $x$  six,  $x$  seven,  $y$  i's  $y$  zero,  $y$  one,  $y$  two,  $y$  three,  $y$  four,  $y$  five,  $y$  six,  $y$  seven now the total quantity in this is eight okay so suppose there is  $y$  three somewhere in the middle so there is an error somewhere in this I have done plus a now we have to see how it is propagating now see what we are doing in this, I will first find out the delta, okay right, if we first find it then the values that will come to us, we know  $\delta y_0$ , okay let us write it in the middle of it, okay  $\delta$  of  $y_1$   $\delta$  of  $y_2$   $\delta$  of  $y_3$  now see what was in  $y_3$  this minus this, okay so here its this minus this so what will we do, add epsilon it becomes  $y_4$  minus  $y_3$  so how much will it come  $\delta y_4$  minus epsilon a  $\delta y_5$  which will be will become  $\delta y_6$  will just come up to, okay so  $7$  minus  $6$   $6$  it became  $6$  minus  $5$  but the error that was here, if we used this quantity that we had, then there would be an error Look, this has been split into two parts and if we look at this, we have one element one and the other minus one. Okay, now we go to the next part and let's see what happens. Now see what is happening in this, we got  $\delta^2 y_0$ , this minus this has become we have  $\delta^2 y_1$

one now this minus this will go, okay, so this minus this will be delta y two plus epsilon. Okay, now next will come delta square this square y cube this minus this will come, minus two epsilon and the next one that will come is delta square y4, this minus this will come, so the quantity of this will become y 4 minus epsilon a plus this was two, this was three, okay, and this will become plus because this minus this will become plus then this minus this y five so we have it, so if we look at this, our error is like this, error epsilon one minus two one, like this we have a vector, epsilon is multiplied with it. The third one will go to delta cube y i what is happening in this delta cube y got zero this minus this minus this will go delta cube y one this minus this so plus epsilon okay next in delta cube y two this minus this so this minus three now comes delta cube y three this minus this so how much is it plus it becomes plus 3 epsilon becomes this minus this minus this is minus epsilon okay so if we see in this then whatever is left of ours epsilon has become 1 minus 3 3 minus one this is here so in this way if we see then it is coming in the form of binomial sum their 1 minus one is always zero one minus two sum of all is zero -3 -1 so their sum is zero so if we see that in each column if we sum up the errors which are the quantities then the sum of them will always be zero of the errors okay so the distributor of errors is equal and equal distributor how is it happening with binomial coefficient with the help of binomial coefficient this is a distributor error and about it, the error about y 3 was about that, so the error about y3 got expanded, then it got expanded further, then it got expanded further, so in this way, the error about it which is symmetric will keep expanding symmetrically, so such error will be expanded with us in this case, so if we ever come to know about an error in any quantity, that we have a quantity and there is some error in it,

**Error Propagation in Difference Tables**

$x_i$	$y_i$	$\Delta y_i$	$\Delta^2 y_i$	$\Delta^3 y_i$
$x_0$	$y_0$	$\Delta y_0$	$\Delta^2 y_0$	$\Delta^3 y_0$
$x_1$	$y_1$	$\Delta y_1$	$\Delta^2 y_1$	$\Delta^3 y_1 + \epsilon$
$x_2$	$y_2$	$\Delta y_2$	$\Delta^2 y_2 + \epsilon$	$\Delta^3 y_2 - 2\epsilon$
$x_3$	$y_3 + \epsilon$	$\Delta y_3 + \epsilon$	$\Delta^2 y_3 - 2\epsilon$	$\Delta^3 y_3 + 3\epsilon$
$x_4$	$y_4$	$\Delta y_4 - \epsilon$	$\Delta^2 y_4 + \epsilon$	$\Delta^3 y_4 - \epsilon$
$x_5$	$y_5$	$\Delta y_5$	$\Delta^2 y_5$	
$x_6$	$y_6$	$\Delta y_6$		
$x_7$	$y_7$			

$\epsilon \binom{1}{-1}$      $\epsilon \binom{1}{-2}{1}$      $\epsilon \binom{1}{-3}{3}{-1}$   
binomial Coefficients



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then how can we define it with the help of this, so we can take the help of an example, like I take an example, like we have a question, some values of x and y are given, okay, so the value of x is given like this, minus one, zero one two 3, four 5 6 7 8, so only 10 and the value of y, we have 7 5 3 7 25 57 115 203 307 sorry 327 and 493 this value is given to us so our question is to locate the error we don't know where it is what will we do its first final result is 5 minus 7 minus 2 3 minus 5 minus 2 7 minus 3 4 25 minus 7 18 okay after this 57 minus 25 32 115 minus 57 58 115 88 out of 203 okay 23 124 out of 327 and this comes to 166 so this is our first difference now second difference from here we are not able to know anything as the quantity will keep decreasing we will start to know now this minus 2 minus 2 so zero 4 minus 2 6 has come okay 18 4 14 has come 14 out of 32 14 has come 52 minus 32 26 88 minus 58 30 36 and this comes to 42 because now if we take that, then 6 minus 0 6 40 minus 6 8 zero here 12 here four here six and time here okay now it looks like something let's see fourth so 8 minus 6 2 minus 8 12 goes from 4 minus 12 minus 8 goes from 6 4 and this zero, this quantity comes, okay so 12 is -8 -8 2 2 so this symmetry is formed here okay so if this symmetry is formed what does it mean if I see from here then we have epsilon and this is being formed one minus two 3 minus 2 one see Okay, I take epsilon like this I will write minus four and six and this minus and it will be written like this and its sum is what is zero okay so what does it mean if the sum of the error here is If zero comes, then from here we can say that epsilon is two. Okay, so it has expanded on this and we have this quantity. This quantity has come symmetric in the form of Binomial. So what does it mean that the error value epsilon value here will be 2. So if there are two values, then we have to see about which value the value is going on. So this y zero, y one, y two, y three, y four. Okay, so we have to see about which value it is going on. What does it mean because this is our y zero, This is y one, this is y two, this is y three. Okay, our quantity is y zero, y two, y three, y four. The fourth one . So if we look at the table, what is happening in our table, y was in y three and y three reached here. Okay, above it there are three quantities, that is it. Okay, so here if we see from this way, we saw that y is 4 and above it there are four quantities, these are four. What does it mean that quantity is 25 has an error. So this shows that the error was 0 1 2 3 4 y 4 plus epsilon is 25. Here it means that y4 is 25 - 2 23. So these quantities have an error here. And the error has expanded to reach here. So in this way we can find out the errors. So now we are fine. So this error has expanded like this.

Examples

Example 4

x	y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	7	-2	0		
0	5	-2	6	6	$2 \rightarrow 30$
1	3	4	14	8	$-8 \rightarrow 21$
2	7	18	14	0	$12 \rightarrow 12$
3	25	32	26	4	$-8 \rightarrow 17$
4	57	58	30	6	$2 \rightarrow 4$
5	115	88	36	6	0
6	203	124	42		
7	327	166			
8	493				

*Handwritten notes:*

- $y_4 + e = 25$
- $y_4 = 25 - 2 = 23$
- $q(1 - 4 \ 6 - 4 \ 1)$
- $e = 2$

*Other notes:* Interpolate error



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So what we did today is to start interpolation. But before that we discussed a little about the difference operators used - forward, backward, central. Today we discussed the relations between them and we saw how we can create the tables that we are going to use in the future. Forward, backward, central. So what we will use in the future is to Find out that we have to do the interpolation, we will use the tables in it, so I hope you liked this lecture and thank you for watching, hello.