Scientific Computing Using Python

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Lecture No. 15

Welcome all of you to Scientific Computing Using Python. So, the topic that we will do today is that we saw that in the previous LU decomposition, the pivot element in the Gauss elimination method has a great contribution. So today, we will do something related to that and explain that if the diagonal elements in the matrix are dominant, then how do we deal with that type of matrix.

So, let's start again. Today's topic is related to diagonally dominant and positive definite matrices. So, as we did in the previous classes, we saw that the matrix that we had — if we want to see that the LU decomposition is converging — then for that, we saw that we keep the pivot element greater than the other elements in that column.

So, if that process is happening, then we do not need partial pivoting. Otherwise, we need partial pivoting, and then we apply LU decomposition or Gauss elimination to it. So, there are some matrices related to this. What we are discussing is diagonal dominant matrix. Now, as the name suggests, diagonal dominant matrix means the diagonal elements will dominate.

But you have to see that diagonal elements can dominate row-wise as well as column-wise. So now, this definition is that if we have a square matrix, and if we say that this square matrix is diagonally dominant, then a ii means the value of the diagonal element — the absolute value — is greater than or equal to all the elements within that row. If we leave the diagonal and take their absolute value and do the summation, it will be greater than or equal to that. Then we will say that our matrix is diagonally dominant, right?

So, in this case, we have the complete sum. We will use this sum, and from there we will get to know that this matrix is diagonally dominant. Now, this inequality of ours — if I make it strict inequality — then what will happen in that case? The matrix is strictly diagonally dominated. If the strict is there, what has happened in that case?

See, in that case, what we have is the same thing. If we see the terms that we have, the equality has just changed. Okay, so now in this case we have this, so we will call it strictly diagonally dominated.

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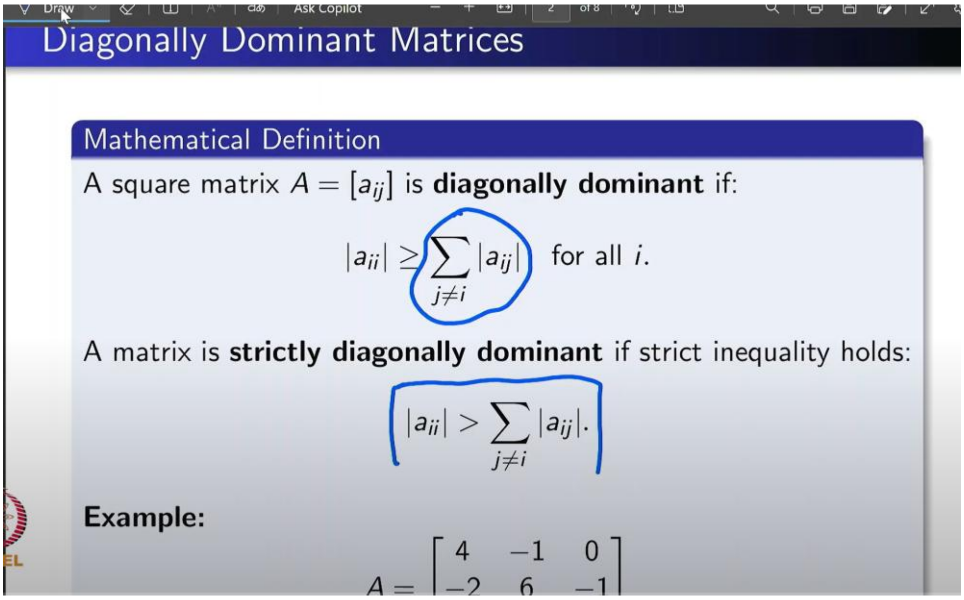

So, as an example — now, if we want to do this example — then to do the example, I have taken this example. Let's see in it. So, we have to check it. So okay, now see, the first diagonal element is this — so I saw that 4 is greater than or equal to -1+0 so 1 — yes 4 is greater than 1.

And the second one, we saw that this is a diagonal element is 6, so the others that are left with us are -2 and -1. Okay, so the value that we have is 3, and our diagonal element is 6, so it is greater than this. Okay, I have taken this diagonal element, so what is happening in this is that the other elements are -1 plus 0, so it has become greater than 5. So, in this case, we will say that this matrix is — now look — it is strictly, if we see in it, then I can say in it that this strict inequality is following it. So, in this case, we will say that the matrix A is strictly diagonally dominant. Okay?

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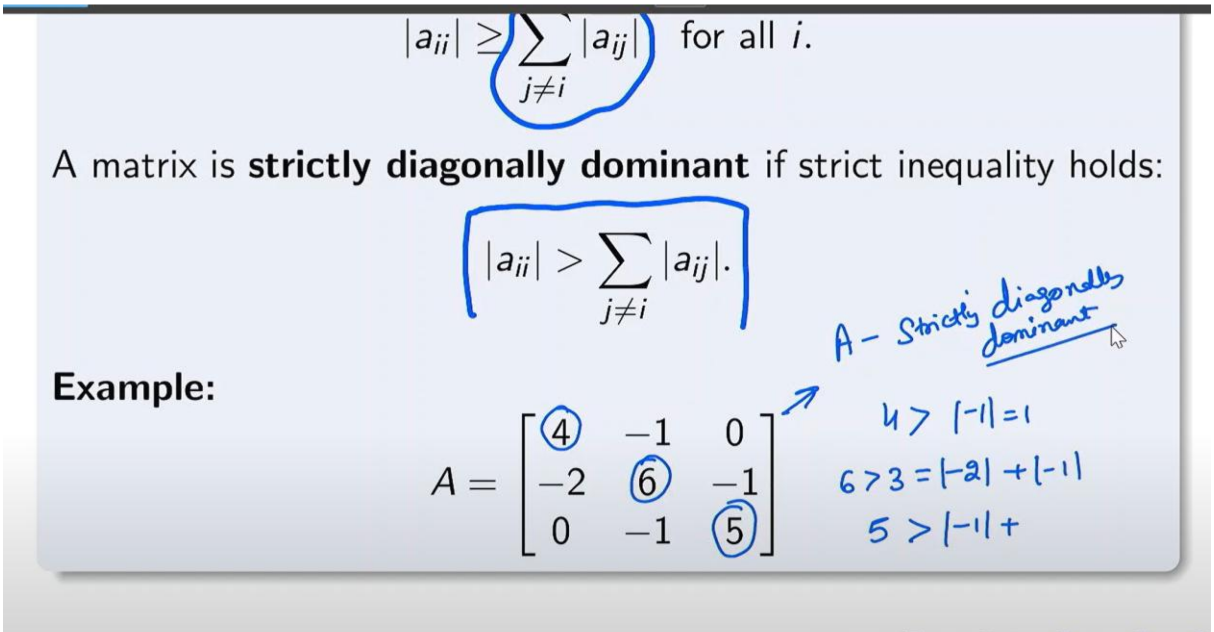

So, if we apply LU decomposition on this matrix, apply Gauss elimination method, then we do not need to do any partial pivoting, and we will get our solution. Okay?

So, in this case, we can say that our pivots are automatically very large. So, if we see in this, then we have talked about diagonal dominant. So, in this case, if you see, we have done the rows. Okay, right?

So, in this case, we are seeing that in the comparison of rows, the diagonal elements should be greater. So, what do we do on this basis? Taking rows, we take the column diagonal dominant matrix.

So, what is the column diagonal dominant matrix?

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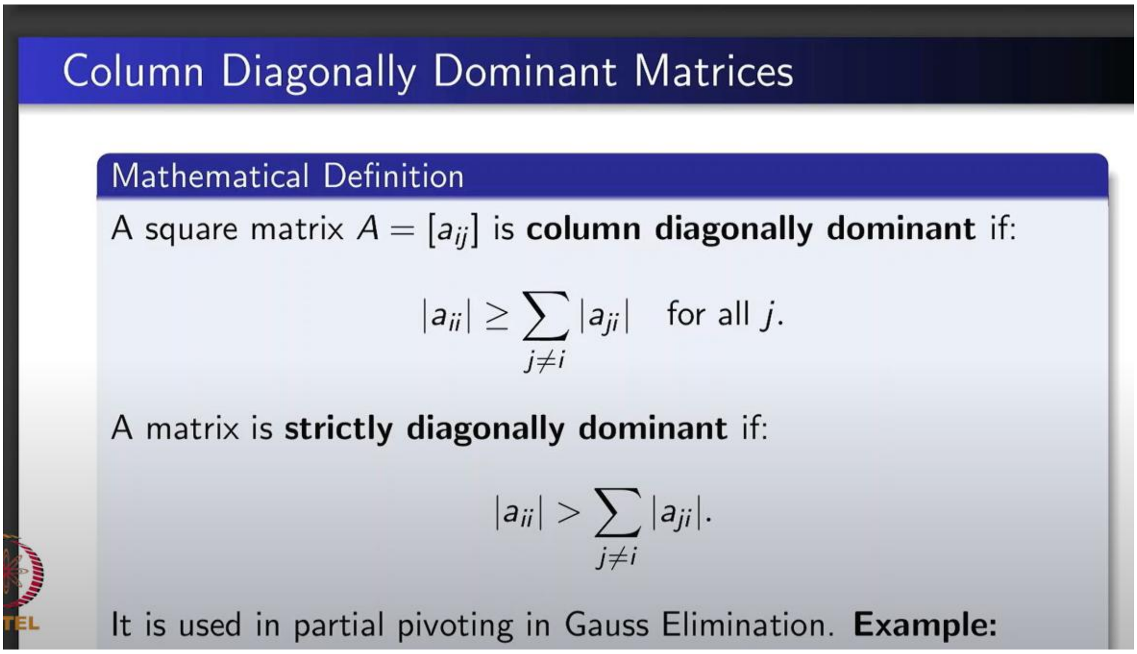

The definition is the same — the square matrix is diagonal dominant. Okay, it will either be diagonally dominant or the column is diagonally dominant. So, if the column is diagonally dominant, it means we are taking inequalities — this one is coming with respect to the column.

See what is happening in it — j is moving for the ii. So, if our first is taken, then this a11 then suppose, I take it like this, then we told that it should be greater than or equal to a. Okay, one is one, otherwise 21 and a31 will go on like this. What does it mean?

We are focusing on the diagonal elements, and same — if that inequality becomes strict inequality — then we will say that the one which is diagonally dominant and strict is a column diagonally dominant matrix.

So, what happens in this is that we have this process. We follow this process for the partial pivoting. We have written that this work is done in partial pivoting. But if the matrix we have is diagonal dominant, then that will also work for us.

So, here we have an example. Now, this was the same example. Now we will see whether this is column diagonal dominant or not. So, we will do the same. This is ours. So, 4 is greater than 2 , then 6 is greater than the 1 plus 1,2 and 5 is greater than 1. So, we can say that this is column diagonal dominant. This matrix is column diagonally dominant.

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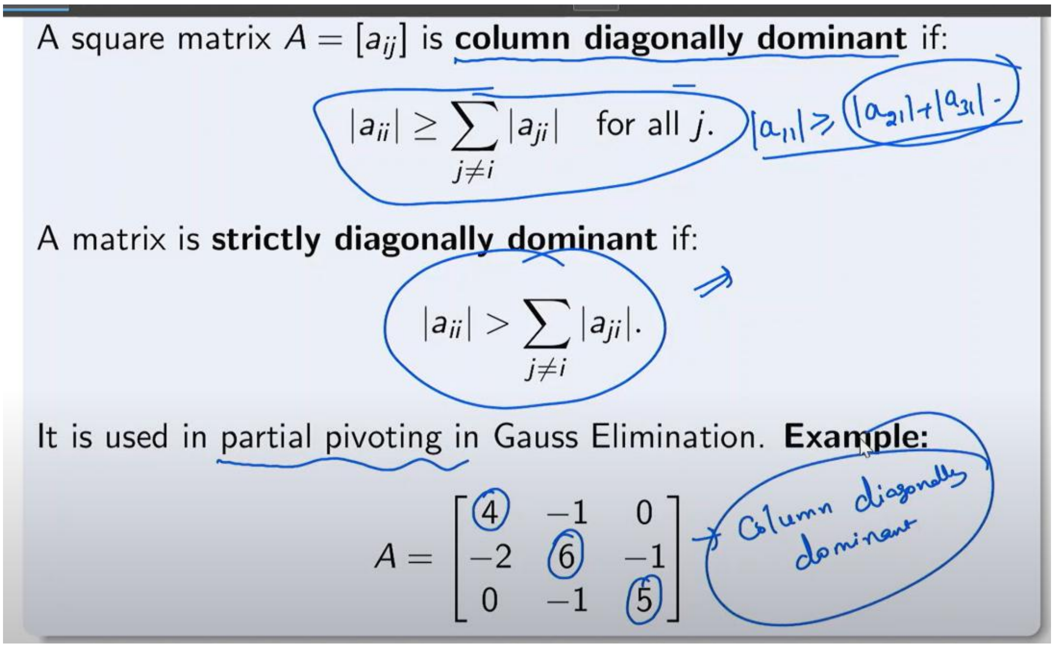

So, in this case, if we apply Gauss elimination, then we will not need to do partial pivoting. So, our solution will also come. So, we use this type of matrix a lot. In the future, we will use it a lot, which will be useful for our purpose. We will discuss the method.

So, if there is convergence in those strict methods, then for that, this diagonal dominant matrix plays a very good role. So, what are the properties of a strictly diagonal dominant matrix? Always non-singular matrix. Okay, any strictly diagonal dominant matrix will always be non-singular.

So, for this, our case has come for strict. Keep this in mind. Like, we have a matrix A, in that, I take the element 5. Okay, 2, 1. I took this. After that, I took 3, 5, -1 after that, suppose I took this 0, 0, 0.

So now, what is happening in this, the first element 5 is following that 5 is greater than 2+1 . In the second also, it is happening , 5 is greater than 3 + 1 — because -1 is coming. But what is happening in the third one, the zero element is equal to 0 plus 0. So, our equality is following.

Isn't it that our equality was greater than or equal to zero? So, if we have a strict — if there is no strict equality — okay, if we talk only about diagonally dominant, then in that case, it is following, and we know that the matrix is a singular matrix.

So, in this case, we will say that the matrix is singular, but it is diagonally dominant. So, for this, we will talk about diagonal dominance. And if we do it strictly, then this thing is not being followed strictly.

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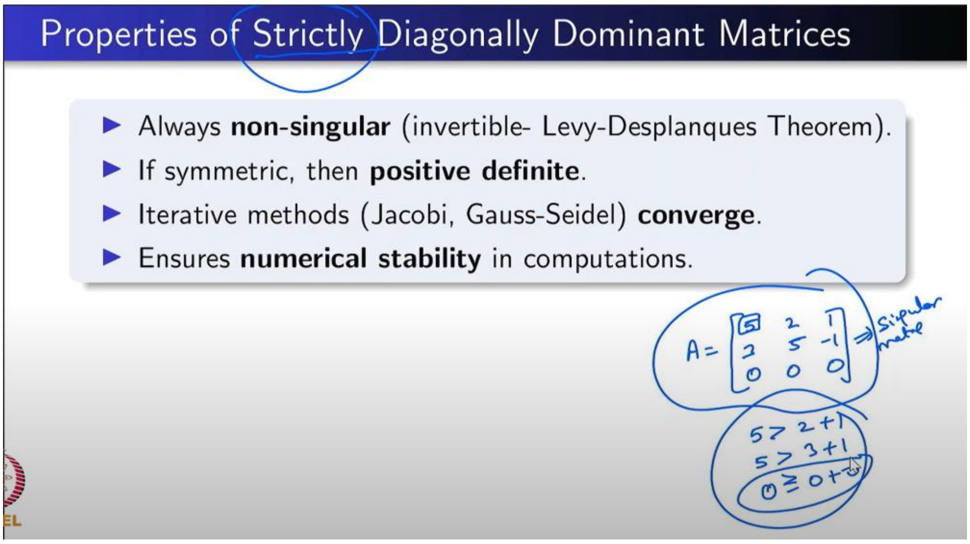

So, if this thing is not being followed, then it means that we cannot say that the matrix will always be singular or non-singular.

So, if it is strictly diagonally dominated, then the matrix will always be non-singular. So, this is discussed by Levy-desplanques theorem. It is explained. So, this has given us the advantage that if we have a large matrix, suppose we have a 10 by 10 matrix or any large matrix, then to check its singularity or non-singularity, we will have to perform many operations because in that, we will have to find out its determinant.

Finding out the determinant is quite computationally expensive. So, what will we do in that case? If we see that our matrix is strictly diagonally dominant, then we can easily say that the matrix is non-singular, and then we can solve it. This is the advantage.

Secondly, if there is a symmetric matrix and it is strictly diagonally then positive definite, then we will discuss it, and its best use is the method by which Jacobi, Gauss-Seidel converges. So, it means that diagonal dominance is ours — we can say that it is a kind of sufficient condition for convergence — then Jacobi and Gauss seidel method, which we will do in future.

And if it is a diagonally dominant matrix — if it is a strictly diagonally dominant, non-singular matrix — then it ensures numerical stability in the computation. So, what happens with this is that numerical stability is achieved. Okay.

So, in this way, we can define all the matrices and check that, because diagonal dominance is very easy to check compared to finding the determinant. Even if we have a very large matrix, okay, like I write 1, 4, 3, 2, 1, 0, 4, 2, 2, 1, and then I take 3, 5, 0, 1, 2, 4, 1, 3, 2, 1, and take one more element, what do I take, 0 ,2 ,3 ,0 ,2.

I have taken this matrix. Now we do not know whether this matrix is invertible or not. So, what will we do? We will find its determinant. So, to find the determinant, you know that a lot of computation will have to be done. But if we see from here — diagonal is a diagonally dominant matrix — then we can easily tell it.

So, see, in the first one, there is one. So, one is smaller than 4, 3, 2, 1. So, this is not strictly a diagonally dominant matrix. So now we cannot tell anything about whether the matrix is diagonally dominant, meaning whether it is singular or non-singular.

Let me change it a little bit. Now what I do is — how many of its elements here — 4 + 3, 7,9, 10 — so here I make it 11. Okay, here I have made it 6. Here I remove it and remove this as well. And look here, how much? Five is there. I make it six. Here, how much is there? 5 plus 3, 8 ,9, 9 plus 2, 11. So I write 12 here. 4 plus 1, 5 plus 3, 8 plus 9. I make it 10, and this is 5. So, suppose here I make 6 — now this matrix will be formed. We have this matrix. I have made it diagonally dominated — strict diagonally dominant matrix. Okay, so now we can check that these diagonal elements — this one, this one — this is the strictly dominant condition.

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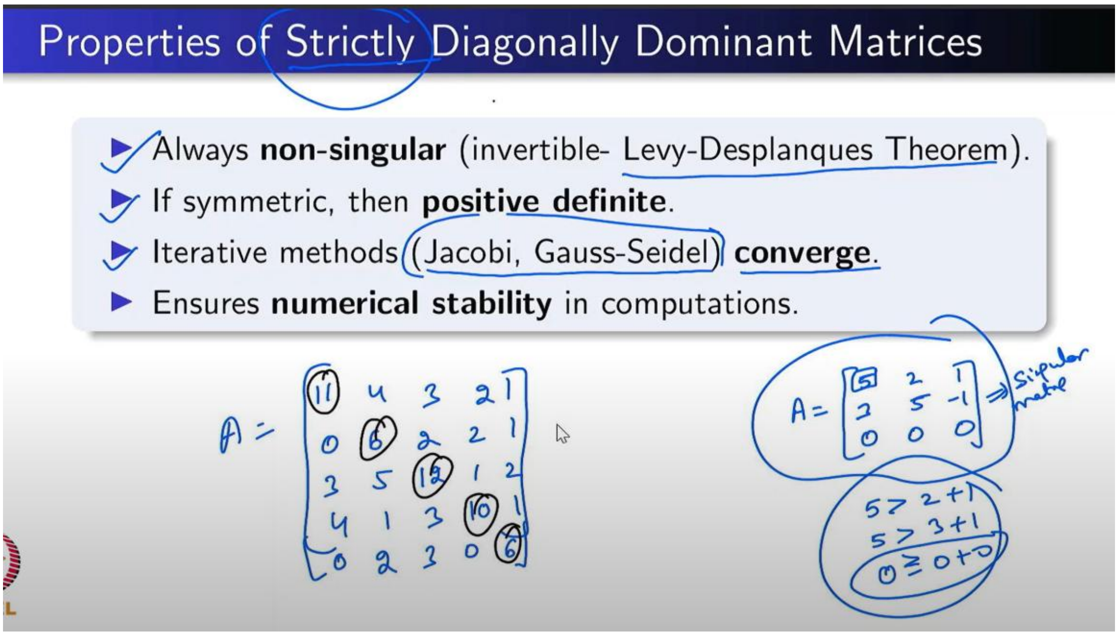

If it follows, then we can say that the matrix A in this case will be non-singular — a non-singular matrix. So, on the basis of this, we can say that this means that this property is very beneficial for us.

And after that, if the matrix is non-singular, then we know that our solution is easily available in dealing with it. So, we can easily solve our system of equations.

So, now our next thing is positive definite. Till now, we had said that diagonal dominant matrix. Now, we have a new word — positive definite. Positive means something related to positive, and define — we have to see what is the meaning of define.

So, a symmetric matrix A is called positive definite or strictly positive definite. So, everything depends on inequality. If this quantity x is transpose Ax, greater than or equal to zero, then we will say positive definite, and if strictly greater than zero, then we will say that strictly positive definite is for all non-zero vectors in x.

So, what does it mean if this quantity is there? Then we will say that our matrix is positive definite or strictly positive definite. What does it mean?

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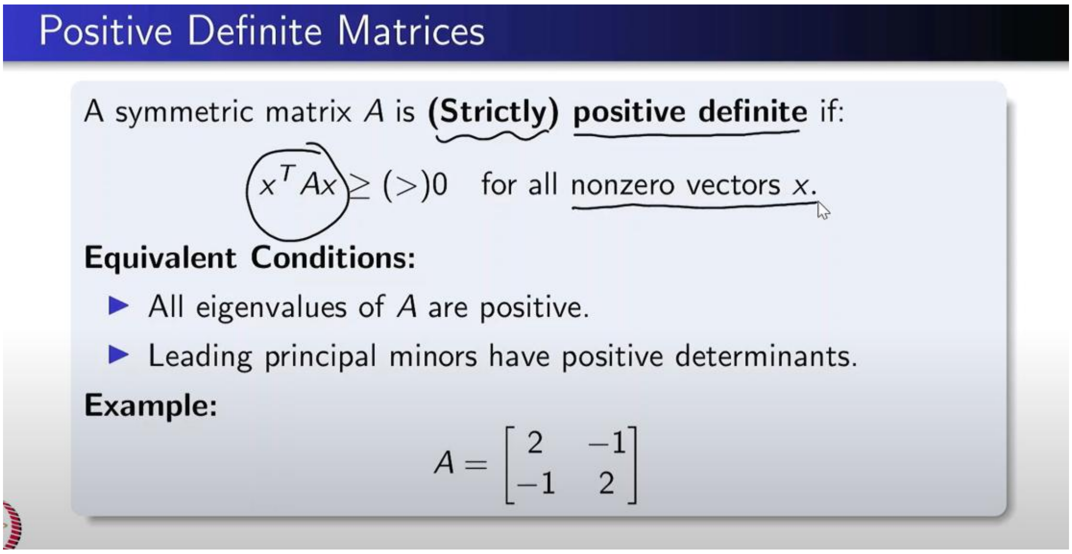

What does it mean that positive strictly definite matrix means that all its eigenvalues will be positive — all of them will be positive. So, if it is so, then we will call it that this matrix is positive definite.

Or, leading minor principal minors have the positive determinant. So, what does it mean that the principal minors are principal minors? Like this matrix — what is the principal minor in this? First element is this, first minor or one by one matrix. Now, this two by two matrix — so these are our principal minors, okay? So, the determinant of principal minors is minus, so these are all matrices. So, if we get their determinant positive, then we will say that the matrix is positive definite. Like in this case — so what did we do? This matrix — this is the determinant of this, assumed that this matrix is A1 has the determinant coming out to be 2. The determinant of A2, 2 by 2 matrix, which is the principal minor, is 4minus 1, 3. So, in this case, we will say that this matrix is a positive definite matrix. Okay, and it is positive definite. And in this case, if we find its eigenvalues, then very soon we will come to know that the eigenvalues of this matrix are positive.

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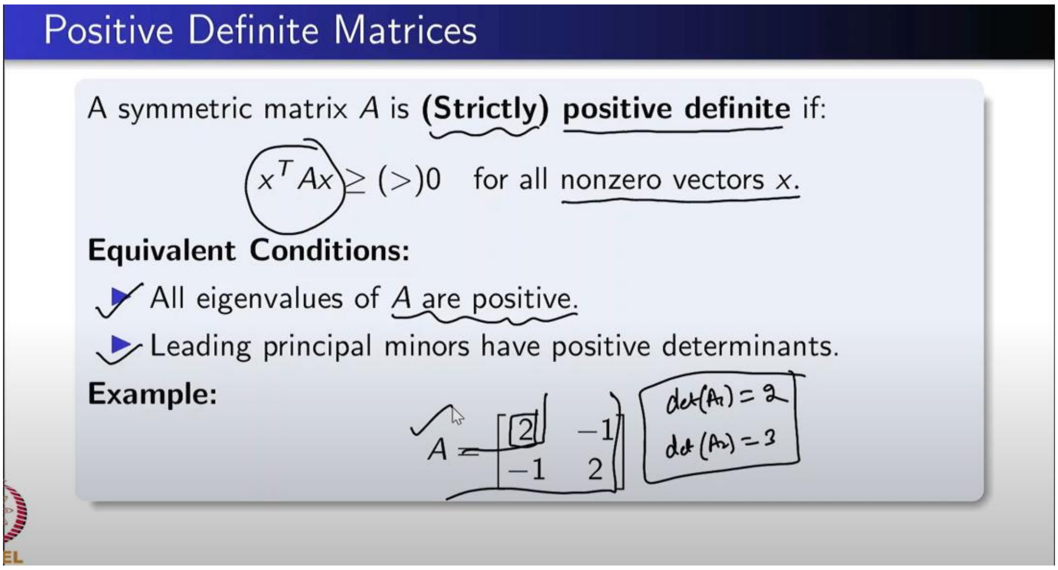

And if we see, our matrix A is symmetric. That means A transpose is equal to A. So, this also follows. So, our symmetric matrix A is strictly positive definite if and only if all its leading principal minors have positive determinants. So, we will start finding the determinants of all the principal minors, like 1 by 2/2 3/3.

If any determinant comes non-positive in between, then we will say that the matrix is not a positive definite matrix. Now, we have to keep in mind that the greater than or equal to zero, or greater than zero, is strictly positively definite or positively definite. We have to keep in mind that we have got this, so we did it.

So, from here, our question arises — what does the definition that we have written there mean? What does this factor on the left side mean? What is this thing? So, if we want to do this a little bit, then we can see from here that one of the things that we have been writing — when we used to write the eigenvalues — because this is related to the eigenvalue. So, if we see, we used to find the eigenvalue like this. If there is a matrix A, n cross n, x, which is a vector, we used to call it lambda x, right? So, if it follows this thing — A x = λ x — so, if it follows this thing Ax= λx , so what we used to say from here is that λ is the eigenvalue is of A , and x is the corresponding eigenvector. Now what will I do, x is a vector and we know x is non zero, so it is non zero so x is a vector, vector means column vector, I will take its transpose and pre- multiply on both sides, means multiply on left side. I have done it like this. See from here, now what is happening is that if there is a vector, its transpose , if we had column vector then its transpose will be row vector. So, if any vector is (x1, x2,…,xn), this vector becomes its transpose and the other vector is x1, x2, xn in itself, so this will be a column matrix and a row matrix. If we multiply them, then we will get x1 square, x2 square and xn square.

This will come to us, and this quantity is always positive. And if we see, the norm that we have is the square of x. So, this quantity will always be positive and will also be non-zero, because x is non-zero. So, its norm cannot be zero — it will also be non-zero. So, we can divide it by λ from here. So, here we can write x transpose x like this.

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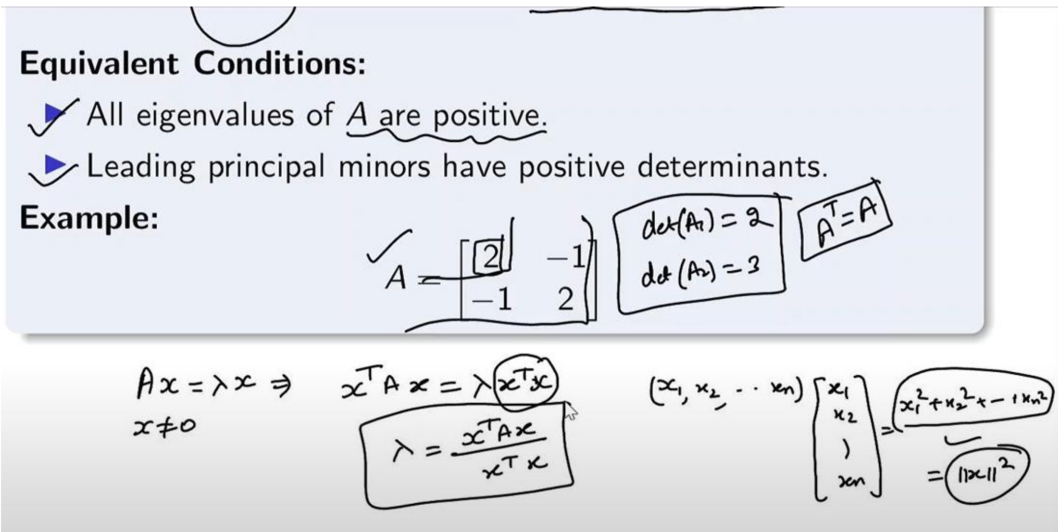

Now, see, this quantity is always positive. From here, we get to know. Now, if I say that the λ is positive — eigenvalue of matrix A — if I say that it is positive, then it means that this quantity should also be positive. If I say λ that it is negative, then it means that if this quantity is negative, then it will be negative.

So, if we have to pay attention to the eigenvalues of the matrix, then we have to pay attention to its sign — what is the sign of this x transpose Ax? If it is positive, then it is positive; negative, then negative. So, it means that this plays a very big role. So, if we come to know from this what our sign is, then from there we can easily tell what is going to be the eigenvalue of the matrix.

So, that is why this quantity plays a very big role. So, this type of equation is called a quartic equation. So, if we solve it, then we get an equation. Like, if I just take an example — we had taken this example. So now, if we see, I have to find out what is the x transpose of this matrix A x. I want to check this.

So, what will I do? This is 2 by 2. So, we have x1, x2 of two dimensions, we have column vector 2, -1, -1, 2 — this. And here we have column x1, x2 — this vector. Now, what will happen if we multiply it? So, if I multiply this, then what will happen?

We have this. See —1 cross 2 to this. This is 2 cross 2, and this is 2 cross 1. So, it is very easily compatible. We can multiply it very easily. So, I have this quantity x1, x2. Okay, let's see here what will happen. So, this will come 2 x1, okay, minus x2. And from here, this will come, -x1 plus 2 x2.

This we have 1 cross 2 to this, and this will come 2 cross 1. Now, if we multiply it again, then it will come 2 x1- x2. We multiplied it by x1, plus x2. And this we wrote, -x1+ 2 x2 . Okay. So, from here we can calculate. If we do this, then it will come out as 2 x1 square plus 2 x2 square.

Now, here we have minus x1, x2, and here also we have minus x1, x2. So, this minus 2x1 x2 comes out.

Okay, so here we have an equation. We call it a second-degree polynomial in x1 and x2 variables, we can also write x1 as x and x2 as y. Okay, so we have this type of second-order polynomial. In our variables x1 and x2, we have this polynomial.

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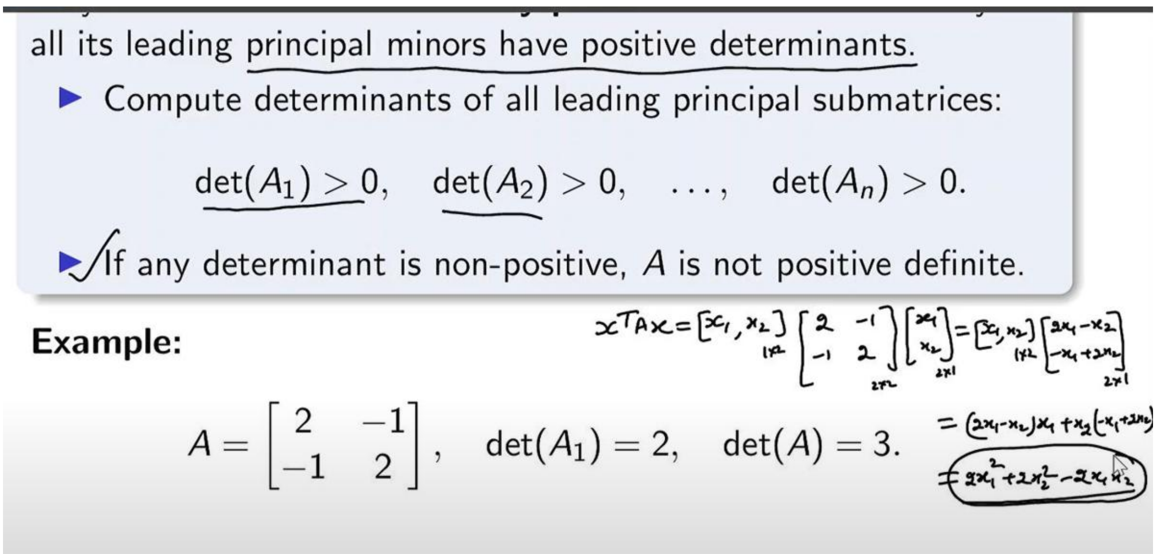

So, if its value always becomes positive for all the values of x1 and x2, then we will say that it will always remain positive. It means it will remain positive definite. It means all the resulting values will always remain positive. And if it becomes negative somewhere in between, then we will say that it is not positively definite. If it will always remain negative, then we can convert this thing and take another definition — that is, negative definite.

So, what does negative definite mean? That all the resulting values are negative, right? So, we will make these quantities less than or equal to all the things. So, what will happen in that? If we want to see negative definite, then what will happen in negative definite? This definition — this negative definition — will become less than for negative definite, right?

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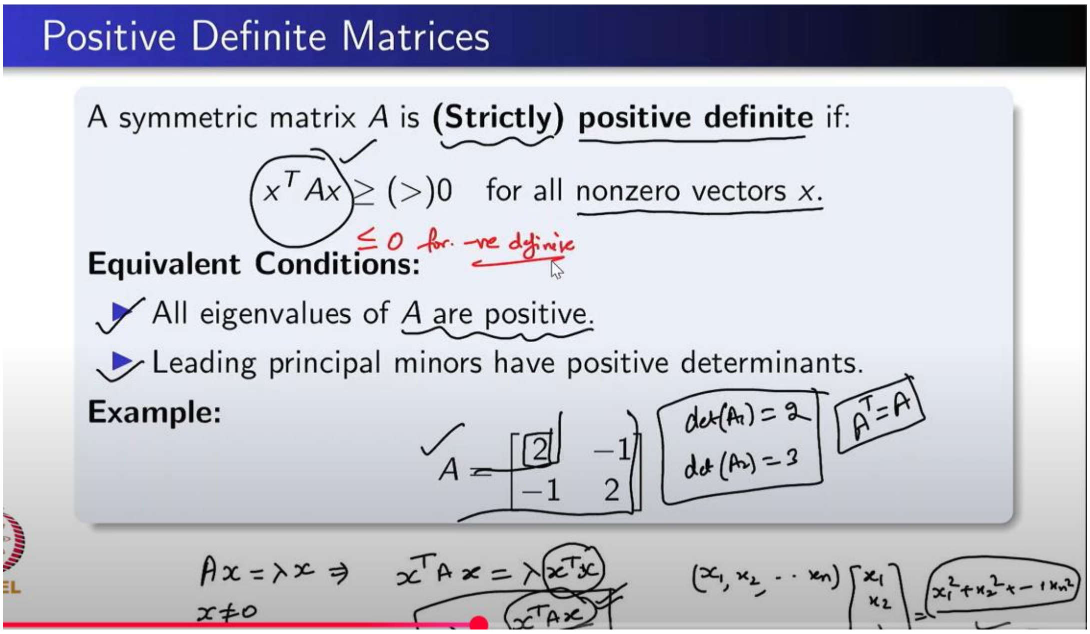

So, this thing in negative definite — we just have to take a little care in that. In positive definite, okay, very easily, we will all become positive, but it is not like that in negative definite. There are interchanges in it. In negative definite, the first one will be less than, the second will be greater than, then the third will be less than, then greater, the fourth will be greater than — it will go on like this, alternatively, depending on what your n is.

If n is odd, then negative plus negative plus negative will go on like this. If it is even, then positive will go on in the last. So, in that case, what we have will become negative definite, right?

So, Sylvester's criterion says that if a symmetric matrix A is strictly diagonally dominant, if and only if all its leading principal minors have positive determinants.

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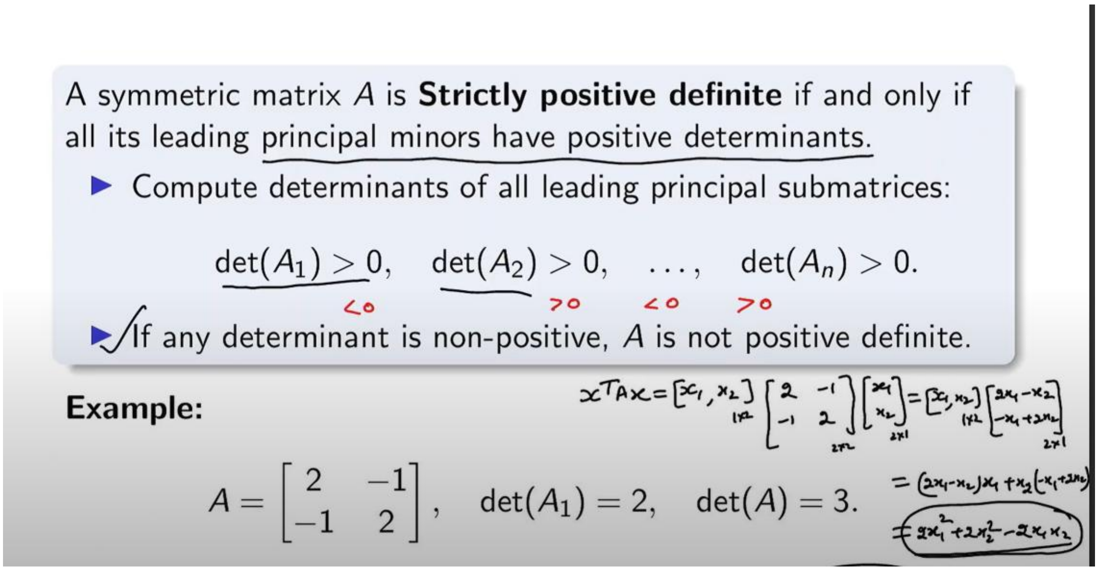

Now, we try to check this — try to apply this to the matrix.

So, now we have to see what is this A matrix. If we see, A is strictly positive diagonally dominant. If we see why — because 2 is greater than 1 — well, it is not here. So, it is not strictly greater than or equal to . So, it is diagonally dominant but not strictly, because of this second row that is coming in the middle.

Now, what we have to do is that to check this, we have to look at its principal minors. So first, see what is it. So, I took the first A1, and its determinant is here. So, the first element is this, the second element is this one, then A2 will be 2, -1, -1, 2. So, 4-1, 3 has come.

Now we have A3 — its determinant — so that is complete. A has come. We have to find its determinant. So, what is its determinant? 2, so 4 minus 1, 3 plus 1. In this, I am taking it from the first row, so the minus will become plus. So, this and this and this goes, then it will come to minus 2 and zero minus 2. Okay, and it is zero. So, from here we get 6 minus 2, 4.

So, you see that all the determinants are positive. So, if they come positive, then we can say from here that A is positive definite. And in this case, it is strictly positive because all the values are coming greater than zero — equal to zero, so no. So, if it is like this, then we will say that this is strictly positive definite.

If I change this, and suppose what I would do — I would make the last column row zero zero — if I change this slightly, okay, and we have this matrix 2, -1, 0, -1, 2, -1, 0, 0, 0 — so now see what is happening in this case. A1 is positive, A2 is also positive, A3, which comes in this case, will be zero, right? So, this means that we will say that it is positive definite — no problem. All its element values are zero or greater than zero.

In this case, we know that if the determinant of the matrix is singular, then we know that its eigenvalues — at least one element is definitely zero, because the product of element values is equal to the determinant of the matrix. So, if the determinant of the matrix is singular, then its determinant is zero. It means that some eigenvalue that will definitely be zero.

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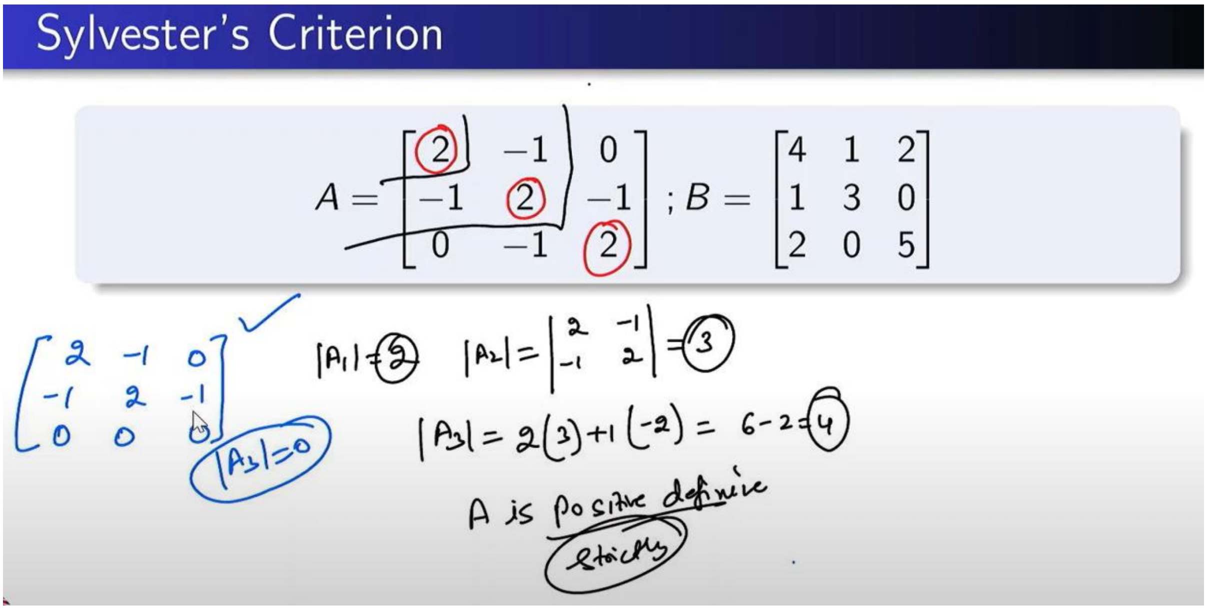

In this way, we can see this B. Now, what is happening in B — like this, we have seen the first element 4, and then see B2 , B2 is this. So, how much is it coming out to be? 12 minus 1, 11. Okay, and B3 is — we saw — how much is it coming out to be 4 so, 15 minus 1, 5 minus 0, 5 plus 2, 12 times 1, 11 — how much is it? 60 minus 5 plus 22. So, it came out to be positive.

So, in this case, we will say that the matrix B is strictly diagonally dominated — strictly positive definite — not diagonal dominant, positive definite. Now, our question can also come that we are talking about positive definite, we are talking about strictly positive definite, but why are we taking our matrix such that it is symmetric?

If you see, A is also a symmetric matrix, B is also a symmetric matrix. So, what role is symmetric playing here — that we are looking only for a symmetric matrix? So, it means that the criteria that we have applied here is — that we know that the symmetric matrix would be — isn't it a property of a symmetric matrix? What is it? Symmetric matrices always have real values. So, in a symmetric matrix, the eigenvalues are always real.

So, if they are real, then only we can say that all its eigenvalues are positive or negative. If it is getting complex, then it does not have ordering, we see all these things that we have matrix is symmetric, by this we get sure that all eigenvalues are real and this criteria applys on real eigenvalues. We can take an example of negative definite as well . If we want, we can also take negative definite Let's take an example of negative definite and check what does negative definite mean. In this case, like I have a matrix, I am taking -2, -3, -1, and after that suppose 1, 1, 0, 0, 1, 1. I have just taken this, I can take diagonal matrix. I have taken diagonal matrix. So in this I have been matrix A so lets check what is this matrix. So what we did? First take the matrix, first principal minor, this and minus 2 come then take second, so this comes 3 into 6 minus 1, 5 come out. So, as this 5 came, okay. Now we will go to the next page. Let's go to A3. Now A3 came to us, so now we will see the determinant of its whole. So, how much did it come? -2, so how much this come 3-2, 1. Okay? -1 ok, I am taking it from here. So, this and this went. So, this came, -1, -1 and 0 so this come -4 plus 1, -3 came, okay.

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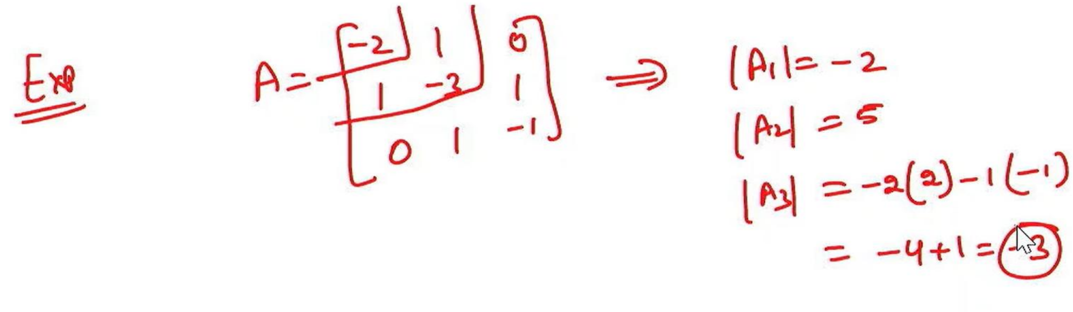

So now look at us. Here negative was coming, but this came positive, but this came negative. So, what will we say? In this case, we had said that if there was negative definite, then everything that would have happened in it is alternative. So, alternative is also coming: negative, positive, negative. So, it means that we can say that the matrix A, matrix A is strictly negative definite. And why are we taking this alternative sign? So, the meaning of alternative sign is that, brother, if we have a one-by-one matrix and that one-by-one matrix has only one eigenvalue, then it has come negative, okay.

If it is a one-by-one matrix, if we have a 2 by 2 matrix, then if both of its eigenvalues are negative, the first one lambda 1 is negative and the second one is also negative, then we know that their product will be positive. The product of both will be positive. And we have 2 by 2 matrix, if we take the product of the eigenvalues, then it will be the determinant, right? So, if we take the product of both, it will come out positive. Now, if there is a 3 by 3 matrix, then the eigenvalues in it will be three. So, if all three are negative, minus this this, then what will be the product of all three? It will come out negative. So, it will come out negative. So, look, this is negative, positive, negative.

So, from here we can say that the matrix is a negative definite matrix, okay, and it is strictly equal to zero. It is not coming anywhere, right? So, in this case, we can say that this matrix is negative definite.

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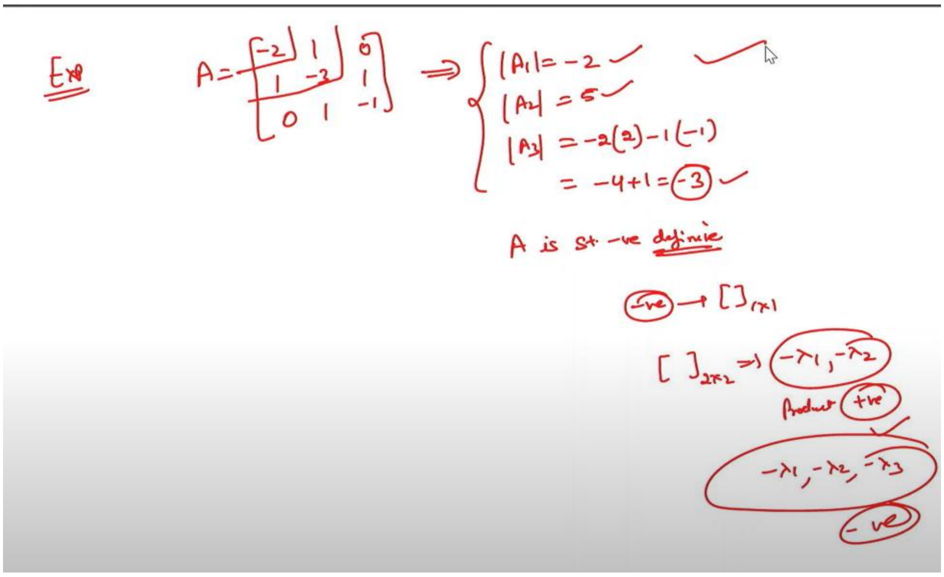

So, we can create many matrices like this, and this is a symmetric matrix. So, we can do this easily. After that, what do we do? Another one which will be used a lot in the future is the ill-conditioned system, okay. So, let us discuss this a little bit so that we know.

Let us see what is the meaning of ill-condition. So, we have given this name "ill-condition." What happens in ill-condition is that we have a system like this. This is the definition. An ill-conditioned linear system or linear system, this will be called system of linear equations where small changes in the input right? So, small changes in the input, if we are making small changes in the input, then what will happen? Which input is x? So, if we input this x, then the result output is input A and B. If we make small changes in them — then I have written — coefficients of the matrix or right-hand side value leads to a change in the solution. So, this makes the numerical computation unstable and prone to the large errors.

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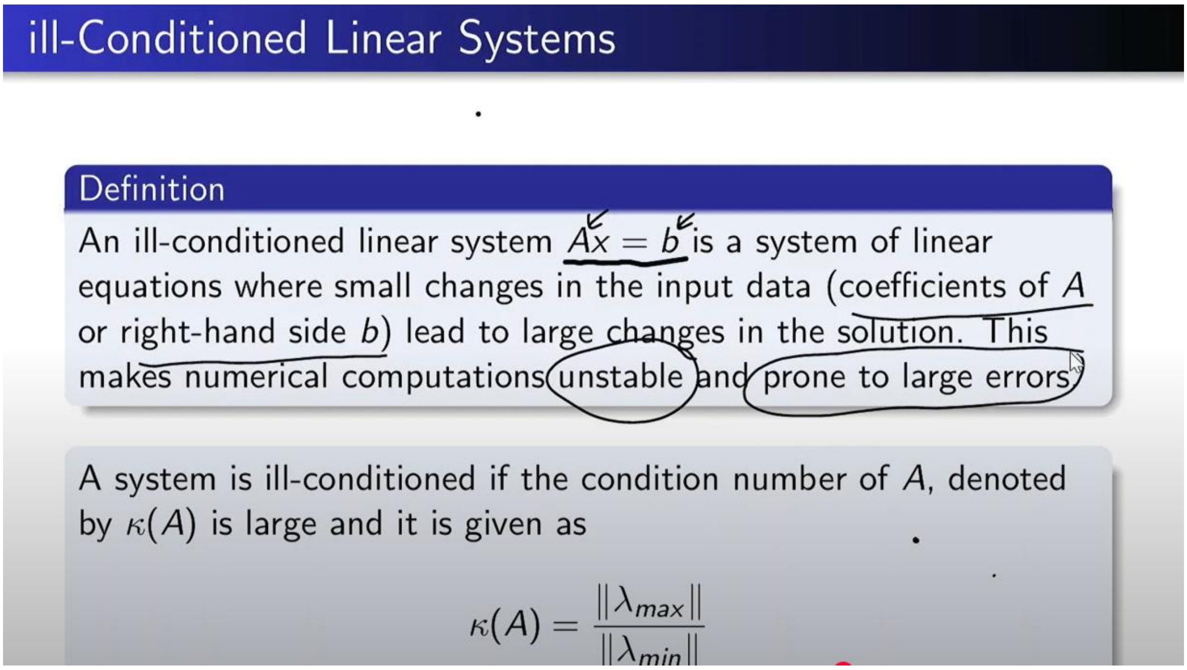

So, this type of systems are called ill-conditioned systems. We made a small change and immediately the solution got a huge change in the solution. So, system ill-conditioned if the condition number of a matrix — then we will define what is the condition number of the matrix — which we will denote as kappa A. Let us denote it. It is large and is given by this one. So, this is one of our norms. We have written it under that category. So, if we see it, it is giving the ratio of the largest amount value divided by the lowest amount value. So, this eigenvalue is just the modulus value. So, it is the largest and the smallest eigenvalue of A.

So, we have to see from where will this come and how can we calculate it. Now we have explained about the ill condition in the linear system.

So now we have to see what is that condition number, what is its definition, real. So, if we go to the definition, a condition number is defined like this. So, the condition number of a matrix A is equal to the norm of the matrix multiplied by the norm of its inverse. So, if we product the numbers of both of them, we call it the condition number.

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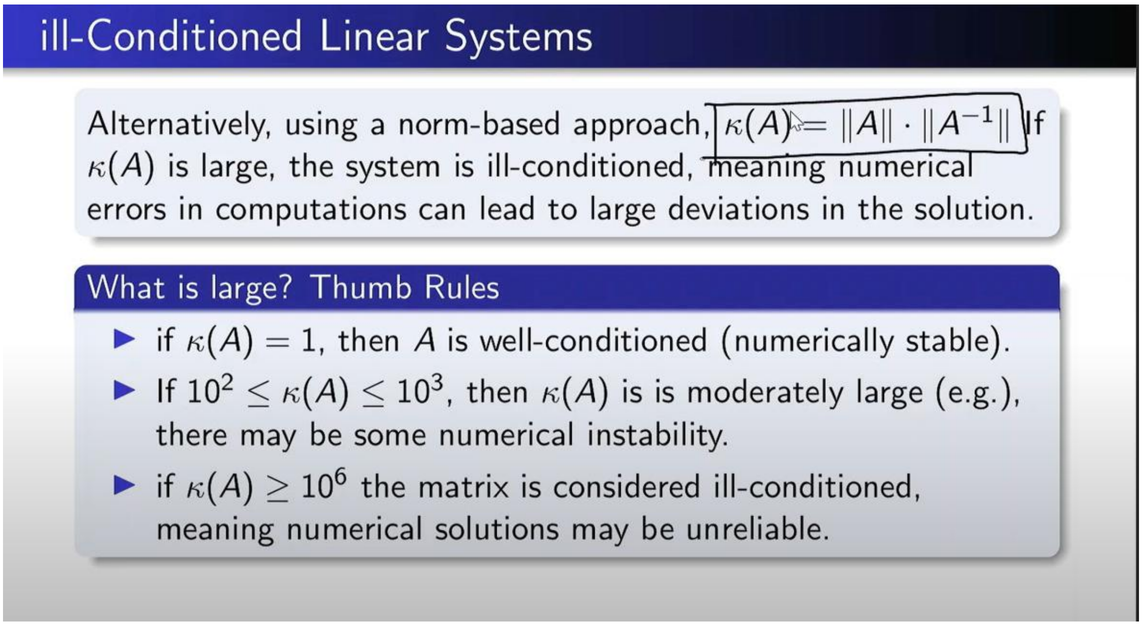

So, if the condition number of the matrix is large, then the system is called ill condition. And what happens in it means numerical error in computation can lead to a large deviation in the solution, okay. Now see, we said that if the condition number is large, then it is an ill condition. If it is small, then it is correct, well-conditioned. What does it mean? Because large and small is a relative quantity, okay. So, we have to see when we say that it is large and when we say that it is small.

So, look, the first thing to do is that if the condition number is one, then the matrix is called the well-conditioned. And which matrix will have condition number one? So, if we want to calculate here, then if we have an identity matrix I, then if we want to take the condition number of the identity matrix, then what will happen? We can always write I as A A inverse like this. It can be any matrix, A * A inverse, right? And we know that it will be I. So, if we have A A inverse and we are taking any matrix — let me take I take it because it will always happen. So, if I am taking the condition number of the matrix I, I means identity matrix, so its condition number is always one. So, the condition number of the matrix I is always one. Then it comes — then the matrix is called well-conditioned.

Now see, the second quantity comes. If the condition number of the matrix is between 100 to 1000, then we say that the condition number is moderately large, means it is a little large, and there may be some numerical instability. There may be some numerical instability. But if the condition number becomes bigger than 10 to the power of 6, then the matrix is considered very ill-conditioned, and the system will be a numerical system. The solution will become unreliable, okay.

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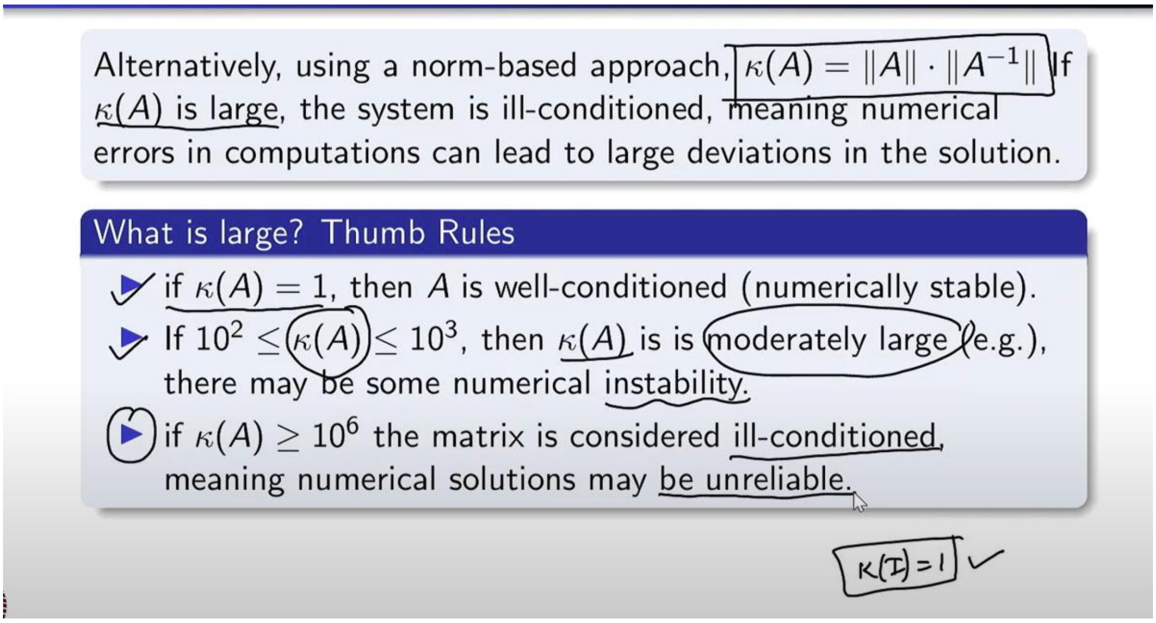

So, we always keep checking that the matrix we have should not have that ill condition, because we have to pay a lot of attention to ill conditions.

So now let's see bhaiya what is — sensitivity is, okay. So, high condition number means change in b leads to large change in x. And if the condition number is greater than the precision of our computer, then the numerical calculation becomes highly unstable. So, this is the double precision of our computer. And what is its application? That in engineering and physics, if the condition number of the matrix becomes greater than 10 to the power of 6, then it is considered problematic. And in highly sensitive applications, if it is greater than 1000, then it might be too large.

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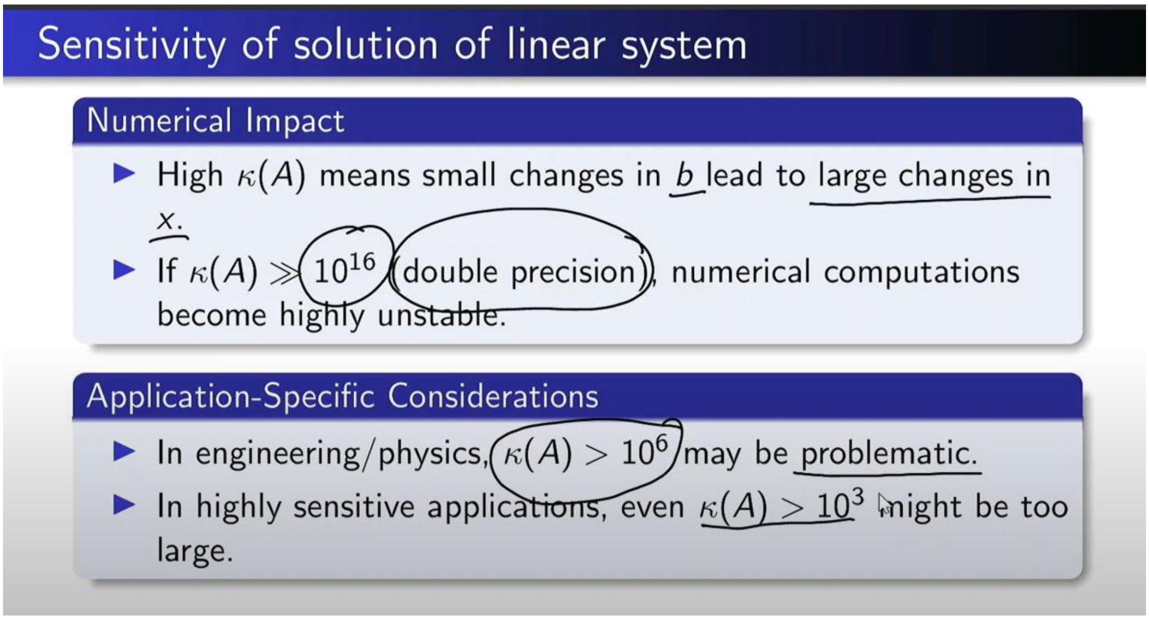

So, this is something we have to take care of a little bit.

So now we see which matrix — if we say that, for example, I take a matrix — suppose I take one matrix A, so I took this matrix. In it, I took the value 1.2, 1.1, okay. Or I took 2.1 — let's take 2.1 — and after that, I did this 1, 2. Or let's take it at a point. Or now this is one. This is our matrix. Now, if I solve it, what will we get? Suppose I have a system. I am taking this system A x = b, b, I took 3.1 and 3. Now what did I do with this matrix? I solved it. So, when I solve it, the matrix A, this is non-singular, and if we see, its solution is coming x = 1, 1, okay. So, this is the solution of this system.

But now what do I do? Suppose I change A a little. So, I take the small change in A because we said that we can change the matrix A or we can change it to b. Now I changed A, and I converted the matrix a little and named it with tilda. And what did I do? I made it 1, 2, 1, 2 like this. See, I made a small change from 2.1 to 2, just a little. Now what happened in this case is that our system became this is equal to b. Now this system has become inconsistent. Inconsistent means — not solvable. Inconsistent is correct, isn't it? Because now its solution does not exist. The matrix that has been formed has become a singular matrix.

So, what happened in this case is that I made a slight change, and see how big the change was that the system became inconsistent. So, what will happen is that we will say that this matrix is an ill-condition matrix, and we will find out its condition number, and then we will check the condition number, and it will come out to be very high.

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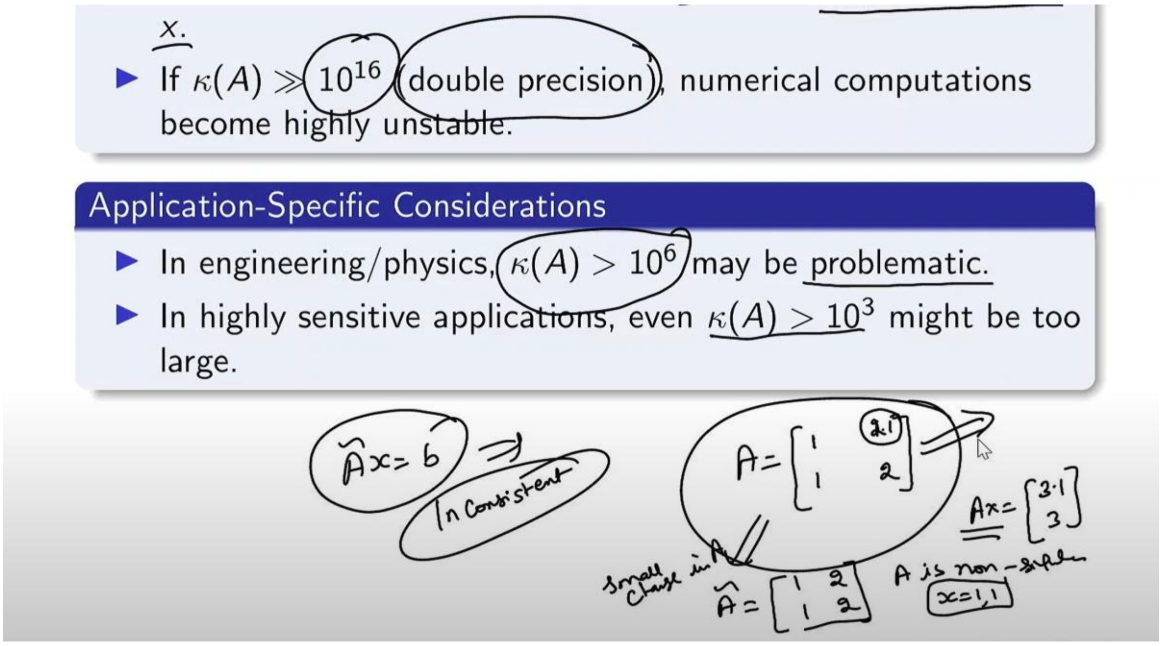

So, we have changed A. In the same way, I can change b as well. It is not that I will just change it and it will be done. Suppose I changed b. Now the system that we have is 2x + 3y = 5. I took this and made x + 2 y = 3, okay, so I solved it. So, if we solve it, the solution that we have is x = 1 and y = 1. This is the solution.

Now what do I do? I can change this system, okay. So, it is possible that I change it. So, this becomes 2 x + 3y = 5, and I write here x + 2y. Now taking one one, I will do this. Let's do this — let's make it zero or make it 3.1. Let's see what happens. 3.1 in this case, okay. What will happen in this case is that the solution will come. No — I will change it like this — it will come out to be 5, okay. So, this system is ready.

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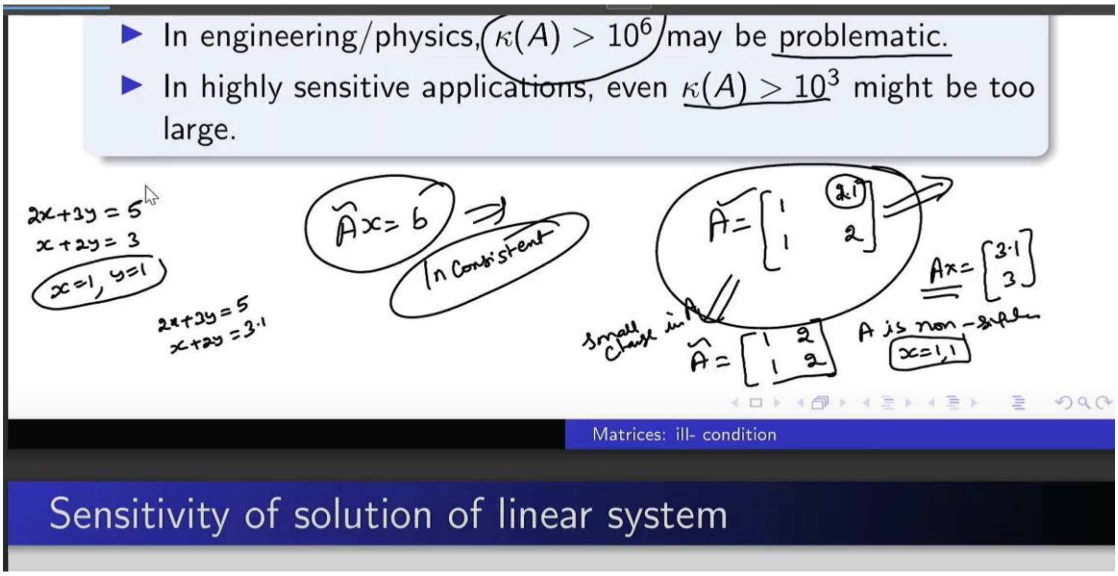

Now I will change it a little bit, and I will make another system. I will make this — I will make 2x + 3y = 5. And I did it — then I made 4x + 6y= 10 system, okay. So, we have a system now, this is our system 2 by 2 and I solved it. So, you will see that its value is x = 1and y = 1. We will solve it. We will do this session now.

What did I do? Let's change this system a little bit. And 2x + 3y = 5, and I wrote 4x + 6y= 9.8 instead of 10, changed it a little. So, now if you look at the system, this system has become inconsistent. This system has become inconsistent. So, this type of systems — we will say that they are very sensitive to the values of A and B. So, this is called the sensitivity of the solution for the linear system.

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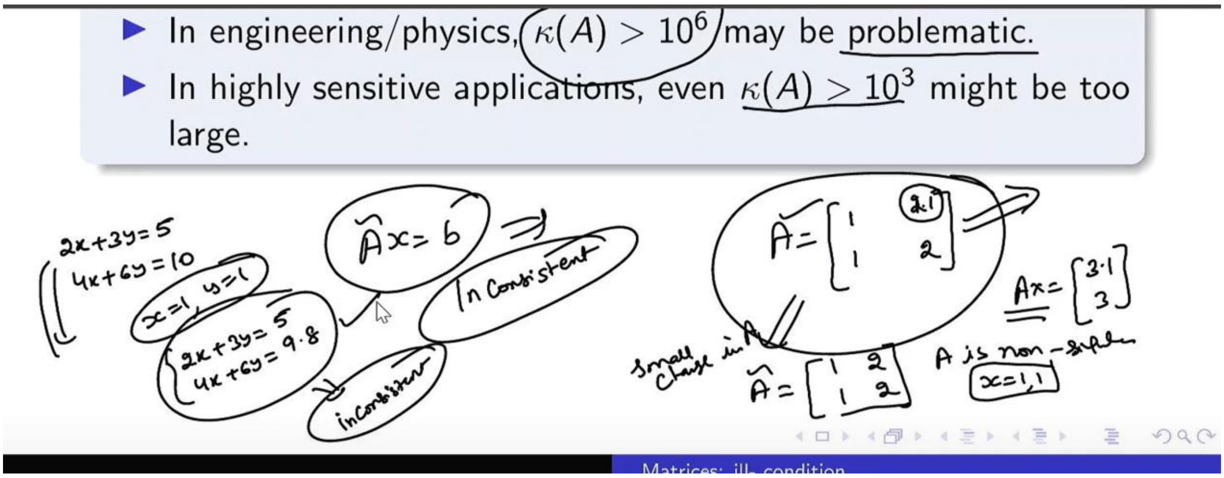

So, if we calculate from here, we will get the inconsistent. So now what we have to do is that we have two things — that we have a system Ax = b. We have to check whether the system we have is sensitive or not — sensitive with respect to the change in A and sensitive with respect to the change in b. So, these will be our inputs, and x is our output.

So now we will check according to this, and we will use its expression. So, now what can we do with this? I can define an example here because, to calculate this, we will have to find the inverse of the matrix. We will have to find out their norm, it can be anything.

So, we have to keep this thing in mind — that everything depends on the condition number of a matrix. We are writing it with kappa. We can also write it like this: condition number and its definition. Now this norm of the matrix can be anything. It can be one norm. It can be two norm. It can be a spectral norm. It can be any name, because we know that we can take different norms.

Now let's take an example. I took the matrix 2.1, 1.8, 6.2, 5.3. I multiplied it by a system. Then A x will come here, and we wrote it equal to 2.1 and 6.2. So now what do we have to do? We have to find out the condition number of the matrix. So, this is our matrix A. So, I have to find out its condition number. So, I want to find out kappa.

Now, what am I doing? Finding the condition number in this case. I was told to find out its condition number using spectral norm. So, what do we have to do for spectral norm? We have to take its spectrum norm — it is also called two norms — and we have to take two norms of A inverse as well. And what are spectral norms? They are related to eigenvalues.

So, if we see, we had defined the definition above. Then this kappa will become maximum eigenvalue. So, if the eigenvalue is one number, then we can take the modulus value, and minimum is fine. So, it means that the values we have will come here.

So, now what will we do? Their eigenvalues and λ maximum and λ minimum — we will calculate it. Because λ maximum is its spectral radius and its maximum eigenvalue will be A inverse. So, that will be the minimum eigenvalue of λ , because we know that the eigenvalues of A inverse is equal to one by of A. So, their inverse — and from here we will get this quantity.

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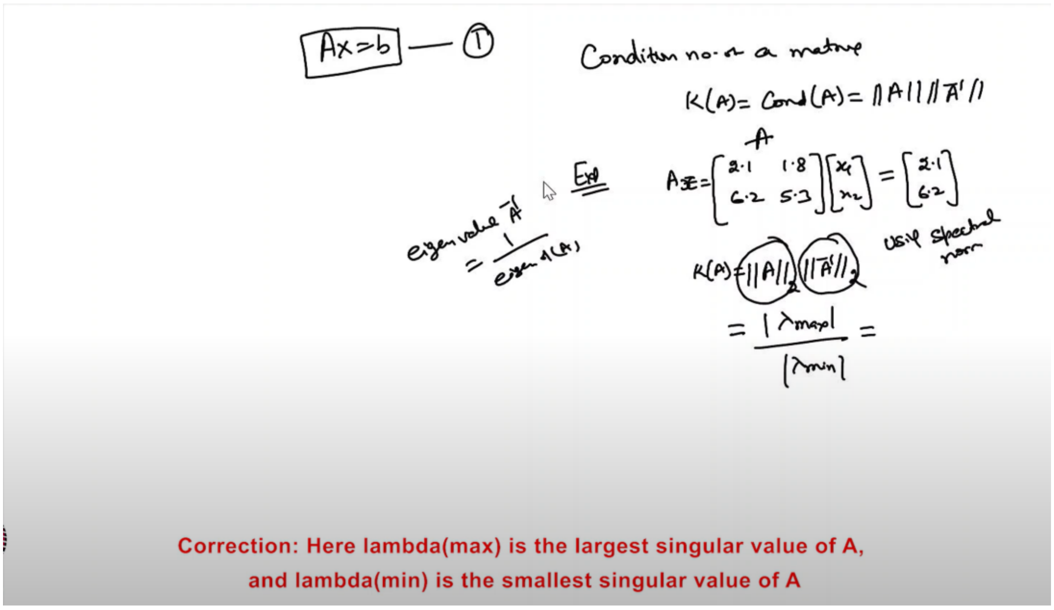

So, if we calculate it, then the values that we get will be — so the eigenvalues that we are getting are 74.179 of the matrix A and write it as λ_1 and write λ_2 as 0.0000123. These are the eigenvalues. So, if we divide 74.17 by this, then we get this value 2472.73. This is the condition number.

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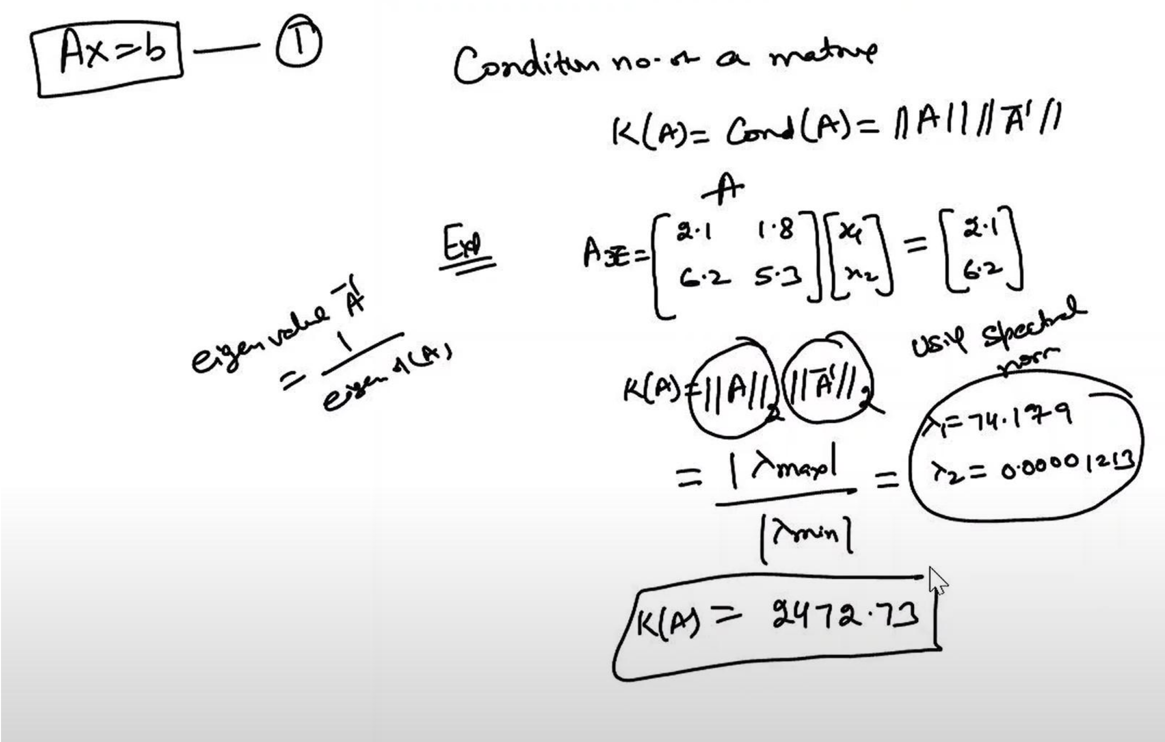

So, if we knew that if the condition number is between 100 and 1000, then it is moderate. If it is above that, then we will say that it is a high condition number. So, in this case, we can say that the condition number is high. This condition number has become high. So, from here, we have found out the condition number.

In this way, I can take any other example. We can find out in that also. Like, I have taken any other example. Let me take another example. Let's take this: A is equal to 1, 4, 9, 4, 9, 16, 9, 16, 25. I have taken it.

So, if I want to find its condition number, then what I am doing now is I am using the, maximum row sum, which means infinity norm, which was one row sum — which was column sum maximum — and infinity norm, which is maximum row sum.

So, what are we doing in it? I am writing it. So, what will we do? We will take maximum row sum. So, what did we do? We calculated it. Now see how much the maximum row sum will be — 25, 16, 25 — 50 will come. So, its maximum row sum is 50.

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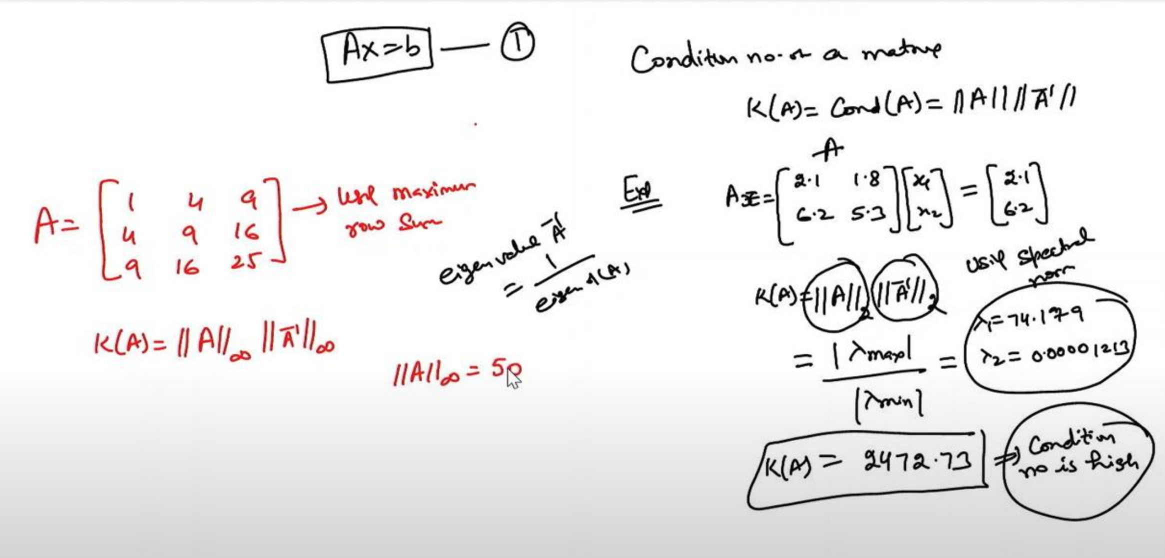

Now we have found the A inverse. So, we will have to find the A inverse. So, we will have to process it. So, if we find the A inverse in this case, then we will have to give it some time. So, it will come to us — 17 by 8. Okay, it is a symmetric matrix, so its inverse will also be symmetric — 56/8 and 7/8 minus 44 by 8. Okay, it has come here, -20/8 — and it will come here also — minus 20/8. And the A inverse which we took as maximum — we got 15.

So, from here, 20 will come from the first and second one — 56 plus 44 plus 20 — so 120 — so 15, 15 came. So, now in this case, if we see, the condition number of the matrix will come as 50 into 15, 750. So, that is the kappa of this matrix.

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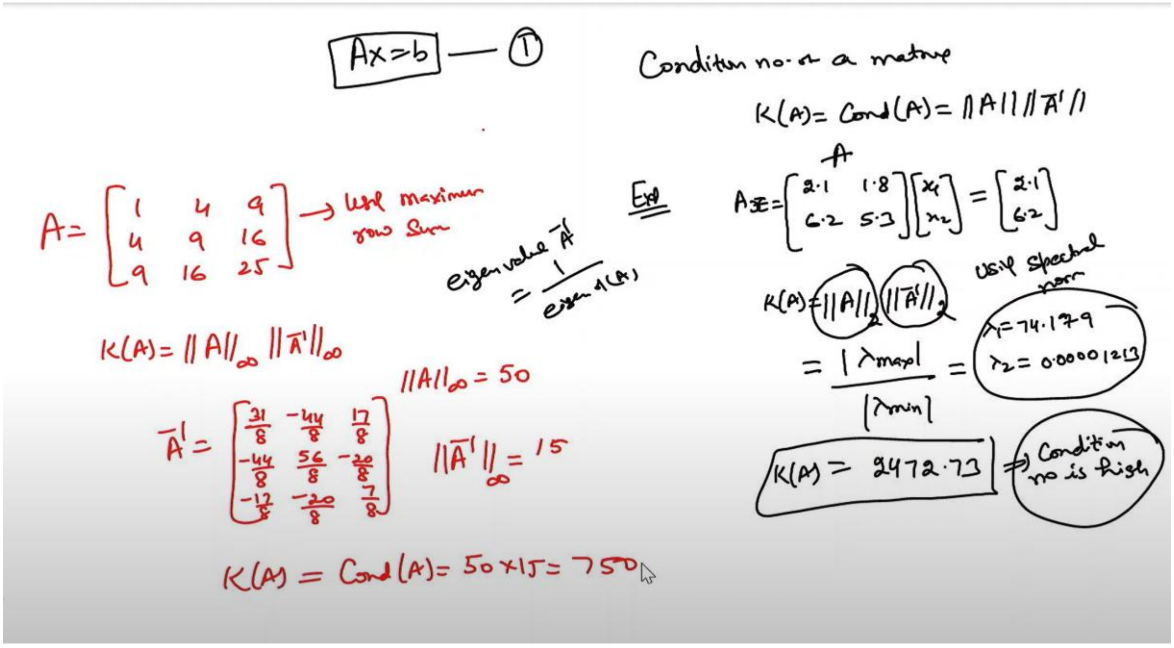

So, in this case, we are moderate. It means it is between 100 to 1000. So, we will call it that our condition number — this is the condition number of moderate value. So, we are neither saying too much nor too less. So, this can mean that a little instability can come in the calculation, but otherwise it is correct. It means it is not that big.

So, today we discussed the diagonal dominant matrix, positive definite matrix discussed, and ill-condition matrix discussed. These things are quite useful, and these error estimates will be very useful in seeing the convergence, and we will use it in the future as well.

So, I hope you liked today's lecture, and thank you for watching this lecture.