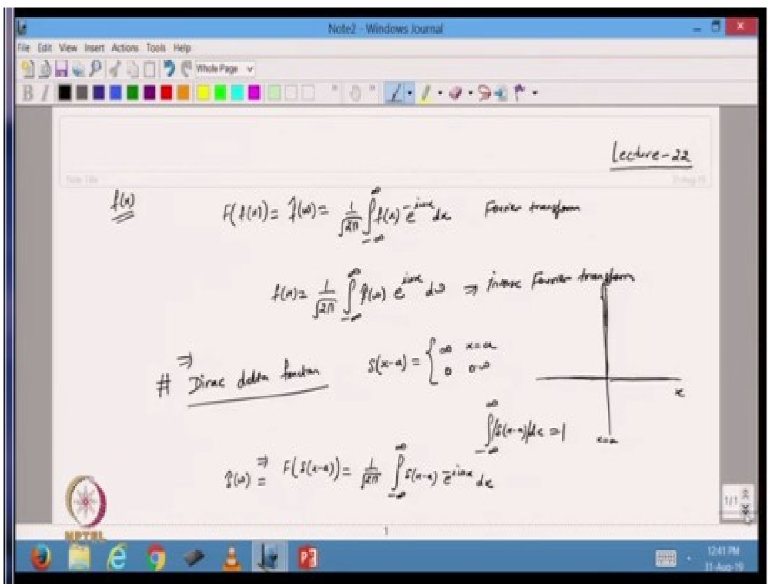
Prof. Vivek Aggarwal and Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology-Delhi/DTU

Lecture-22
Fourier integral and Fourier transform (contd...)

Hello viewers welcome back to this course again, now in the last lecture we have this started with the Fourier integral and then we have defined the Fourier transform. So now we will go further with the Fourier transform.

(Refer Slide Time: 00:36)



So in the last lecture I know that is the function f(x) is satisfying the condition that it is piecewise continuous and the absolute integrable then I can write my Fourier transform. So if I function is there and I want to write the Fourier transform with this function f(x), so that we have written as omega is 1 over under 2 pi - infinity to infinity then defining this function f(x) e raise power - i omega x dx, so this in my Fourier transform.

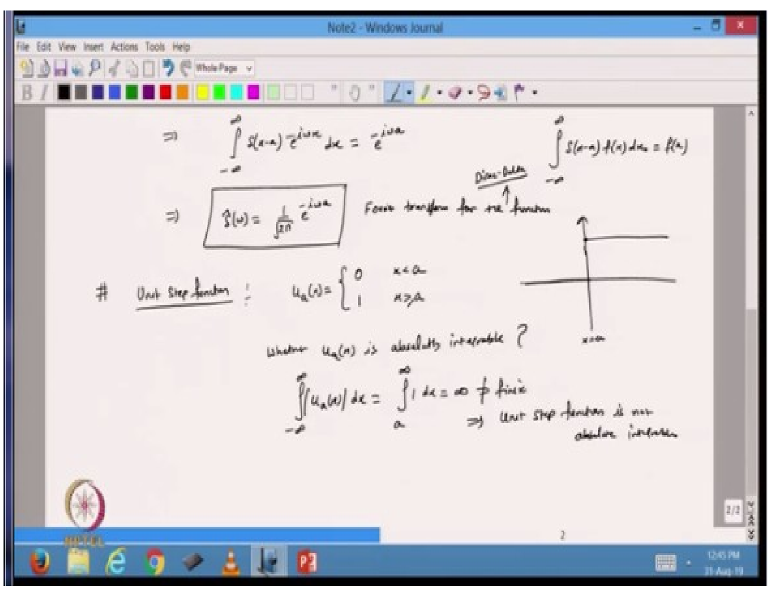
And then from the Fourier transform or from the omega space, if I want to come back to the my x space then I have defined that this function can be written back as, from here I will get my f omega e raise to power i omega x and then d omega. So this expression is called the inverse Fourier transform, now I want to define some so let us take the Fourier transform of some functions. So the function we have started the very important function is the Dirac delta function.

So what is the Dirac delta function, I know that the function delta x – a, so this is equal to infinity when x = a and 0 otherwise, so this is a function we know that we may x and I am taking my this function as x = a. So here the function just has a infinite amplitude otherwise the value the function is 0, so let us see that whether because if I want to take the Fourier transform for this function it should satisfy the condition.

So let us check that whether it is satisfying the condition or not, the first one is that the function is piecewise continuous in the interval yeah. So this function is almost 0 except 1 point, second one is that whether the function is absolutely integral then we know that from the theory of Dirac delta function that if I take the integration from - infinity to infinity for the Dirac delta function then it is value is 1. So in that case I can say that this function is having the absolute value, so I can put like this one here and that value is a finite value.

So from here I can say that this also satisfying the second condition that it is absolute integral, so from here after if satisfying this one then I can define my Fourier transform. So this is my Fourier transform going to write 1 over under 2 pi from - infinity to infinity e raise to power - i omega x dx, so this is I will write it omega.

(Refer Slide Time: 04:32)



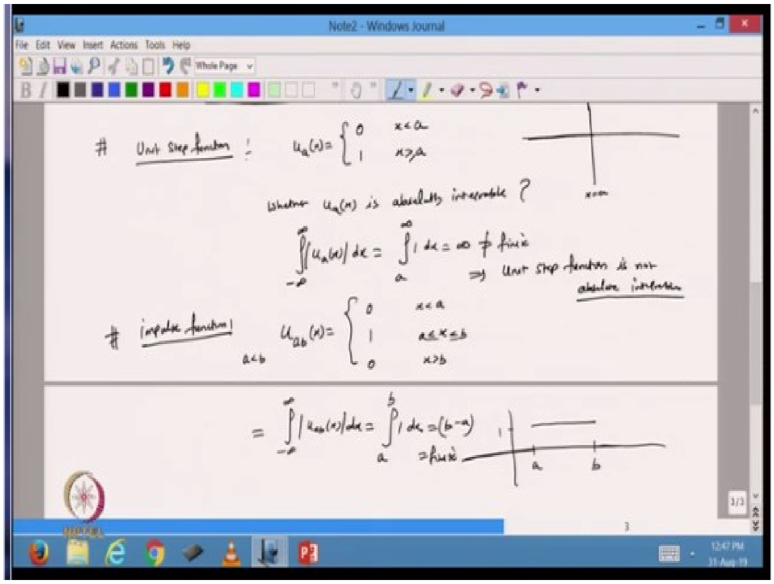
Now from here I know that how to solve these type of integral the Dirac delta function into e to the power - omega x dx and if you remember that if I have some function from minus infinity to infinity I have a dx. Then if I want to take the integration and I know that this function is well defined only for x equal to a then if you remember then it is value is coming equal to f a. So if I use this expression then from here I can write that the value of this expression would be e raise to power - i omega.

So if this integral have this value, so from here I can write that the Fourier transform for the Dirac delta function will be 1 over under root 2 pi into e raise to power - i omega a, so this is the Fourier transform for the function for the Dirac delta function. So this is the Fourier transform, another one I want to take let us check that I define the unit step function, so in the unit step function I know that I defined my unit step function about the point a x = 1 when x is 0, when is 0 when x is less than it and 1 when x is greater than or equal to 1.

And if you remember then this was my expression that when suppose this is my x = a and if I go x a less than a it is value is 0 and if I go for x is greater than equal to a then it is value is becoming 1, so that is what is called the unit step function. Now we check that if I want to take the Fourier transform then whether it is satisfying the condition or not the sufficient condition. So the first one is that it is continuous in the finite interval yeah, so that is there.

If I take any finite interval from - l to l, this function is a piecewise continuous in that interval. The second thing is that I want to check that whether u a x is absolutely integrable that is the we want to check. Now if you see from here I want to define from - infinity to infinity and this is the function i u a x mod value and dx. So this will be equal to from a to infinity and value will be 1 dx and that is itself infinity, so it is not finite. So from here I can say that unit step function is not absolute integrable, so in this case it is not the absolute integrable.

(Refer Slide Time: 08:51)

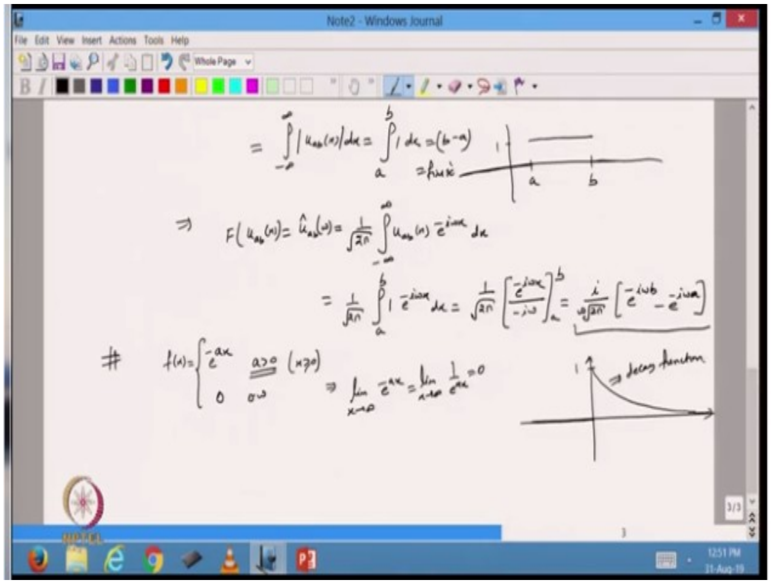


But suppose this in not there then suppose I take the impulse function another function I want to define, so now it is not absolute integrable, so I can now take the Fourier transform for the unit step function. Now I want to define the impulse function, so if you remember impulse function like this one u a b x and this quantity is so here I am considering that the a is less than b. So this quantity we say then this is 0 when x is less than a and when x is lying between a and b it is value is 1 and when x is greater than b it is value 0.

So in that case if I want to plot this function, so this is suppose my a and this is suppose my b then from here that the function will be 0 before a, it is value will be 1 whenever this is lying between a and b and after that also it is value 0. So let us check that whether it is satisfying the condition or not. So in this case if you see that this is piecewise continuous, so first condition is well satisfied in a finite interval and about the second absolute integrable.

So from here if I see clearly that most of the time the function is 0 except in the finite interval, so if I want to take the integration over this function of the finite interval then definitely it is value is going to be 1 or going to be finite value. So if I from here if I just check the absolute convergence, so from here I can say that this will become a b 1 dx and that will be b - a and that is finite. So from here I can say that the impulse function satisfying all the conditions.

(Refer Slide Time: 11:06)



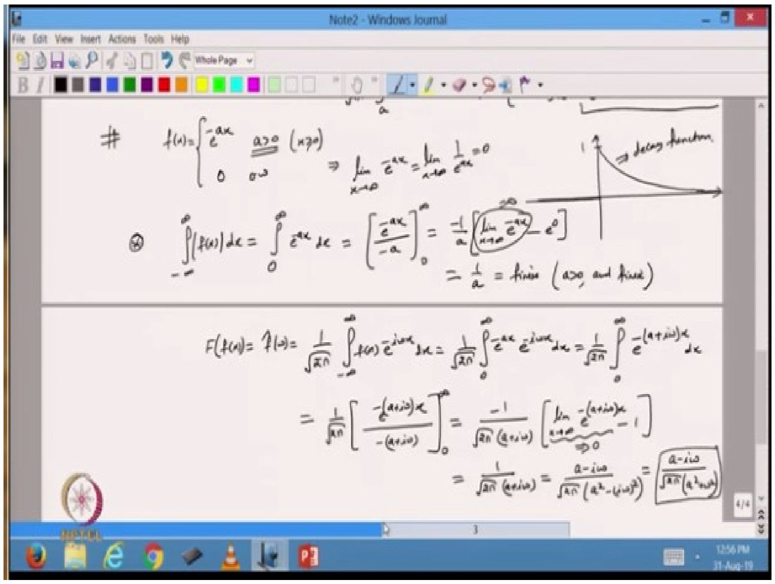
And then from there I can define the Fourier transform of the function u a b, so this is my u a b hat omega, so this is equal to 1 over 2 pi under root - infinity to infinity u a b x u just power - i omega x dx. And if I take this quantity, so from here it will be only from a to b because otherwise it is value 0, so it will be 1 - i omega x dx, so this will be equal to under 2 pi, so this is - i omega x over - i omega from a to b.

So from here I can take that this will becomes i over omega under root pi e raise to power - i omega b - e raise power - i omega a. So that will be my Fourier transform for the impulse function and that is defined from a to b, so yes from here I can say that I am able to find the Fourier transform for the impulse function. Now what about this function I take a function f(x) as e raise to power – a x where a is positive and I also define that x is also positive, so this is my function.

So from here I can define that this is equal to this 0 otherwise, so when x is positive or equal to 0, you can tell when x is greater than equal to 0 this function is value a is always positive and 0 otherwise. So from here you can see that if you just check this function when x is less than equal to 0, it is value is 0 and when x = 0 it is value will be 1. So as suppose this is my 1 and you from there if you see that if x axis increasing the value will be decreasing. So if you see this one this function we will look like this one and keep going like this one.

And from here also I can say that the limit when x tends to infinity e – a x, so this should be limit 1 over e x and a is positive, so this will be equal to 0, so it is going toward the 0. So the rest check that whether it is satisfying the condition or not, so in this case what about this, so this is basically a decay function we call it the decay function. So I want to check that whether I am able to find the Fourier transform for this function or not ok.

(Refer Slide Time: 14:49)



So let us try to find that now the first condition whether it is piecewise continuous in the finite interval yes this is a function is well defined function differentiable function. So in any finite interval this is a piecewise continuous function, second one is the main property that I want to check whether it is absolutely integrable or not. So let us check this one, so I want to find from - infinity to infinity dx.

So this quantity I am taking, so my function is e raise to power – a x from 0, so I should write this as only defined from 0 otherwise it is value is 0. So modulus value will be same because this function is positive function, so they no need to write modulus value and this will be include dx. So this quantity will be e – a x divide by – a, 0 to infinity, so from here I can write that 1 over a – so I can put from here the limit x tends to infinity e – a x - e raise to power 0.

So from here now this quantity just now we have checked that this is equal to 0 the function is decaying and as x tends to infinity this value is coming 0. So from here and this is 1, so from here I can write that this will be equal to 1 over a and that is finite because a is positive and finite. So I am chosen that a is positive and finite, so from here I can say that this is satisfied so this condition is satisfied.

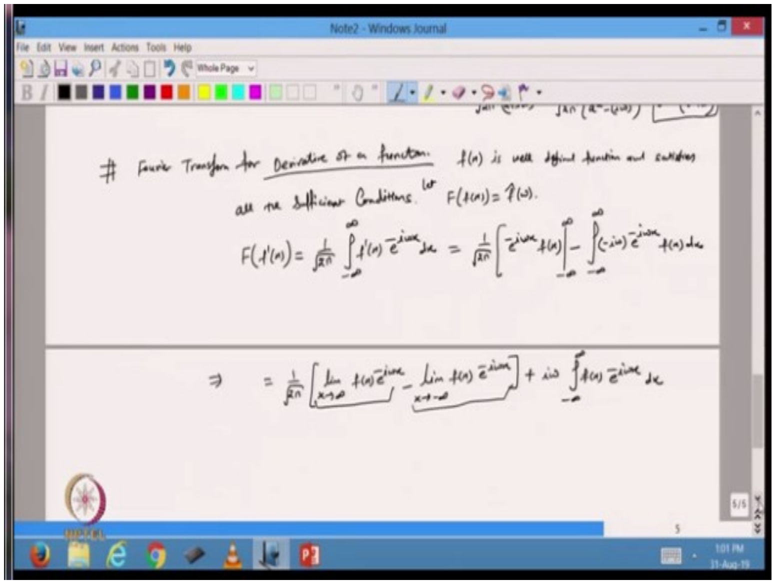
And from here I can say that I am should be able to take the Fourier transform, so let us take the Fourier transform of this function. So this will be equal to omega 1 over under 2 pi and then my function from - infinity to infinity f(x) e raise to power - omega x dx. So now I plug in the value of the function here, so it is value will be from 0 to infinity only. So it is e - a x e raise to power - i omega x dx which can be further written as from 0 to infinity - a + a + i omega x dx which can further be written as.

So the integral for this one will be a + i omega x divided by - a + i omega from 0 to infinity. So this quantity will become - 1 over under root 2 pi a + i omega I can write here and then we put the limit x tends to infinity e - a + i omega -, so this should be 1. Now this is already know that this quantity will tends to 0 because both are the decay function. So from here this is the frequency positive frequencies, so this is always a decay function.

And from here I can write that this quantity will tends to 0, so from here I can write that this expression will be equal to 1 to pi a + i omega because - 1 and - will cancel out and this further we can solve it by multiplying this by a - i omega. So from here I can say that under pi from here I can say a square - i omega square which can further be written as a -i omega under root pi and this is will be a square + omega square.

So that is the basically the Fourier transform for the function that is the decay function. So basically you can say that as we have defined the Laplace transformation for the exponential form. Here we are taking the Fourier transform for the decay function, so for the decay function this is valid.

(Refer Slide Time: 20:11)



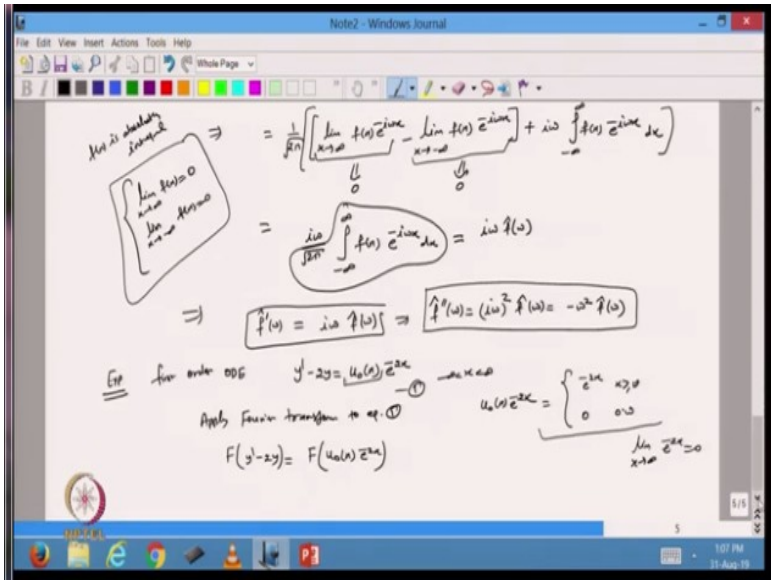
Now I want to take the Fourier transform for derivative of your function as we have defined the Laplace the same thing we are going to do with for the Fourier transform. So let my f(t) or f(x) is the function is well defined function, well defined means and also satisfying and satisfies all the sufficient conditions. So let the Fourier transform of this is equal to this is well defined, now I want to take the Fourier transform of f dash (x) but this one I want to find out.

So let us do, so this will be equal to from - infinity to infinity f dash (x) e raise to power - i omega x dx, so by the definition it should be this one. Now I take the integration of this one by parts, so from here I can write I can take the first function this one and the second function this one. So from here I can write e raise to power - i omega x and then integration of f dash (x) is f (x), so this is my limits - infinity to infinity.

So this is the function I want to take the derivative this function, so if you take the derivative it will be - i omega and e raise power - i omega x and then integration of the function f(x) and this is the dx. So this is I have just written, from here now I will try to find out the limits, so this becomes 1 over under 2 pi, so I apply the limits. So limit will be limit x tends to infinity f(x) e raise to power - i omega x - limit x tends to infinity f(x) is – infinity.

So this will be x tends to - infinity f(x) and from here I can add + i omega and the expression dx, now so let us see what will happen to this one and what will happen to this one also. Now from the analysis we know that if the function is absolutely integrable.

(Refer Slide Time: 24:10)



Then we know that, this will be always true when x tends to infinity f(x) will be 0 and x tends to - infinity the function will be 0. So these 2 of the consequence or the results of the function if my f(x) is absolutely integrable because you it is not possible that if the function is not going to 0 for very large value of x. In the case of x tends to infinity and very small value for the x tends to - infinity because if that is there then you cannot have that there is the value of this integration will be finite.

So this is the results from the absolute integrable, so I will apply here and if I apply here then this will be equal to 0 and this quantity also going to be 0. So from here I can say that this expression I can take from here and this will becomes i omega over - infinity to infinity f(x) e raise to power - i omega x dx and if you see then this is the Fourier transform of the function f(x). So from here I can write that this is equal to i omega Fourier transform of the function.

So from here I can say that f dash omega = i omega f hat omega, similarly I can from here I can define now I have just taken the first derivative of this one. So that was the - i omega and then with this expression becomes i omega, now I can from here I can say that if I want to take the

Fourier transform for the second derivative then I have to take 2 time. So after getting this one the same thing will apply here, so it will be i omega square and then it will be this.

So this will be - omega square f hat omega because i square is – 1, so from here I will get the expression for this one ok. So from here now if I know the Fourier transform of the function then I can define the Fourier transform for it is derivative first derivative, second derivative and so on. So if I am able to take the Fourier transform for the derivative then I should be able to solve the differential equation using the Fourier transform.

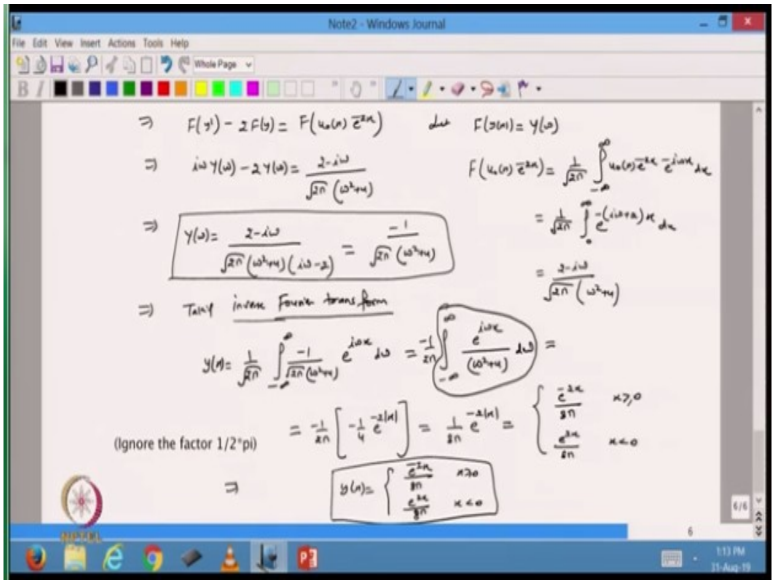
So let us solve the try to solve the a differential equation using the Fourier transform, so let us do take one example I have a first order differential equation first order ODE. So I take the simple one y dash - 2y = u0 x e raise to power - 2x and x is from - infinity to infinity but just now we have seen that we are unable to take the Fourier transform for the unit step function. But what about this function u0 x e raise to power - 2 x and e raise to power - 2 x is a decay function.

And we were able to take the Fourier transform, so if you see this one then this be equal to e raise to power - 2 x when x is greater than equal to 0 and 0 otherwise. Because u0 x will be 1 when x will be greater than equal to 0, so then in that case I will get this value otherwise this is the value 0. So this function I know that it is satisfy all the conditions for this one and it is a decay function and I know that the limit x tends to infinity e raise to power - 2 x a 0.

So this function satisfy the absolute integrable property and from that property I can verify this condition also that whether it is satisfying or not. Because when x tends to infinity it is value is coming 0 and when x is - infinity it is value is already 0, so it is satisfying. Now I write it, so this in my differential equation I should call it the differential equation 1, Now let my y also satisfying all the property.

So I am should be able to take, so take or apply take apply Fourier transform to equation 1 both side. So I will get the Fourier transform of y dash – 2y is equal to again the Fourier transform of u0 x e raise to power - 2 x.

(Refer Slide Time: 29:50)



So from here, now I know that the Fourier transform means a linear and it satisfy linear property. So from here I can write my - 2 times f of y is equal to the Fourier transform of u0 x e raise to power - 2 x now let the Fourier transform of y dash of Fourier transform of y I should call it x = y omega capital Y omega so let us do it this one. So from here by the just result I know that this value can be written as i omega Y omega – 2 times Y omega and then the Fourier transform of e raise to power - 2 x.

So let us do this one, the Fourier transform in just taken for u0 x e raise to power - 2 x will be - infinity to infinity u0 x e raise to power – 2 x this one. And this quantity will be again from 0 to infinity e raise to power - 2 x, so it will be - i omega + 2 x dx. So just now we have seen, so this expression using the previous result I can write because this was e raise to power – a x, so instead of a I just put, so it would be 2 - i omega.

So this will be 2 - i omega over + omega square + 4 because a is just 2 so a square, so from here I can write then this quantity will be equal to 2 - i omega omega square + 4, so this is my expression. So after doing this one, from here I will take the this common, so this will become 2 - i omega omega square + 4 and this quantity will be again i omega – 2, so this will cancel out, from here I can write that this is equal to under root pi omega square + 4.

So that is the quantity I got, now from here now using this one I got this value, now this is the solution in terms of omega the frequency, but I started with my differential equation that is a function of x. So in this case now I take that now taking inverse Fourier transform, so what is the inverse Fourier transform. So I now after applying this one I will get the solution in the terms of x, so this is 1 over under root 2 pi from - infinity to infinity.

So this is the my value I am going to get - 1 over under root pi omega square + 4 into e raise to power i omega x d omega. So if I solve this further, so this will be 1 over 2 pi - i I can take outside from - infinity to infinity it will becomes i omega x over omega square + 4 d omega. So if you want to solve this quantity and so if we take the integration of this expression then from there I can write this expression as that this will be equal to - 1 over 2 pi and then this quantity will be same as - 1 by 4 e raise to power - 2 mod x so this integration will get this value.

So from here I can say that the value of this will be 8 pi and 2 e raise to power – 2 mod x which can be further written as e raise to power - 2 x by 8 pi when x is positive and it is value will be e raise to power -, so - - will be +, so it will be 2 x by 8 pi when x is less than 0. And when x = 0 it is value will be just 1 over 8 pi, so that is the solution for the equation, so from here I get the solution y x, 8 pi when x is less than 0.

So in this way we can find the solution for various first order or second order maybe high order differential equation. And then once we get the solution Fourier transform for the differential equation then we can take the inverse Fourier transform and then we can get the solution in terms of the variable in which the differential equation is defined. So that is about the Fourier transform.

So today we have done taken the Fourier transform for very famous function that is Dirac delta then we are also seen that this is applicable or the Fourier transform is applicable for the decay function. And then we have solve the first order differential equation, so maybe in the next lecture we will try to do some more examples related to this one. So I hope you enjoyed this one, so thank you very much for this.