Basic Real Analysis.
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Lecture 46
Optimization in Several Variables – Part 1.

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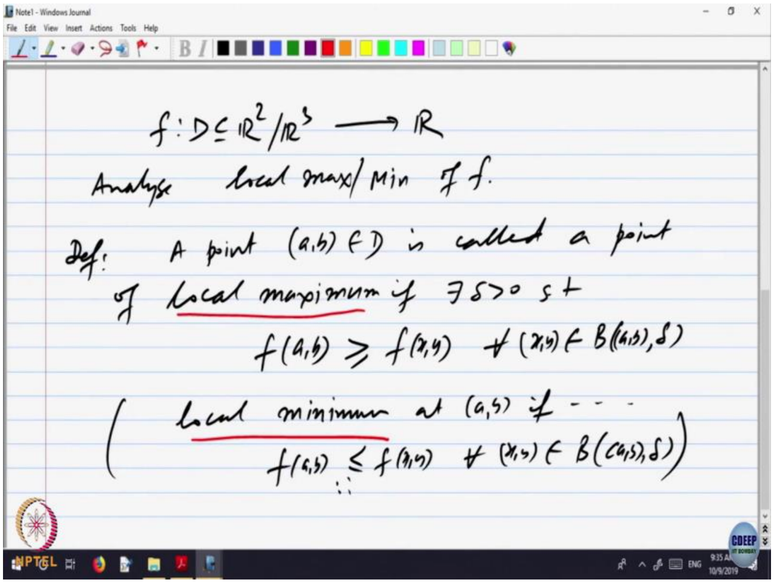

So, what we are going to look at it is functions f defined in a domain say R2 or R3 taking values in R, we want to look at we want to analyze local maximum minimum of f. So, to analyze that let us first define what is local maxima minima for functions of two variables. So definition, so A point say a b belonging to d is called a point of local maximum, so it is a sort of maximum, say it is largest and local means in a neighborhood.

So, if there is some delta bigger than 0 such that the value of f at a b is bigger than or equal to value of f at x y for every point x and y in a neighborhood, so let us write a ball around the point a b of radius delta. So, that is local maximum, and you can define similarly minimum if it is inequality is less than or equal to. So, local minimum at a b, if all at as above f of a b should be less than or equal to f of x y for every x y, in that neighborhood.

So, local maximum the name itself says, so it is a the value at that point is largest in the neighborhood and minimum says it is the smallest in that neighborhood. So, the problem is, how do we identify points where the function as local maximum or local minimum and how do we verify that indeed these are points of local maximum or local minimum.

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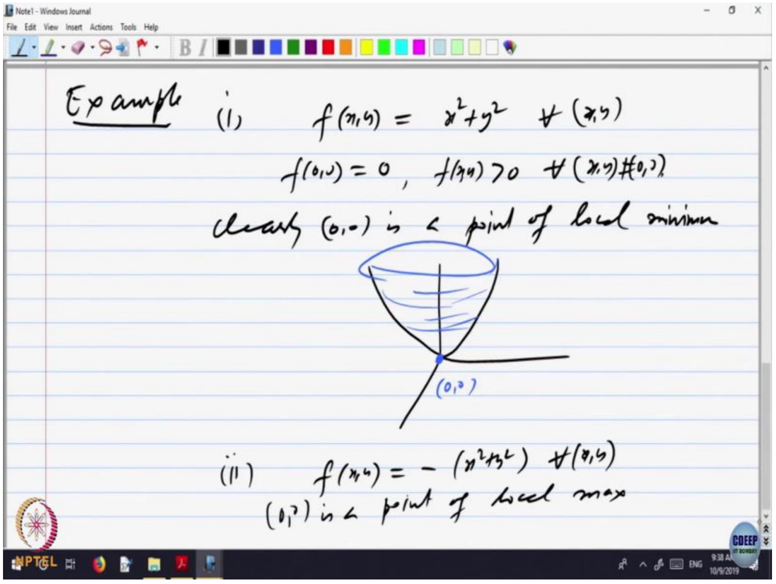

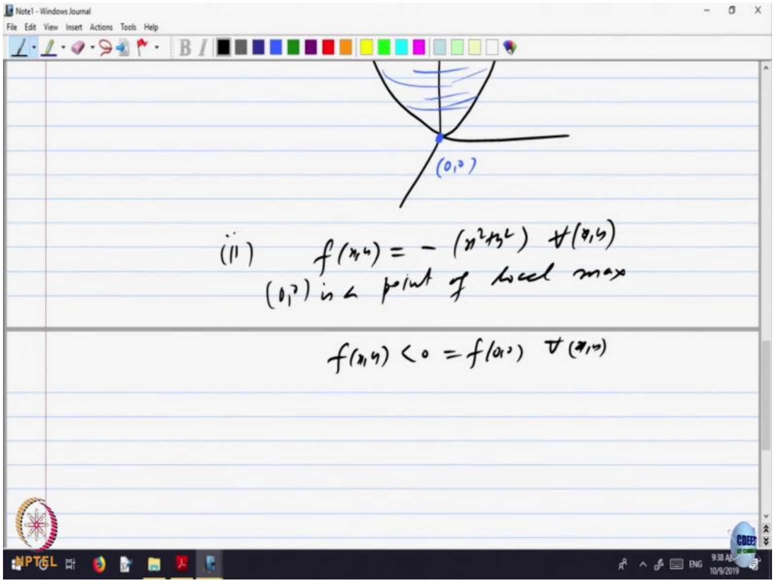

So, let us for example let us look at some examples. The simplest would be for example let us look at f of x y is equal to x square plus y square for every x y. So, geometrically if you look at this, so this is the square of the distance so f at say 0, 0 the value is 0 and for every other point f of x y is bigger than or equal to 0 for every x y other than of course equal to the strictly bigger than 0, is bigger than 0 for x y not equal to 0 0.

So, clearly 0 0 is a point of local bigger than local minimum. Can you visualize this geometrically? So what does it look like geometrically? Geometrically if we try to visualize the graph of the function is a surface and every section of the surface x square plus y square is equal to some number if a z is fixed then this is a circle. So, it is going to be, kind of so that is the surface looks like.

So, that is and obviously it is a minimum at the point 0, 0 geometrically also. So, here we are just looking analytically and that this is the minimum. For example, if you change f x y equal to minus of x square plus y square for every x y, then it is just inverting this surface. So, 0, 0 is a point of local maximum at every point the value is less than or equal to 0 and 0 the value is 0. So, that is a point of local and minimum.

So, f x y is less than 0 which is equal to f of 0, 0 for every x y, but that is not good enough for us these kind of examples we would like to find out like in one variable we said that if you want to find points of local maximum minimum, we got a condition that the derivative at that point must be equal to 0. So, that gave us that necessary condition gave us a collection of points where possibly the function can have local maximum or minimum those were the critical points.

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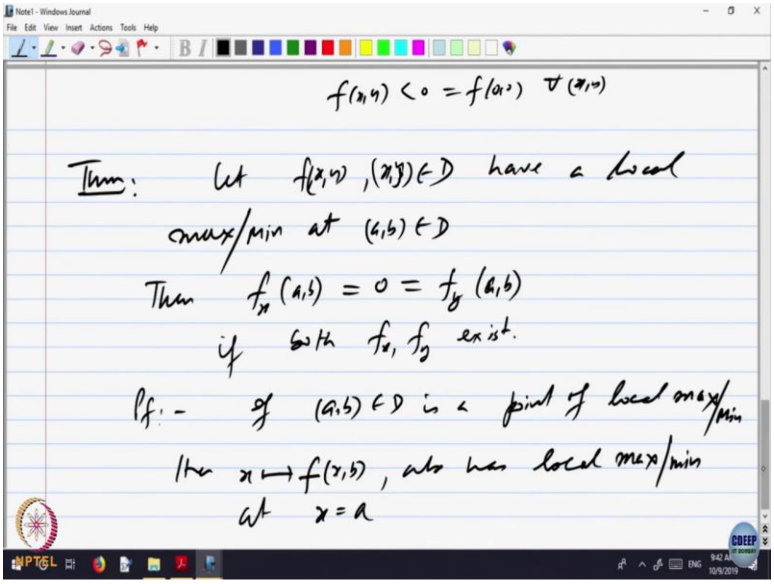

So, something similar can be done here for example let us look at the following. So, let us write it as a theorem, so let f x, y xy belonging to D have a local maximum or minimum at a point a b. So, suppose if a function has we want to find a necessary condition like function of one variable. So, let us put this back on, put this back on one variable. So, then the partial derivative at the point a b with respect to x should be 0 and should also be equal to partial derivative with respect to y at the point a b if both f x fy exist.

The reason for this is obvious that if you have function of two variable and it has a local maximum at a point, then one variable fix also it is a point of local maximum if at a point the value is largest in a neighborhood, and if a fix one of the variables say x or y, then as a function of one variable also it is a point of local maximum or local minimum at that point, and if the derivative exist with respect to x, then the derivative by that theorem of one variable that should be equal to 0 and similarly with respect to y.

So, proof is just saying that if a b is a point of say local maximum or minimum, then let us consider the function say x going to f of x, x going to varying x, so x, b. Also has local maximum or minimum at x is equal to a, is that clear to everybody? That if in the domain a b is a point in the domain where it has a local maximum say then in a neighborhood in that domain it is a point of.

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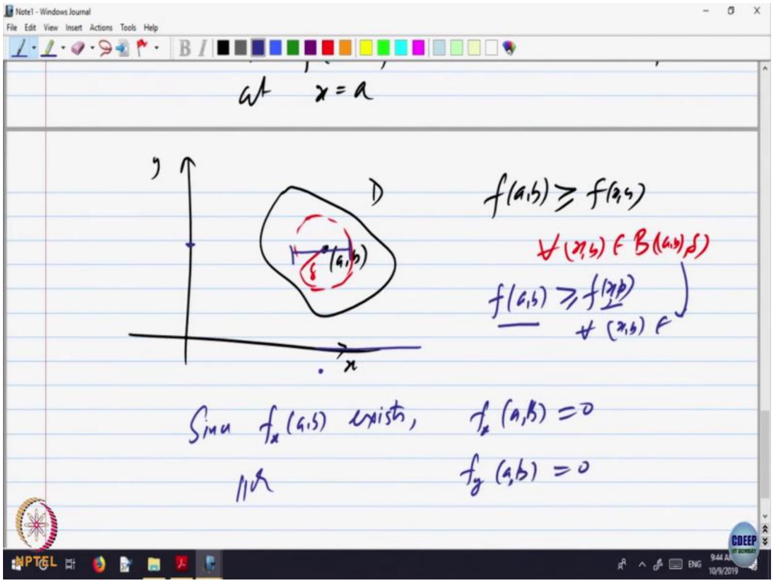

So, let us just draw a picture and understand what we are saying, so this is the domain and this is the domain D and this is the point a b where the function has so f of a b is local maximum, so is bigger than or equal to f of x y for every x y in a neighborhood. So let us draw a neighborhood, so, this is the neighborhood so in a neighborhood of radius delta for every x y belonging to ball at a b of radius delta.

Now if I fix, if I fix the y equal to B and let take x vary. So, y equal to b is fixed and x is varying. So, you are moving along this, so in this part of you are moving along this values. So, in that part we will have f of a b is still bigger than or equal to f of x b for every x b, belonging to that ball. So, as a function of one variable it has a local maximum at that point. So, f of a b as a function of variable x is bigger than or equal to the value at the point x b in that neighborhood.

So, since f x a b exist f of x a b should be equal to 0. So, as a function of one variable it has a local maximum the derivative exist, so that derivative must be equal to 0, and similarly f of y a b should be equal to 0. So, that is just a conclusion we are putting back, so this is a necessary condition, so this gives a necessary condition. So, let us define the points.

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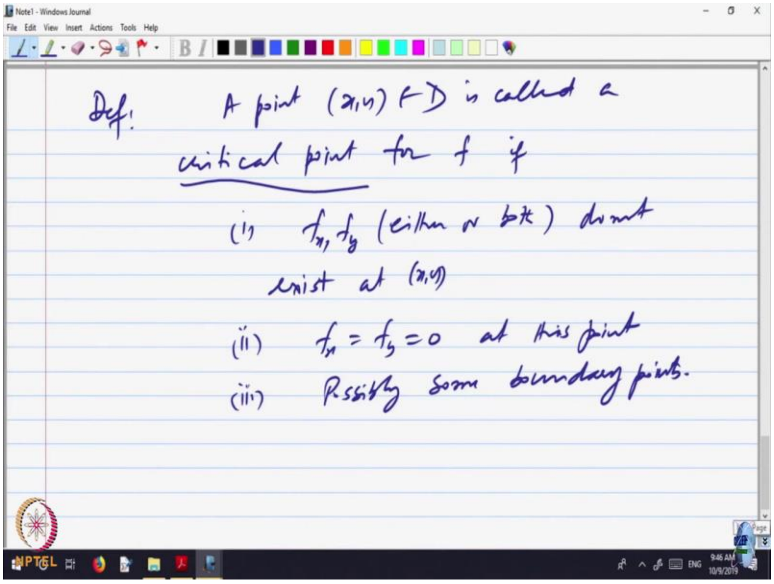

So definition, A point x y belonging to D is called a critical point for the function f if, like in one variable what are the possibilities? Either of the partial derivative or both the partial derivative do not exist. So, one fx fy either of this let us write either or both does not matter even if one does not exists, both do not exist at the point x y. So, those are points of non-differentiability like in one variable.

Two, fx equal to fy equal to 0 at this point derivative exists, and are equal to 0 at that point and possibly some boundary points. So, these are critical points are the possible points where the function can have local maxima or minima as a function of two variables. And then again we will have to find sufficient conditions to check whether these are points of local maxima or local minima. So, let us look at some examples to understand this bit more.

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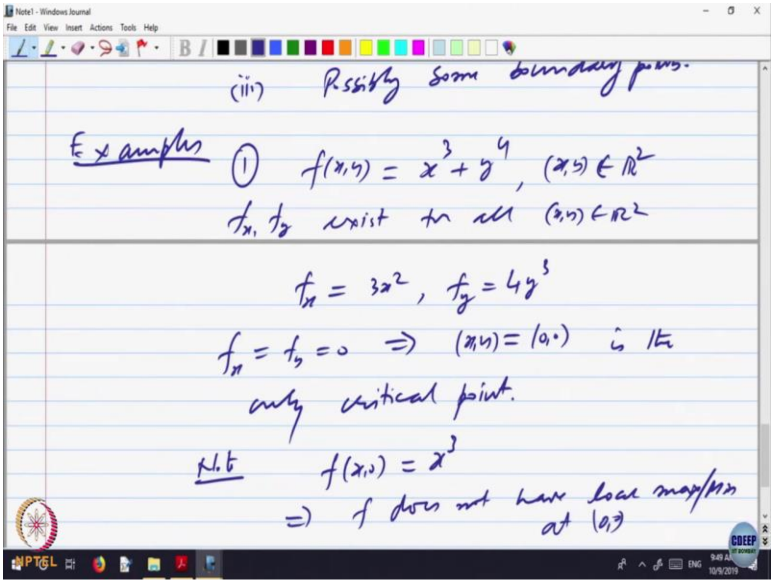

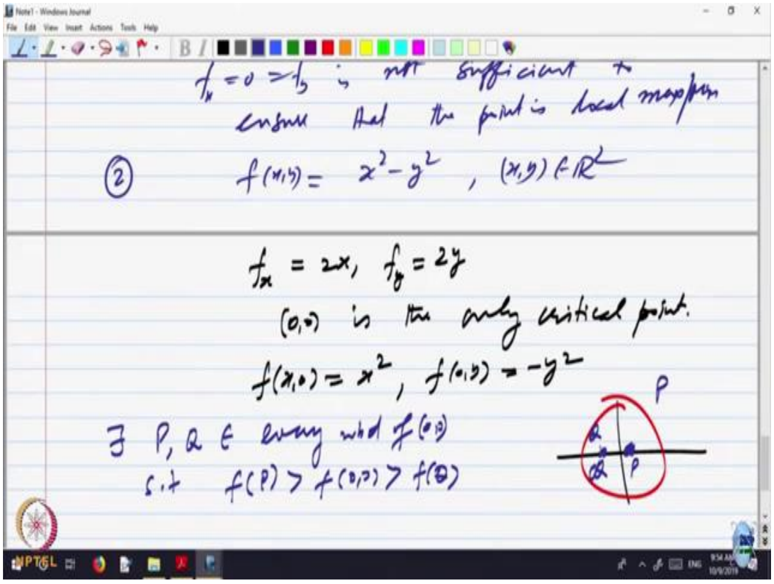

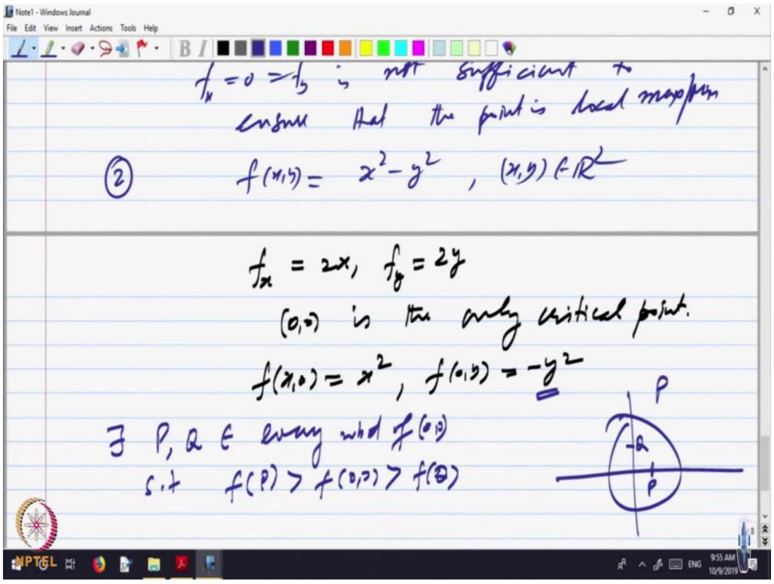

So, we have already seen that example of x square y square, so let us look at some other example. So let us look at example of f x y a simple one, x cube plus say y 4 x y belonging to all of R2 domain is the whole of R2. So, partial derivative exists everywhere, so f x, f y exist for all x y belonging to R2, and what is f x? The partial derivative is equal to 3 x square f y is equal to 4 of y cube. So, what are the points? f x equal to f y equal to y equal to 0 implies the point is x y equal to this will give you x is equal to 0 and that is the only critical point.

So, for this function the only critical point is 0, 0. So, can we say at 0, 0 the function has a local maximum or a local minimum by looking at the function. So, let us note at 0, 0, so if I look at as a, let us put y equal to 0. So, f of x 0 as a function of one variable, it is equal to 3, it is equal to x cube, it is equal to x cube. So, derivative at that point 0, 0 exist and is equal to 0, but we know that f of x is equal to x cube as a function of one variable does not have local maximum or a minimum at the point 0, 0.

On the left side of x negative x negative it values negative on the positive side value is positive at value 0 is 0. So, that implies that f does not have local maximum or minimum at 0 0. Though the partial derivative exist at 0, 0 but the function does not have local maximum or local minimum. So, that like in one variable even if the function has both the partial derivative exist and are equal to 0 that is only a necessary condition.

That the point may have local maximum we have, local minima it may not be anything like in this example. So, let us look at another example, so, fx equal to 0 equal to fy is not sufficient to ensure that the point is local maximum or minimum. Let us look one more example, so let us look at say f of x y, so x square minus y square for every x y belonging to R2. So, what is partial derivative f x? That is 2x f y is equal to 2y.

So, 0, 0 is again is the only critical point as before if look at f of say x 0 that is x square and f of 0 y is equal to minus y square. So, as the function of the variable x y fixed as 0 it is x square which has a minimum at the point 0 as a function of one variable, and f of 0y has a maximum local maximum actually global maximum at the point 0. So, I cannot say for the two variables the functions has a local maximum or local minimum.

But something more interesting is happening that whichever neighborhood of 0 I take there is a point where the value of the function is less than the value of the function at that point and there is another point where the value is bigger than the value at that point. For example if you look at 0, 0 for x very small, so x, 0 will be a point where the value of the function is positive which is bigger than the value at 0, 0 that is 0.

So, and similarly for if I fix x is equal to 0, the value is negative I can make it as close to 0 as I want. So, whichever neighborhood of 0, 0 I take, so in the picture you can think it as in this picture for whichever neighborhood I take there is a point, there is a point say P, and there is a point Q, the point P is actually in this one there is a point here and there is a point Q here, very close whichever you want.

So, here is a point P here is the point Q the value at this point is positive the value at the point Q is negative whichever neighborhood you choose you can always find points close to 0, 0 with these properties.

So, the conclusion is, there exists points P and Q belonging to every neighborhood, such that of 0, 0 such that, the value at P is bigger than the value at 0, 0 is that is bigger than the value at the point Q, which is negative. So is it clear what we are saying?

This function has a special property not only the point 0, 0 is not a point of local maxima or local minima, in fact something more is happening that whichever domain whichever neighborhood I choose of 0, 0 I can find a point how so ever small that neighborhood maybe, I can find a point where the value of the function is positive.

So, that is bigger than the value at 0, 0 and some other point where the value is negative.

Student: (())(22:20)

Professor: No, no not x comma that is point Q is here, it is in the y axis, so y x is 0, so in the picture let me draw a correct picture point. So, here is the neighborhood here is the point P and here is probably the point Q. A point on x axis close to 0, 0 value the value is positive a point close to 0, 0 on the y axis where the value is negative. So, that is less so such points are called saddle points.

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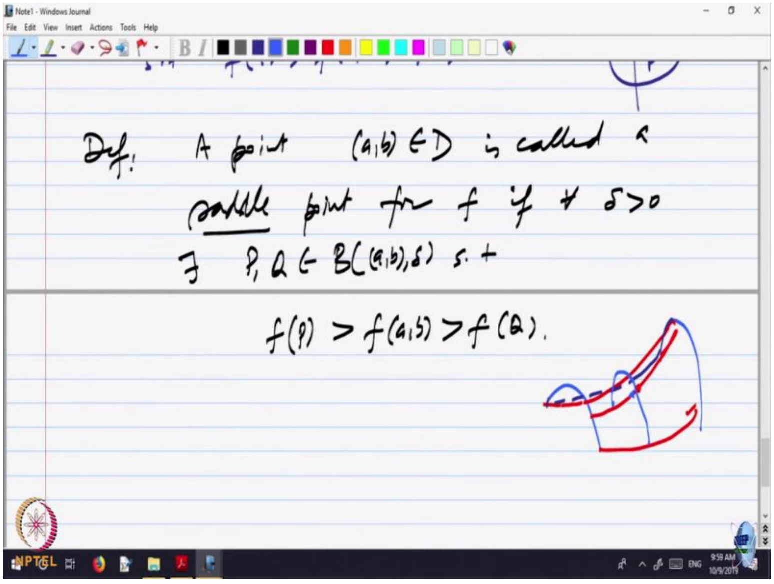

So, let us put it as a definition A point say a b belonging to D is called a saddle point for f if for every delta bigger than 0 there exist P and Q belonging to ball at a b of radius delta, such that the value of the function at the point P is bigger than or equal to, not equal to we should say strictly bigger because otherwise every point will be is bigger than the value at the point a b is bigger than the value at the point Q.

So, such points are called saddle points, why are they called saddle points? If you have, so let me draw another picture this is the domain, so let us look at the graph of such a function what does it look like, what does saddle point mean. So, let me see if I can, I should not take it 0, 0. So, let us skip that so let us let me draw, now let us draw. Look at this kind of surface, this is something that you put on a horse, if you want to ride the horse you put a seat on the horse.

You do not just jump on the horse and ride, you put a seat and that seat is called saddle, you seat into that and that looks like this. So, it is a curve like this and there is a curve the other way around. So, if you look at along the curve if you look at this red ones, so there is a minimum at in between and if you look at these ones blue ones then that point is a point of maximum.

So, along some curve there is maximum along, some other curve there is a minimum passing through the same point that is why it is not a point of local maximum or local minima because as closed to that point there are points where the value is bigger and there points where the value is smaller, but not only that you can for every neighborhood if you can find points like this, then it is called a saddle point.

So, there are three possible, possibilities for a critical point, one it can be point of a local maxima, it can be a point of local minimum, it can be a saddle point. It may not be anything you may not be able to conclude anything about it that critical point.

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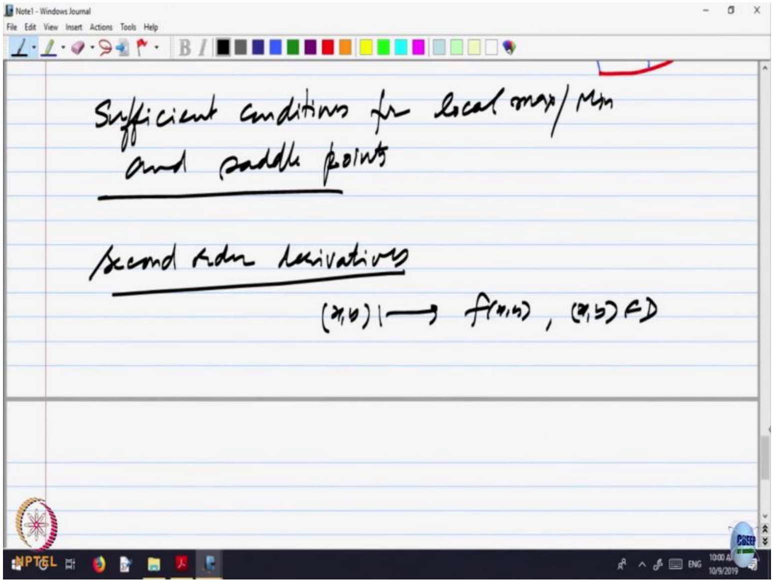

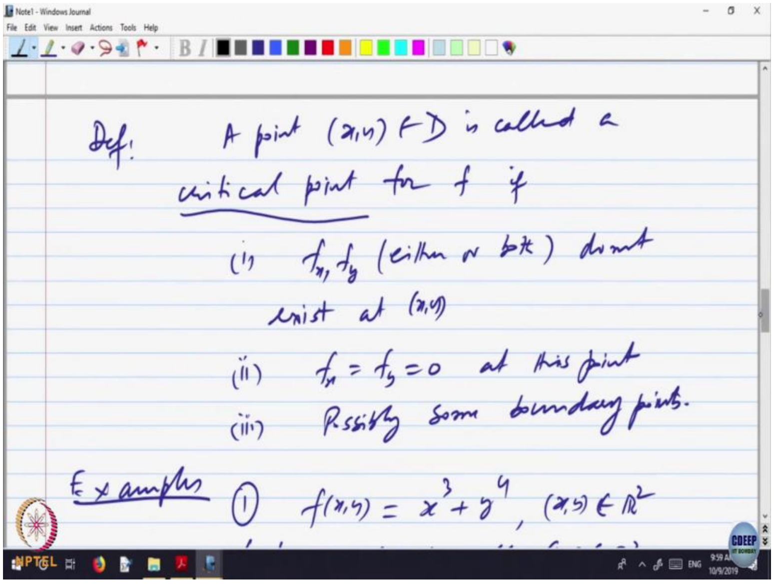

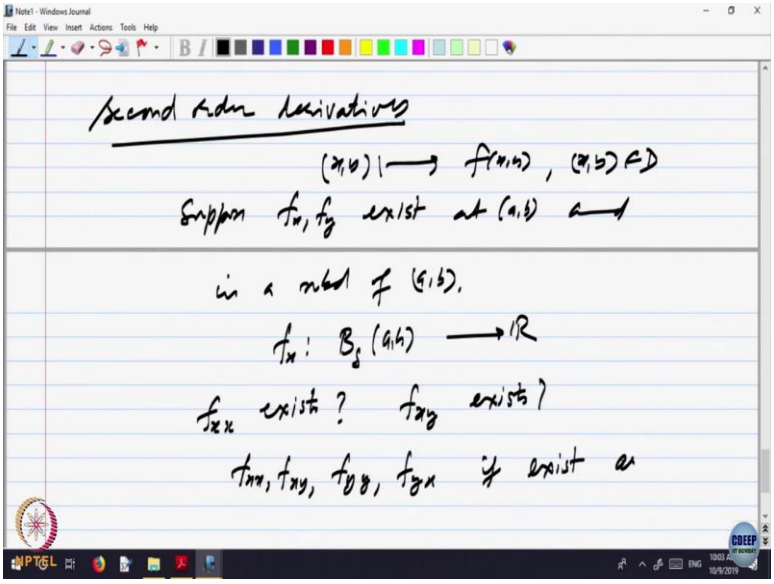

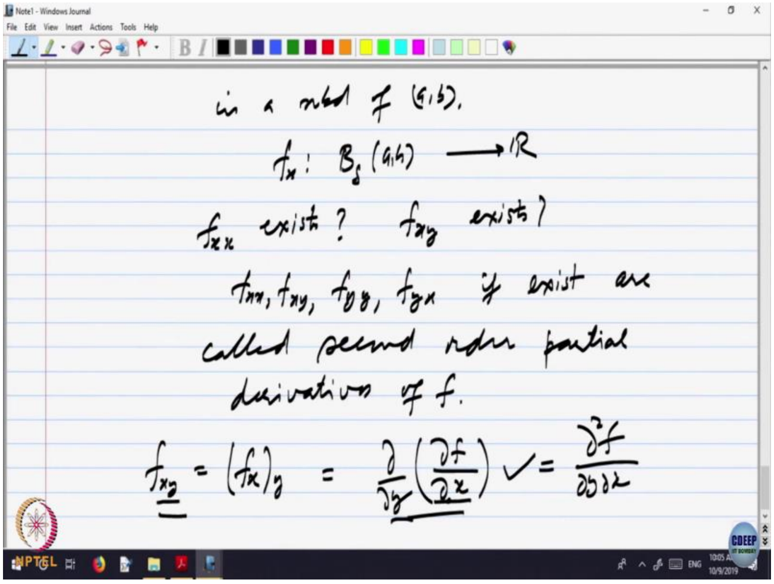

So, one would like to find out what are called sufficient conditions or local maximum minimum and saddle points. So what are, we can find critical points by looking at those collection of points namely where the derivative exist and are equal to 0 and looking at points where the derivative do not either or either both are the partial derivative do not exist. But out of this, how do you find out which are the points of local maximum or local minimum?

Like in function of one variables we had various test, we had a test of continuity test we had first derivative test, then we had the second derivative test which was sufficient conditions to ensure that the function has a local maxima or a local minimal. So, for that we need to go to second order derivatives for functions of two variables.

So, let us look at what are the call second order derivatives. So, we have a function x y going to f of x y, x y belonging to domain D. So, let us say that the partial derivative is this, so suppose fx fy exist at a point a b like function of variable, the first derivative exist then we can ask whether the second derivative exist at that point or not. So, similarly we are going to ask partial derivatives with respect to x and y exist at a b.

Whether we can definition again with respect to, so note that fx and fy are defined at a point a b taking values in R they are constant. So, if you want to talk about the second order derivative, derivative of the partial derivatives, then the functions fx and fy should be defined in a neighborhood first of all, otherwise you cannot talk about the derivative, a b and in a neighborhood of a b.

So, let us assume they both exist in a neighborhood of a b, that means the partial derivative are defined as functions in a neighborhood. So, let us say a ball of radius delta at ab they are functions. Like for one variable if the derivative exist, then the derivative is itself function of a variable at that point. So, now supposing fx exist in a neighborhood, then it is a function of two variables so we can ask whether fx x exist and fx y exist.

Because it is a function of two variables, so it can you can ask, whether it has partial derivative with respect to x and with respect to y both, you can ask for both. So, they may exist they may not exist, so these are called, so fx fxy and similarly fy y and fy x if exist are called second order partial derivatives of f they are called second order partial. So, for a function of two variable there are four partial derivative of second order.

If you have a three variable function, how many there will be? fx fy fz first order derivatives, then fx with respect to y and then with respect to z or fx with respect to z and then with respect to y. So, order may change, so for each one there will be two of them. So, there will be totally six mix partial six second order partial derivatives of function of three variable. Now, a question arises, if I take the function f, find its derivative with respect to, so what is this? Find the derivative with respect to x and then find the derivative with respect to y.

So, this is what we have written as f x y. Sometime let me also give that notation partial derivative sometime also is written as with respect to x, this also notation also we had introduced and then partial derivative of this with respect to y. Now, keep in mind here we are going x y and here it looks like y x, so in this notation you are going from right to left and this one you are going from left to right.

So, first you differentiate with respect to x and then with respect to y, first with respect to x and then with respect to y. So, this is also denoted at some time written as this instead of writing brackets and all that. So, this is same as this, these are different notations used for second order, so they are second order partial derivatives but there is no condition there is no surety.

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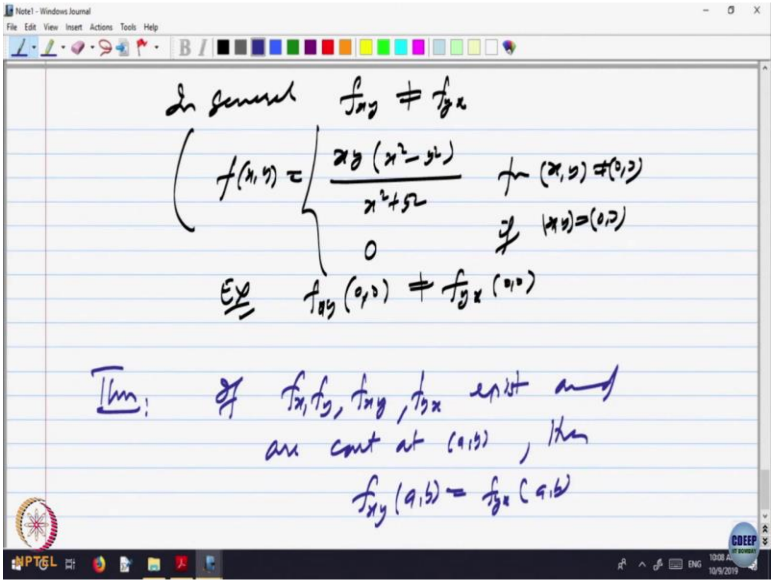

So, in general f xy mean not be equal to f yx, if you first differentiate with respect to x and then with respect to y then it may not be same as differentiating with respect to y first and then with respect to x, in general that may not happen the two may not be same. I think one example let me state here probably you can try it in later on yourself. So, for example if you look at f x y equal to xy x square minus y square divide by x square plus y square for x y not equal to 0, 0 and 0 if x y is equal to 0, 0.

So, I leave it as a exercise for you to check that this function has got. So, f x y at 0, 0 is not equal to f y x at 0, 0. So that is an exercise, that means it has partial derivatives with respect to x and with respect to y at 0, 0 at every point near, near 0, 0 also you want. So, second order partial derivative also exist but check at 0, 0 the two values are not equal. So, one wonders is there any conditions, so one proves the theorem, we will not prove that.

So, if fx, fy, fxy, fyx if they exist and are continues at a b, basically you want for fx fy need differentiability so you need continuity will come automatically. So the condition is the mix derivative fxy and fyx if both are continues at that point, then they will be equal, then fxy at a b will be equal to f yx at a b. So, they will be then they are equal we will not prove that and we will not be a sort of having much use of it, so most of our functions will have that properties so it will be okay, but in general this is the condition one should verify to I am sure that the mix partial derivatives are equal.