INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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Lecture No. – 35

Prime Ideals in Polynomial rings

Today I want to just tie up certain things about dimension and some more example of a computation with a dimension, with the polynomial ring. So I want to just recall, if you have a field K maybe I have a polynomial algebra over a field in n variables, and in linear algebra we have the corresponding objectives, vector space of dimension n, K^n , this is a K vector space of dimension n, this is usually called affine n space, and in commutative algebra, algebraic geometry that correspond to this polynomial algebra.

And we have also seen that if you assume more K is algebraically closed then this affine n space also you can realize it as a maximal spectrum of this polynomial algebra, so this is SPM $K[X_1,\ldots,X_n]$, so there is the identification here, and but note that when you have a, if you have a usual field like real numbers of complex numbers usually on this K^n one takes usual topology, but here we have a Zariski topology so that is weaker than an usual topology.
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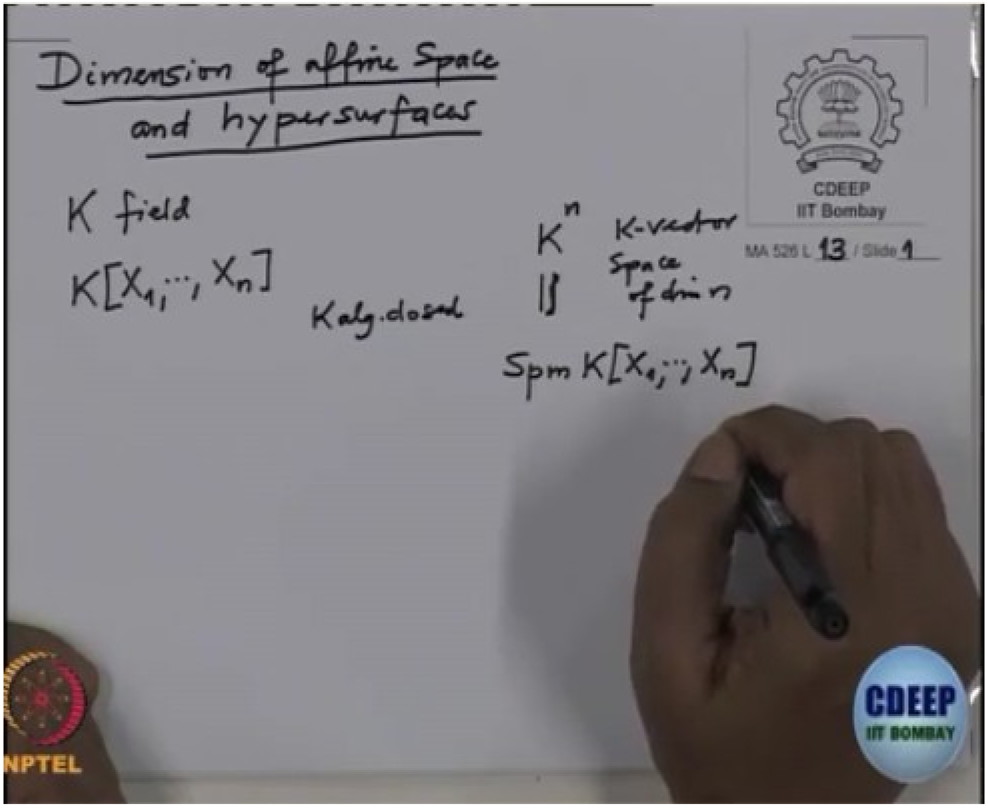

And this n therefore in this set up it correspond to the Krull dimension, this is n,

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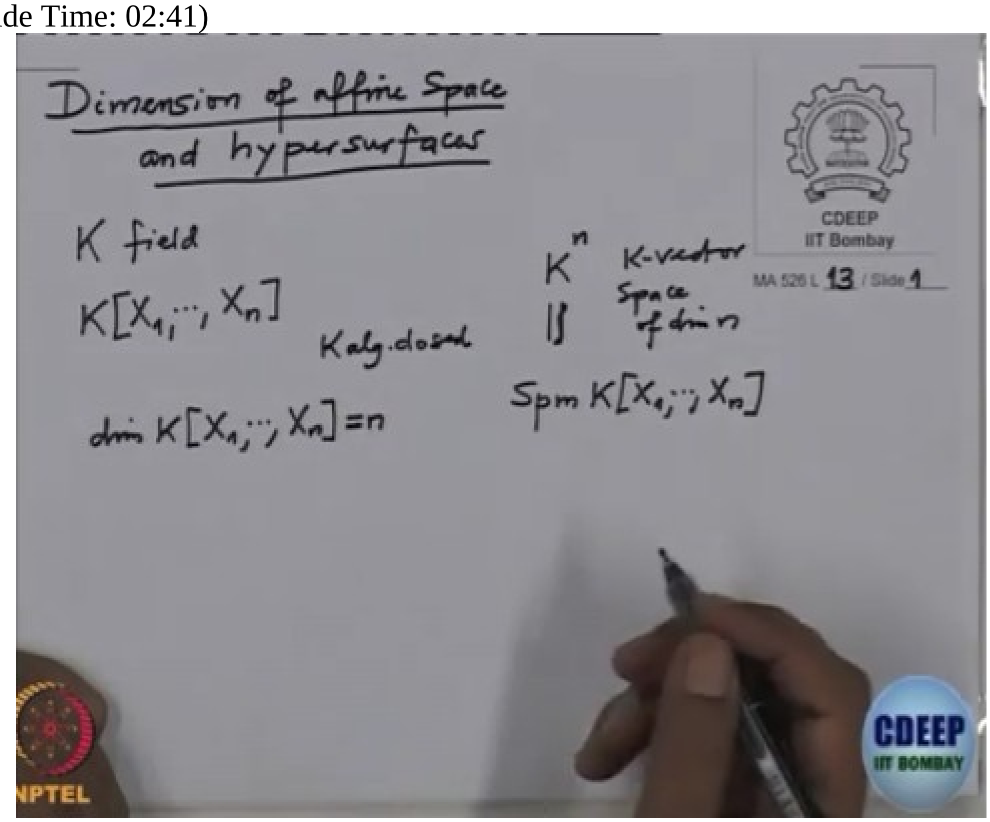

and for this reason one might call this $K[X_1]$, this spec of $K[X_1,\ldots,X_n]$ is called affine n space over K, and it is n dimensional.

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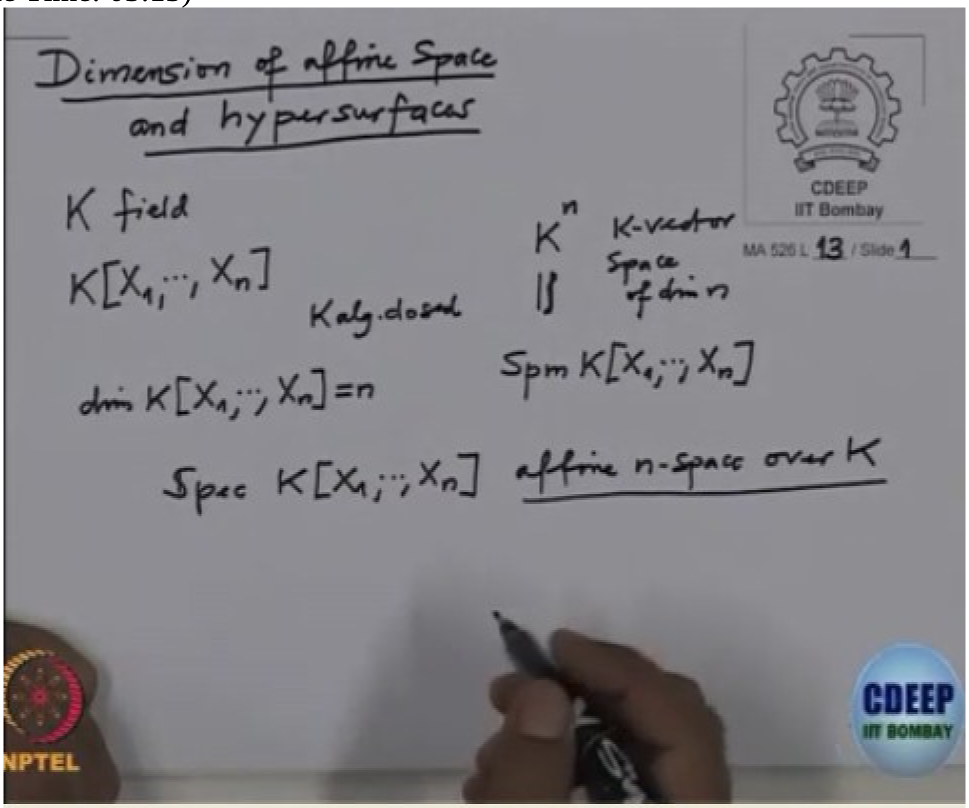

More generally if you have arbitrary Noetherian ring, A is Noetherian then for the similar reason if you take spec of $A[X_1,\ldots,X_n]$ this is called affine n space over A, or one should actually say not A but spec A, and also we have proved that the dimension of this ring, dimension of this and therefore dimension of this spectrum that will be dimension A + n.

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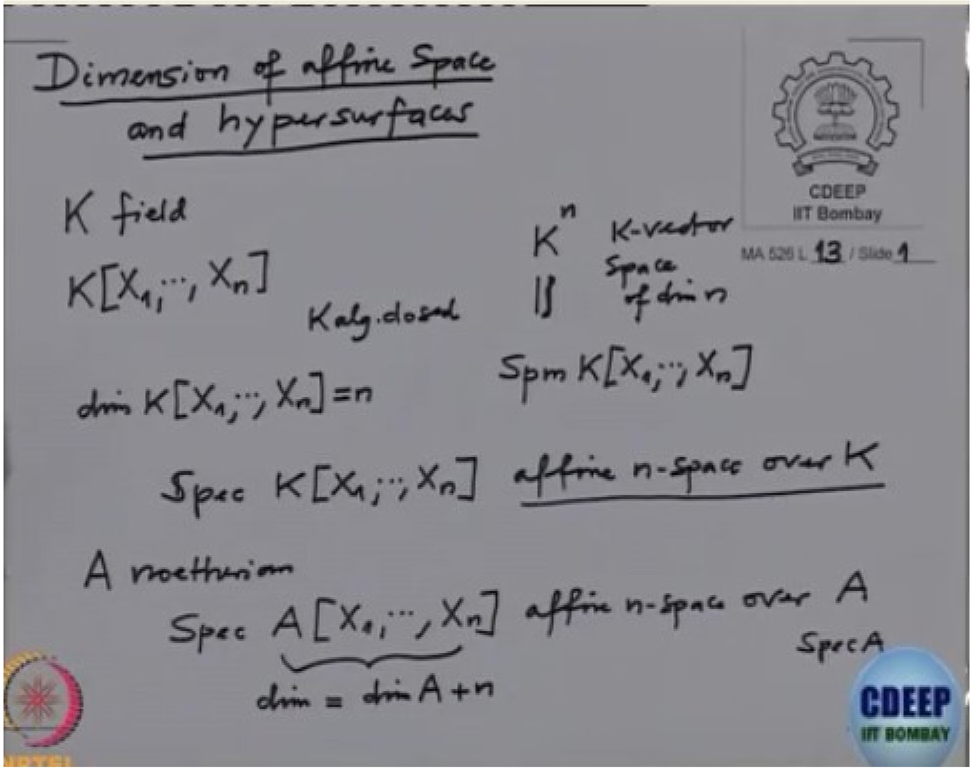

Note that the over a field it will be easier to realize or K algebraically closed it is, you can realize this points with this, but to understand prime ideals or even maximal ideals in A it's rather complicated in general, and I just wanted to recall that when we proved this dimension of the polynomial algebra is more than the dimension A, we have postponed one proof, of one proposition that we will complete today.

So let me recall that and then we will complete that proof, and then the next we will go to hyper surfaces. Hyper surface is the analog of this, but then how to get equations in this case that will be the next theorem, okay.

So this is what we have left to proof last time, so this proposition we have not finished complete proof, I'll state it completely for the sake of completeness, there are three parts and only one was not proved, and the other two were proved, but for the sake of completeness let us note, so A is a ring, mostly Noetherian I would like to assume, A is a ring, and this polynomial extension and we wanted to study prime ideals here and prime ideals here and their heights and their relations and so on right, so let's call this to B, okay.

And Q is the prime ideal in B, and P is the contraction of Q, Q intersection A, so this simply means if you take the natural map from spec B to spec A, there is a P here, and there is at least this Q is lying over that, so this means the fiber over P is nonempty, and this Q is one element in the fiber, okay.

The first assertion was height of $\frac{Q}{P[X]}$, so note that this $P[X]$ is also prime ideal in $P[X]$, this means extension of P to B, this is PB, this is also prime ideal in B because when you take $\frac{A[X]}{P[X]}$, that is $\frac{A}{P}$ and then X, so polynomial commutes with the mod operation so therefore this is a polynomial ring over integral domain, therefore integral domain so therefore $P[X]$ is a prime ideal, so this height plus, okay and at P there is a residue field $K(P)$, and at Q there is a residue field $K(Q)$, this means go mod P and then take the quotient field, this is a $\frac{B}{P}$, $\frac{B}{Q}$ and then the quotient, okay.

And it is clear that this one, this one is an extension of $K(P)$, so the transcendent degree of $\frac{K(Q)}{K(P)}$, this height plus this transcendent is equal to 1.

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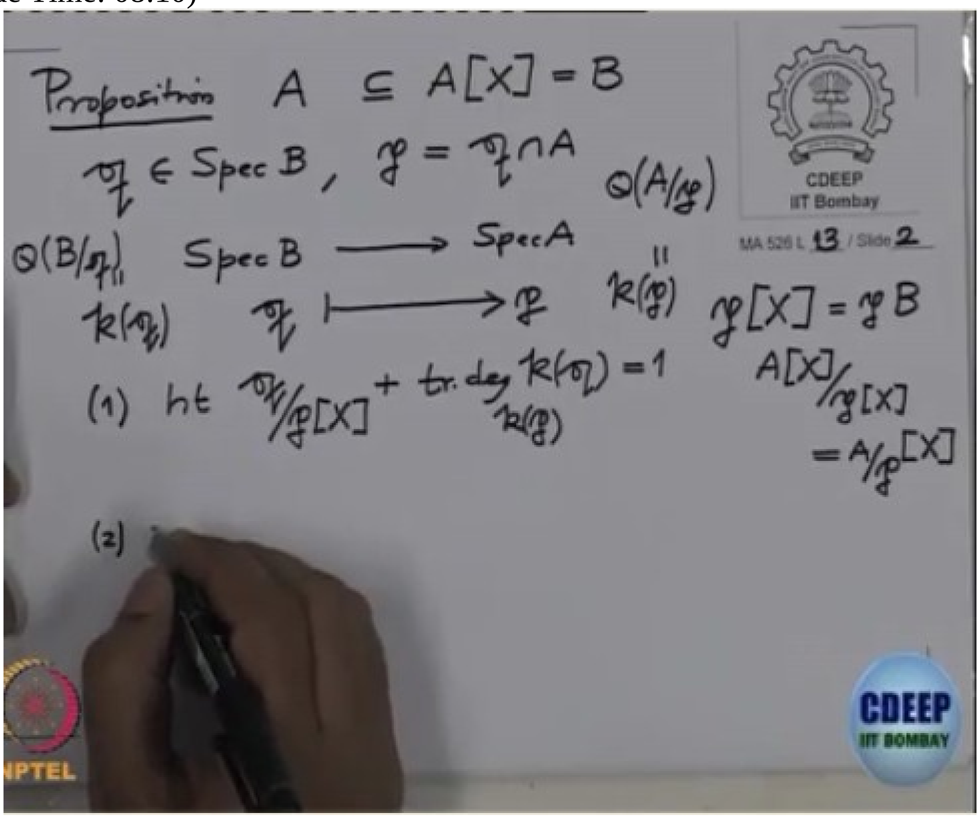

Two, height of Q, see ultimately we want to relate height of Q and height of P, so height of Q = height of P + height of $\frac{Q}{P}$ [X], this is a main interest today to prove, because this is what we have not proved last time, and then when you plug it from 1 this equal to one minus that, so this will be equal to height P + 1 – transcendent degree of $\frac{K(Q)}{K(P)}$.

And the third was there are two cases height of Q will be equal to height of P or it will be 1 more than the height of P's, this is the case when $Q = P[X]$ and this is the case when Q is not $P[X]$, this was it,

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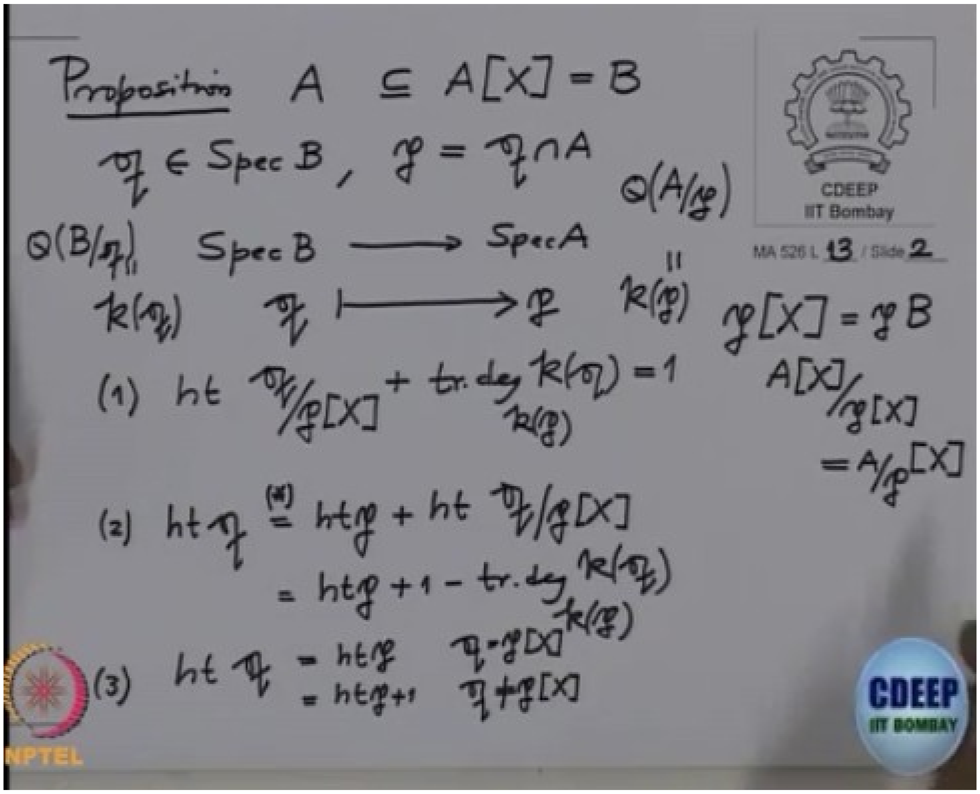

and this will therefore imply that for a Noetherian ring if you take a polynomial ring it will have exactly one more height than the dimension, dimension of A polynomial ring over a Noetherian ring will be one more than the dimension of the base ring.

And we have proved one, we have proved three by using one and two, and this equality just follow from this one so we just have to prove this equality today, this is what we want to prove today, okay.
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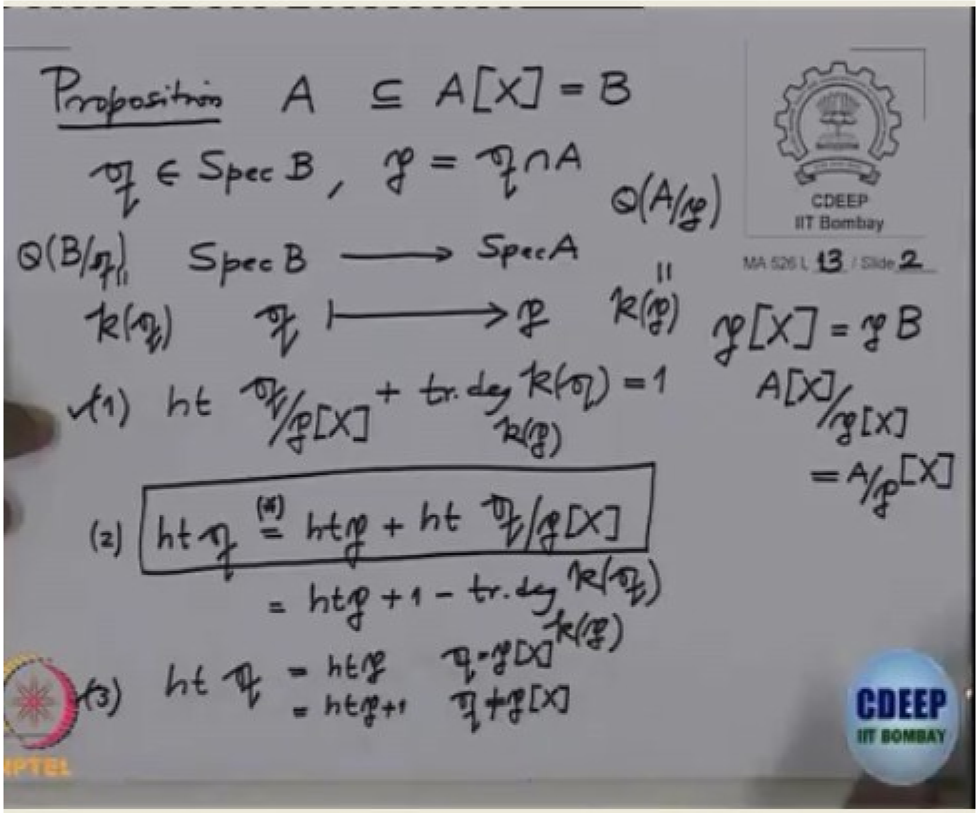

So proof of 2, so first of all note that if I take height of $P[X]$, this is definitely bigger equal to height P, this simply follows from the fact that if I take away chain of prime ideals of length R which ends at P and extend those to B, then $P_0[X]$ contained in, this would be a proper chain of prime ideals in B, and this obviously contains, it contained in Q therefore height of Q will be at least, at least R, so therefore this for every chain it happens, so therefore height supremum so on.

Okay now therefore height of Q will be equal to height of $P[X]$ + height of $\frac{Q}{P[X]}$, this is because you take this, so it's clear, because if you take this side that is coming from a chain of length, some length ending at P, and we extend that length and then $P[X]$ will be contained in Q, so but parts, so height is bigger equal to.

Now I want to prove the equality here that is what, so this is equal to height P + height of $P[X]$, so we want to prove the other inequality,

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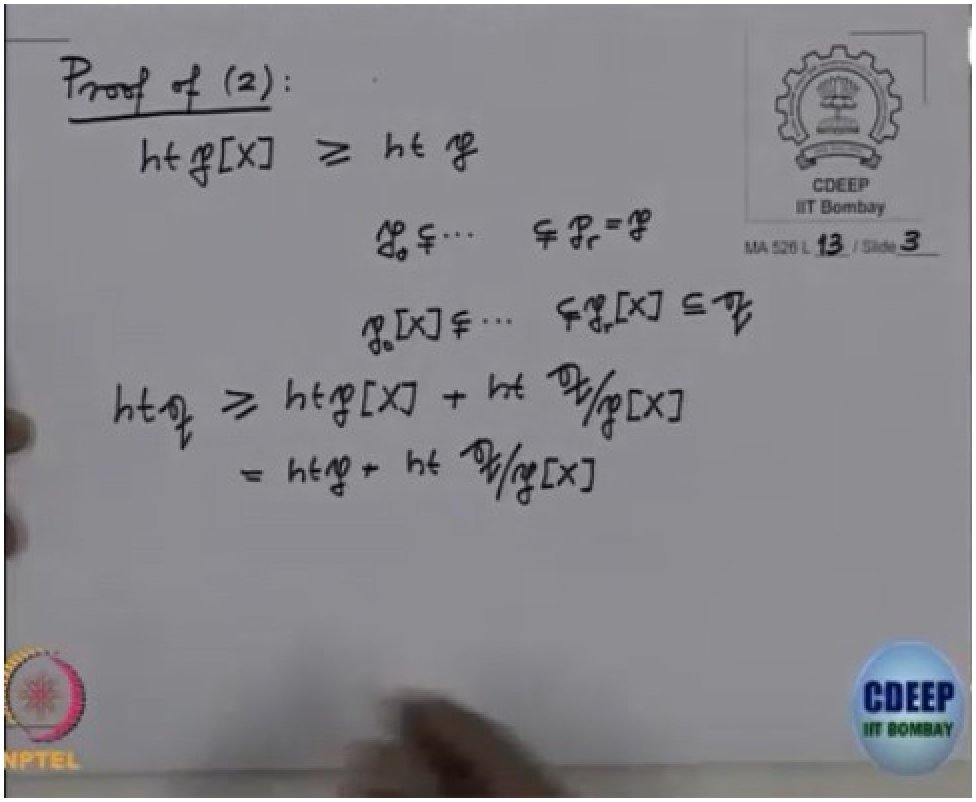

so for that let us put localize, see all the chain in all that contain in P and contain in Q, so I want to come down to the local situation, so let us put B = A localize at P, and C equal to $A[X]$ localize at Q, so now both these are local rings and maximal ideal M here is PAP, and the maximal ideal here is QC let us say, QC the maximal ideal, let us call it N.

So we have this two local rings, okay, and let us call d to be the height of P which is in the dimension of B, and also let us call e to be the height of, so this I want to call it height, e = height of $\frac{Q}{P[X]}$, so this two heights and we want to prove $d+e$,

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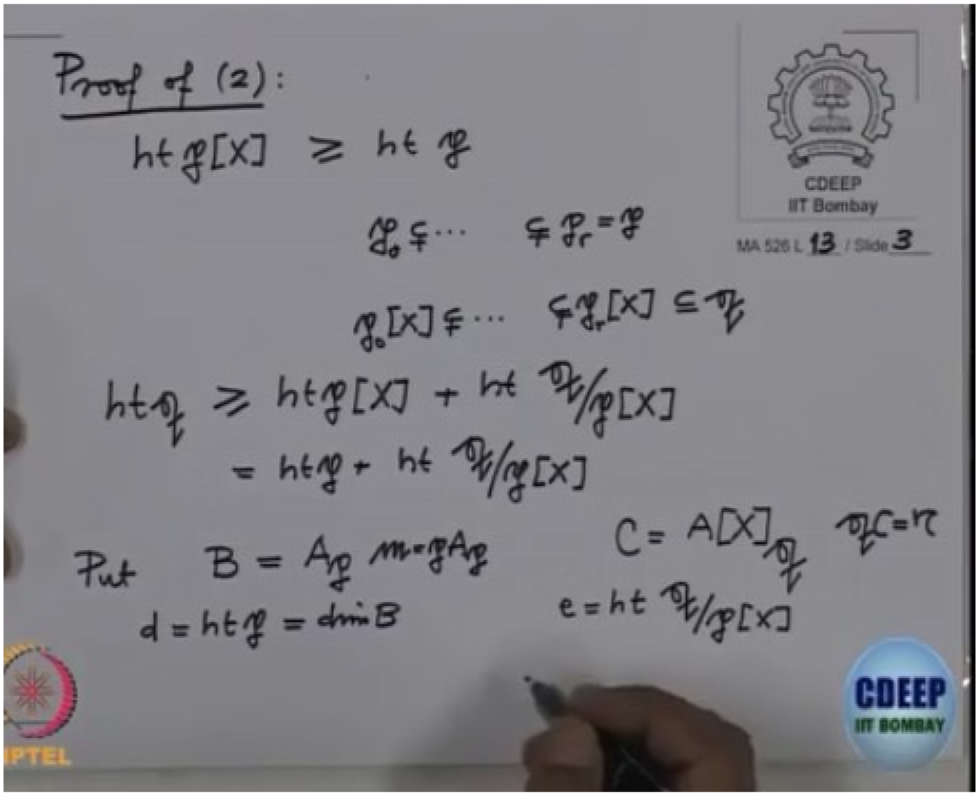

so to prove, we want to prove that $d+e$ is less equal to height of Q, this is what we want to prove which is the dimension of C.

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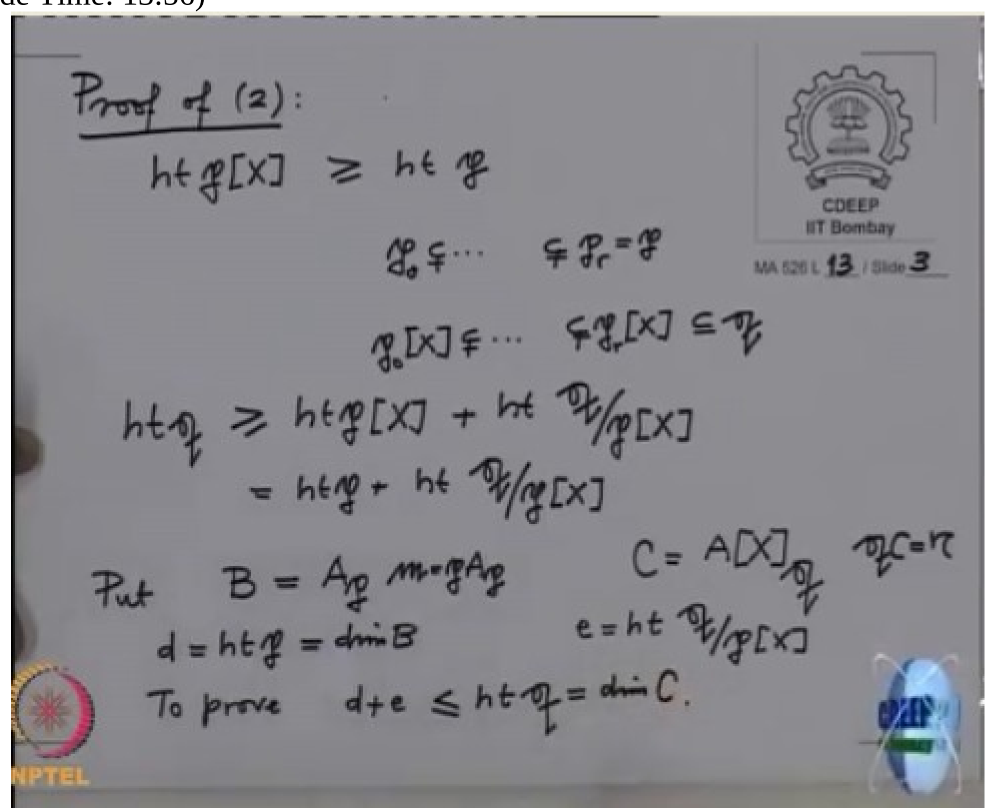

Okay so now we are in this B the local ring, B is a local ring, local of dimension d and we have proved that local ring there is a system of parameters generated by d elements, that is Chevelley dimension, so therefore there exist d elements in B, so $a_1,\ldots,a_d$, $a_d \in m$ such that this $a_1,\ldots,a_d$ is a system of parameters, system of parameters for B that simply means that it's a primary ideal for so that is it generates, that is this $a_1,\ldots,a_d$ is m primary ideal in B, that means some power of this m is contained in this, so let us say m^P is contained in ideal generated by $a_1,\ldots,a_d$.

Okay now this in C by, on the other hand this C is local and we are interested in actually the ring $\frac{C}{PC}$, this is same as, so in this the maximal ideal will be now corresponds to $\frac{Q}{P[X]}$, remember C was polynomial ring over A localize at Q, and when you go mod P will go to this, so this is the maximal ideal in this, so this is like when you read, so this is like $\frac{C}{mC}$, see B to C there is a homomorphism, and this m it goes to mC , so this ring, if this ring, this is a maximal ideal so the maximal ideal because local ring this C is local therefore this is local, therefore they were system of parameters, so system of parameters for C I will denote by $c_1,\ldots,c_e$ that is what we have called that E to be the height of this, which is the dimension of this ring, so $c_1,\ldots,c_e$ choose system of parameters for, you either call $\frac{C}{PC}$ or $\frac{C}{mC}$, this is same as $\frac{C}{mC}$.

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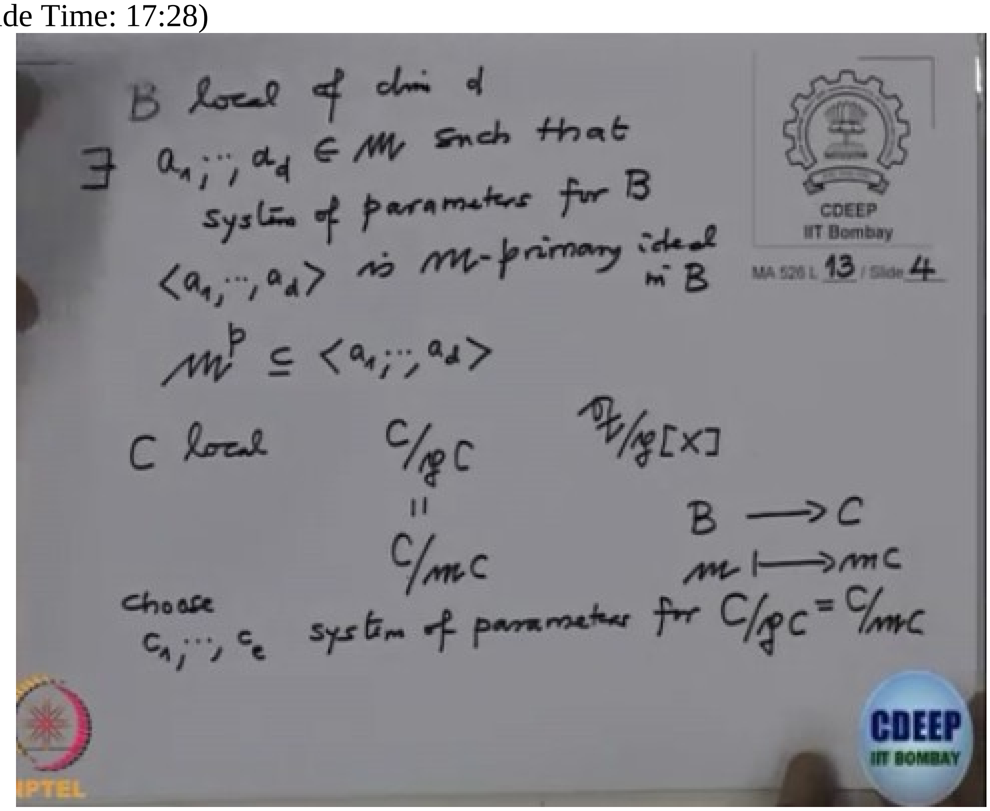

So because it is a system of parameters the power of a maximal ideal will be contained in this ideal generated by $c_1, \ldots, c_e$, so that means N power, what do I call it now? Some Q will be contained in ideal generated by $c_1, \ldots, c_e$, but this means m, so this means N power, N is the maximal ideal in C, but maximal ideal is generated mod in C generated by this N, so therefore that will imply that N power P, this is in $\frac{C}{mC}$, so N power if I further read it to N^p , that m^p if I further raise this to Q, this will be contained in $a_1, ..., a_d$, along with $c_1, \ldots, c_e$.

So when you say the power of a maximal ideal is contained in the ideal generated by this, the maximal ideal here is $\frac{N}{mC}$, so therefore when you lift it you have to go $N+mC$ and then when you raise enough power and then shift this into the other side this is generated by, it's contained in ideal generated by this, that simply means that the height, that simply means the dimension, so this is therefore a primary ideal, this is N primary ideal in C, because it contains the power of the maximal ideal,

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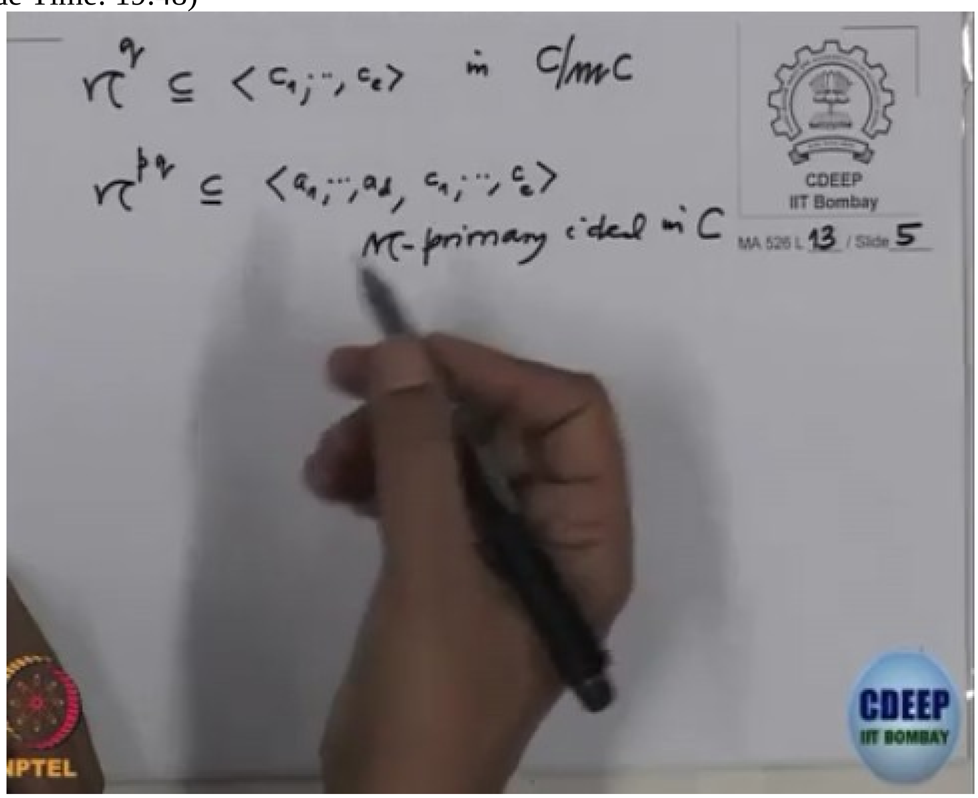

therefore this number $d+e$ has to be bigger equal to, less equal to, bigger equal to see its primary ideal, so the minimal, the system of parameters is the minimal number of generators for the ideal, when you go mod the length becomes finite, but primary ideal will make that, but we want the correct, so that means infimum of those, remember this system of parameters the chevelle dimension S of A was by definition infimum of the number r so that length of $\frac{A}{\langle a_1, \ldots, a_i \rangle}$ is finite, where A is the local ring, so this was infimum and this is one ideals so that when you go mod that length becomes finite, because this is N primary, so that implies the height of N is less equal to d+e, this is the dimension of C, but this is same as height of Q, because we are localize it, so that proves this inequality and that therefore finishes the proof of that inequality, equality that we wanted to prove, therefore we now justified that when you take Noetherian, A Noetherian then we know dimension of $A[X]$ is one more A+1, and more generally when you take dimension of induction, this dimension of $A[X_1,\ldots,X_n]$ more than the dimension of A.

And Noetherian is very important otherwise this equality may not be equality,
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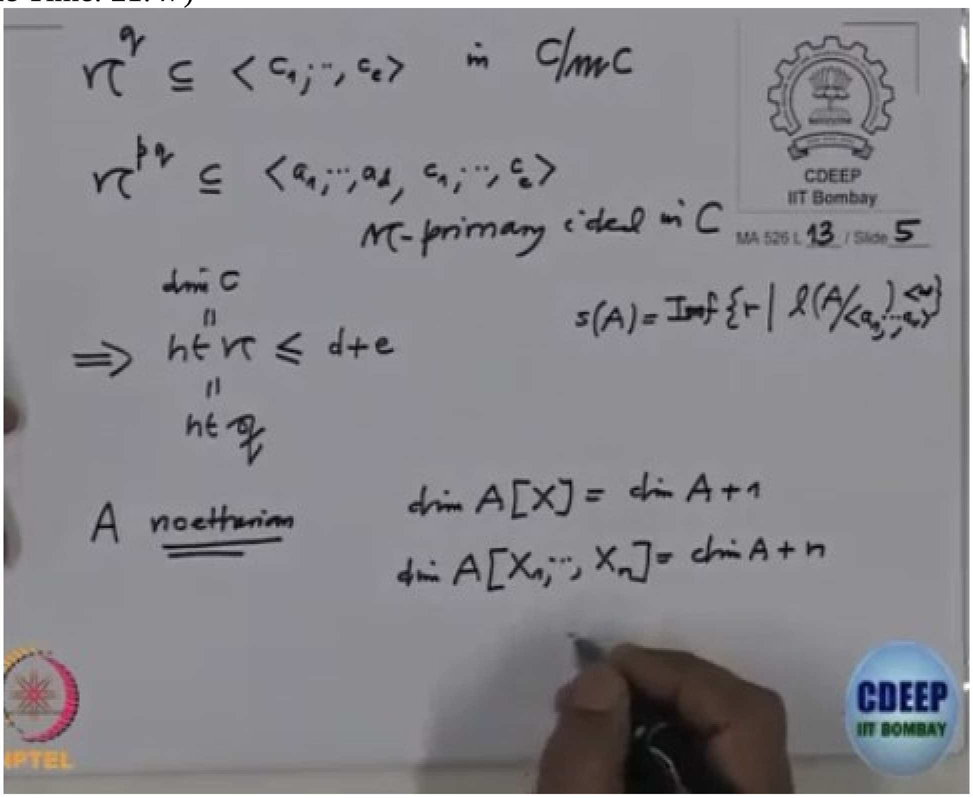

we should see example in maybe in the tutorial section next time, so if justifies also that this is called the affine n space over A, now it's the dimension like N, okay, then our next object that we usually study in linear algebra for example or hyper surfaces, and hyper surfaces are given by one equation, so if this is analog of that I want to prove in this set up today, but before I do I just want to recall that when we say, when X is a Noetherian space, do you know the definition and so on, okay, so I'll assume and so you know that Noetherian space is a union of finitely many irreducible and irreducible components make sense and so on.

Suppose $Z_1,\ldots,Z_n$ are irreducible components of X, so I want to, when do you say X is equidimension, X is called equidimensional that means all of this irreducible components have the same dimension and that dimension will be equal to then dimension X, if dimension of X equal to dimension of Z_i for each i, then you call it equidimensional, so that means so typically you think of, so usually one should think X as spec of A, and A is Noetherian, so for example if your ring is d dimensional that means the supremum of dimension of the irreducible components will be equal to d, but some of them maybe lesser, so for example if you imagine a point and a line, poly line some curve, so this one has a dimension 0, this one has dimension 1, but if you take together so dimension will be 1, so like that so the irreducible components here corresponds to the minimal primes, so irreducible components of spec A they corresponds to the minimal elements in the associated primes of A,

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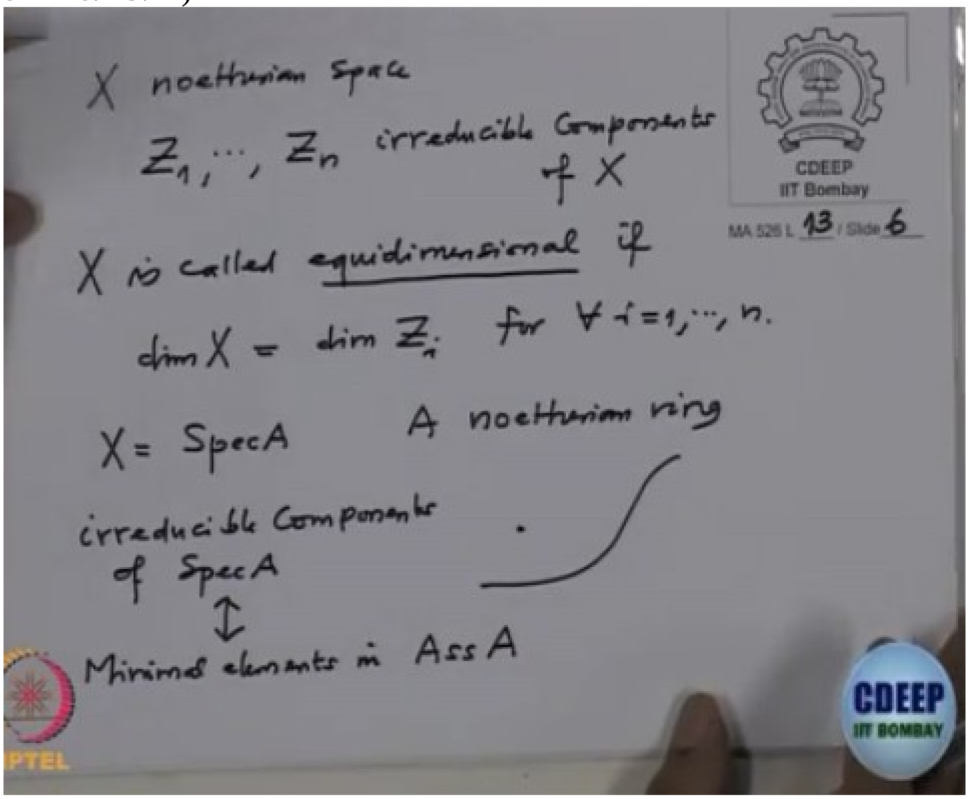

and they are finitely many here so finitely many minimal elements therefore finitely many irreducible components, all of them may not give the dimension some may be equal, some may not be equal, if all of them have the same dimension we call it equidimensions

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