Introduction to Operations Research
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Module – 02
Graphical and Algebraic Methods
Lecture - 04
Algebraic Method (Maximization)

In this class, we will look at the Algebraic method to solve minimization problems. We consider the same minimization problem that we used, the same problem that we solved using the graphical method.

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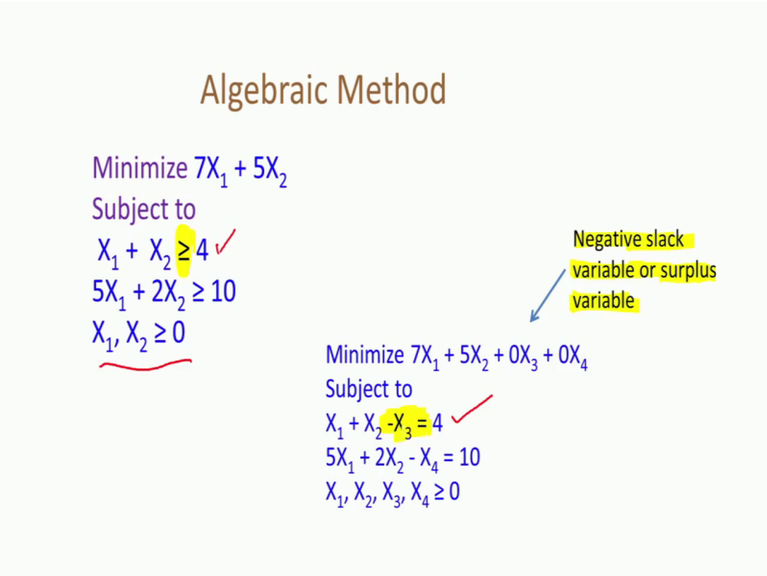

So, the problem minimizes 7 X 1 plus 5 X 2 subject to X 1 plus X 2 greater than or equal to 4, 5 X 1 plus 2 X 2 greater than or equal to 10, X 1, X 2 greater than or equal to 0. Now, once again, we will follow the same procedure, where we convert the inequalities to equations and we try to solve the equations. So, as we did before, we will consider the two constraints, we will at present leave out the objective function and the non-negativity.

So, the two constraints are X 1 plus X 2 greater than or equal to 4 and 5 X 1 plus 2 X 2 greater than or equal to 10. Now, let us look at X 1 plus X 2 greater than or equal to 4 and convert it to an equation and we do it as follows. Now, this constraint X 1 plus X 2 greater than or equal to 4 is now written as X 1 plus X 2 minus X 3 equal to 4. Now, X 1 and X 2 are greater than or equal to 0 and we want X 1 plus X 2 to be either equal to 4 or more than 4.

So, we introduce the variable X 3, such that, if X 1 plus X 2 is more than 4, then X 3 would take a positive value and then, make it equal to 4. So, that X 1 plus X 2 minus X 3 is equal to 4, if X 1 plus X 2 is exactly equal to 4, then X 3 will take value 0. Now, X 1 and X 2 have to be such that, they ordinarily should not be less than 4. Therefore, X 3 does not take a negative value. So, X 1 plus X 2 greater than or equal to 4 is now rewritten as X 1 plus X 2 minus X 3 equals to 4.

Now, this X 3 is called a surplus variable, it is also a slack variable, but because X 3 has a negative sign, it is also a called a negative slack variable or it is called a surplus variable. So, when we have a greater than or equal to the constraint like this, now that would result in a negative slack or a surplus variable. In a similar manner, 5 X 1 plus 2 X 2 greater than or equal to 10 is written as 5 X 1 plus 2 X 2 minus X 4 equal to 10.

Now, the minus X 4 comes because, when 5 X 1 plus 2 X 2 exceeds 10, then X 4 will take a positive value to make it equal to 10. If 5 X 1 plus 2 X 4 is exactly equal to 10, then X 4 will take 0. We also should have X 1, X 2; such that, 5 X 1 plus 2 X 2 is greater than or equal to 10, 5 X 1 plus 2 X 2 is not less than 10 and therefore, X 4 does not take a negative value.

So, when we have a greater than or equal to constraint, we have a negative slack variable or a surplus variable introduced. X 3 and X 4 have been introduced and importantly, X 3, X 4 are greater than or equal to 0 and they will have a negative sign if we have a greater than or equal to constraint. In the earlier class, when we had a less than or equal to constraint, we introduced X 3 and X 4 as positive slack variables. Again, X 3 and X 4 were greater than or equal to 0, but X 3 and X 4 had a plus 1 sign appearing.

So, when we convert an inequality to an equation by adding a slack variable, we could add a positive slack variable, if the constraint is less than or equal to type. And a negative slack, if it is greater than or equal to type. But, what is important is that, the slack variables are defined as greater than or equal to 0 in both the cases, just as the decision variables are defined as greater than or equal to 0.

So, once again, we observe like in the previous example that with the addition of the two slack variables, we now have two equations, but we have four variables. We also know that if we have two equations, we can only solve for two variables. Therefore, we follow a similar manner like, what we did in the previous class to try and solve this system that now has four variables and two equations.

Just as in the previous example, the negative slack or surplus variables do not contribute to the objective function and therefore, they contribute a 0 to both X 3 and X 4. We now go on to solve this system that minimizes 7 X 1 plus 5 X 2 plus 0 X 3 plus 0 X 4, subject to X 1 plus X 2 minus X 3 equal to 4, 5 X 1 plus 2 X 2 minus X 4 equal to 10, X 1, X 2, X 3 and X 4 greater than or equal to 0.

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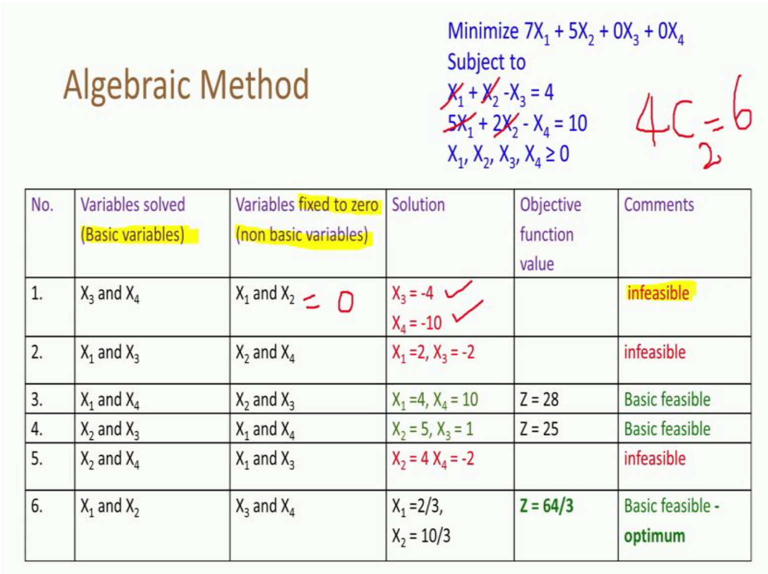

No.	Variables solved (Basic variables)	Variables fixed to zero (non basic variables)	Solution	Objective function value	Comments
1.	X_3 and X_4	X_1 and X_2 = 0	$X_3 = -4$, $X_4 = -10$		infeasible
2.	X_1 and X_3	X_2 and X_4	$X_1 = 2$, $X_3 = -2$		infeasible
3.	X_1 and X_4	X_2 and X_3	$X_1 = 4$, $X_4 = 10$	Z = 28	Basic feasible
4.	X_2 and X_3	X_1 and X_4	$X_2 = 5$, $X_3 = 1$	Z = 25	Basic feasible
5.	X_2 and X_4	X_1 and X_3	$X_2 = 4$ $X_4 = -2$		infeasible
6.	X_1 and X_2	X_3 and X_4	$X_1 = 2/3$, $X_2 = 10/3$	**Z = 64/3**	Basic feasible - **optimum**

Now, the linear programming problem that we have to solve is to minimize 7 X 1 plus 5 X 2 plus 0 X 3 plus 0 X 4, subject to the two constraints and non-negativity. Since, we have four variables and two equations. We can only solve for two variables at a time and these two variables from four given variables can be chosen in 4 C 2 or it can be chosen in six ways. So, it can be chosen in, so it is be chosen in 4 C 2 ways or it can be chosen in six ways.

Now, these six ways of choosing the two variables are shown here, for example, one of them would be to solve for X 3 and X 4; another one could be to solve for X 1 and X 3 and so no. Now, when we choose the two variables in anyone out of the six ways, we have the other two variables, which we are not solving for and these variables have to be fixed to some arbitrary value.

Now, they can be actually fixed to any arbitrary value, which means, they can take infinite values, but as we did before in the previous example, we fix those variables to 0. So, these variables which we are fixing at 0 or these variables, which we are fixing, variables fixed to 0 are called non basic variables and those variables that we are solving are called basic variables.

So, at a time, we take two variables out of the six and solve for them, by fixing the remaining two variables to 0. So, now, let us go back and look at all the six possible solutions. So, first let us look at the case, where we are going to solve for X 3 and X 4 by fixing X 1 and X 2 to 0. Now, when we solve, when we fix X 1 and X 2 to 0, I am kind of deleting X 1 and X 2. So, we are going to solve for X 3 and X 4. So, we solve for minus X 3 equals 4 minus X 4 equals 10, which gives us a solution X 3 equal to minus 4 and X 4 equal to minus 10, it gives us this solution.

Now, when we consider this solution, where X 1 equal to 0, X 2 equal to 0, X 3 equal to minus 4 and X 4 equal to minus 10. We have this solution, where X 3 equal to minus 4 and X 4 equal to minus 10, violates the non-negativity restriction and therefore, this solution is infeasible. So, this solution becomes infeasible, because it violates the non-negativity restriction.

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Algebraic Method

Minimize $7X_1 + 5X_2 + 0X_3 + 0X_4$

Subject to

$X_1 + X_2 - X_3 = 4$

$5X_1 + 2X_2 - X_4 = 10$

$X_1, X_2, X_3, X_4 \geq 0$

No.	Variables solved (Basic variables)	Variables fixed to zero (non basic variables)	Solution	Objective function value	Comments
1.	X_3 and X_4	X_1 and X_2 = 0	$X_3 = -4$, $X_4 = -10$		infeasible
2.	X_1 and X_3	X_2 and X_4 = 0	$X_1 = 2$, $X_3 = -2$		infeasible
3.	X_1 and X_4	X_2 and X_3	$X_1 = 4$, $X_4 = 10$	Z = 28	Basic feasible
4.	X_2 and X_3	X_1 and X_4	$X_2 = 5$, $X_3 = 1$	Z = 25	Basic feasible
5.	X_2 and X_4	X_1 and X_3	$X_2 = 4$ $X_4 = -2$		infeasible
6.	X_1 and X_2	X_3 and X_4	$X_1 = 2/3$, $X_2 = 10/3$	Z = 64/3	Basic feasible - optimum

Now, we look at the second instance, where X 1 and X 3, we are going to solve and we are going to fix X 2 and X 4 to 0. So, we are going to, if we take the second instance, we are fixing X 2 and X 4 to 0 and we are going to solve for X 1 and X 3. Since, we are fixing X 2 and X 4 to 0, I am leaving them out. So, I am going to solve X 1 equal to 4, 5 X 1 is equal to 10. So, 5 X 1 equal to 10, X 1 minus X 3 is equal to 4.

So, 5 X 1 equal to 10 gives me X 1 equal to 2 and when I substitute X 1 equal to 2, I will get X 3 equal to minus 2, so that X 1 minus X 3 is equal to 4. So, we have a solution X 1 equal to 2, X 2 equal to 0. So, X 2 equal to 0, X 3 is equal to minus 2 and X 4 is equal to 0. Now, because X 3 is equal to minus 2, it violates the non-negativity restriction and therefore, this solution also becomes infeasible.

Now, we look at the third out of the six problems, where we are going to solve for X 1 and X 4 and we are going to fix X 2 and X 3 to 0. Since, I am fixing X 2 and X 3 to 0, I am removing X 2 and X 3 from this solution. So, this gives me a system, where X 1 is equal to 4, this gives me a system, where X 1 is equal to 4 and 5 X 1 minus X 4 is equal to 10.

So, X 1 is equal to 4 is shown here that is the solution and when I substitute 5 X 1, 5 X 1 becomes 20, therefore, X 4 becomes 10, so that 5 X 1 minus X 4 is equal to 10. So, I get a solution X 1 equal to 4, X 4 equal to 10. So, X 1 equal to 4, X 2 equal to 0, X 3 equal to 0, X 4 equal to 10, satisfies the non-negativity restriction and therefore, it is feasible. It is a basic solution because, I am solving for two variables, because I have two equations. So, this solution is both basic and feasible, therefore, it is basic feasible.

Now, I compute the value of the objective function for X 1 equal to 4, X 4 equal to 10. So, 7 into 4 plus 5 into 0 gives Z equal to 28, the value of the objective function is 28. Now, we move on to the 4th out of the six solutions. Now, in this solution, I am going to solve for X 2 and X 3 and I am going to fix X 1 and X 4 to 0. So, I am going to fix X 1 and X 4 to 0, so this goes, this goes, this also goes.

So, the second equation becomes 2 X 2 equal to 10, from which I get X 2 equal to 5, from which X 2 is equal to 5 and if I substitute X 2 5, I get 5 minus X 3 is equal to 4 therefore, X 3 equal to 1. So, I have a solution X 1 equal to 0, X 2 equal to 5, X 3 equal to 1, X 4 equal to 0 and this solution is feasible, because it satisfies the non-negativity restriction, it also satisfies the two equations. It is also a basic, because I am solving for two variables X 2 and X 3 and I have two equations.

So, I have a basic feasible solution, therefore, I try to find out the objective function value. So, the objective function value would be 7 into 0 plus 5 into 5, which gives me Z equal to 25. Now, I move on to the fifth of the six problems, where I am going to fix X 1 and X 3 to 0 and I am going to solve for X 2 and X 4. So, I fix X 1 and X 3 to 0. So, X 1 and X 3 to 0.

So, the first equation gives me X 2 equal to 4, which is shown here and when I substitute X 2 equal to 4, 8 minus X 4 is equal to 10, which gives me X 4 is equal to minus 2. Now, when X 4 is equal to minus 2, it violates the non-negativity restriction and therefore, this solution is not feasible. So, it becomes infeasible and this is written as in feasible.

Now, we go to the last of the six situations. Now, the last one, I am going to solve for X 1 and X 2, I am going to fix X 3 and X 4 to 0. So, I am going to fix X 3 and X 4 to 0, so this goes, this also goes, I have to solve for X 1 plus X 2 equal to 4 5 X 1 plus X 2 X 2 equal to 10, I multiply the first equation by 2 to get 2 X 1 plus 2 X 2 is equal to 8. Subtracting, I will get 3 X 1 is equal to 2 X 1 is equal to 2 by 3, I will get X 1 is equal to 2 by 3, which is shown here.

Now, when I substitute X 1 is equal to 2 by 3 in X 1 plus X 2 equal to 4, I will get X 2 is equal to 10 by 3. So, that X 1 plus X 2 is 12 by 3, which is 4. Now, this satisfies the non-negativity restriction and therefore, it is feasible, it is also a basic, because I am solving for two variables, since I have two equations. So, it is a basic feasible solution. Now, among the… I calculate the value of the objective function for this, so 7 X 1 plus 5 X 2 will become 7 into 2 by 3, 14 by 3 plus 5 into 10 by 3, 50 by 3, which will give me 64 by 3, which is equal to 21.33.

Now, out of the three solutions which are basic feasible, the one that has the lowest value minimizing the objective function, the one that has the lowest value is optimum, it is the best solution. So, the solution 2 by 3 comma 10 by 3 is optimum to the minimization linear programming problem. In the next class, we will see some more aspects of the algebraic method and we will try to relate the algebraic method to the graphical method.