Selected Topics in Decision Modeling
Prof. Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur

Lecture - 28
Constrained NLP: KKT Conditions (Contd.)

So, in our course Selected Topics in a Decision Modeling, we are in our 28th lecture.

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We are discussing Constrained Nonlinear Programming specifically on Karush-Kuhn-Tucker Conditions. Is it alright?

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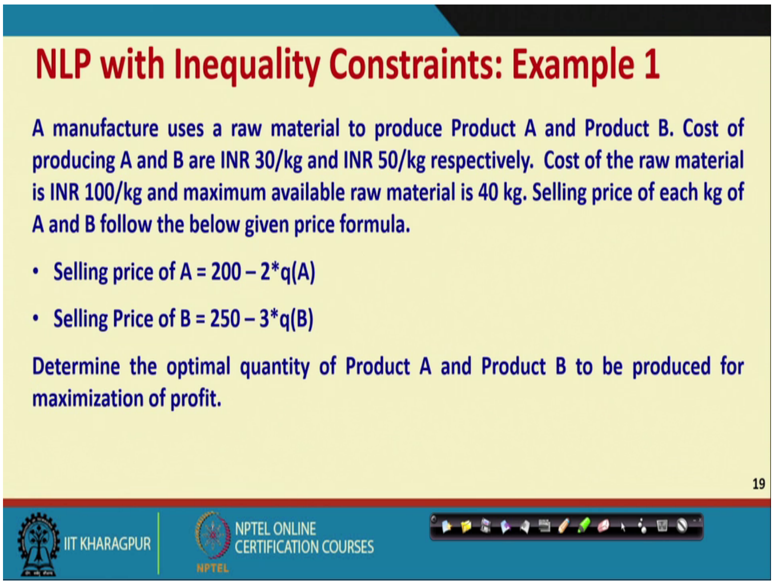

So, now we are in the middle of a problem, that was and non-linear programming with inequality constrains. So, in this particular problem, we have seen that 2 products are being produced product A and product B and a raw material is required for that, which is available around 40 kg and the selling price is given as per a formula, which is dependent on quantity which is going to have an effect on nonlinearity of the problem, and we have to determine the optimal quantity of product A and product B for maximization of profit.

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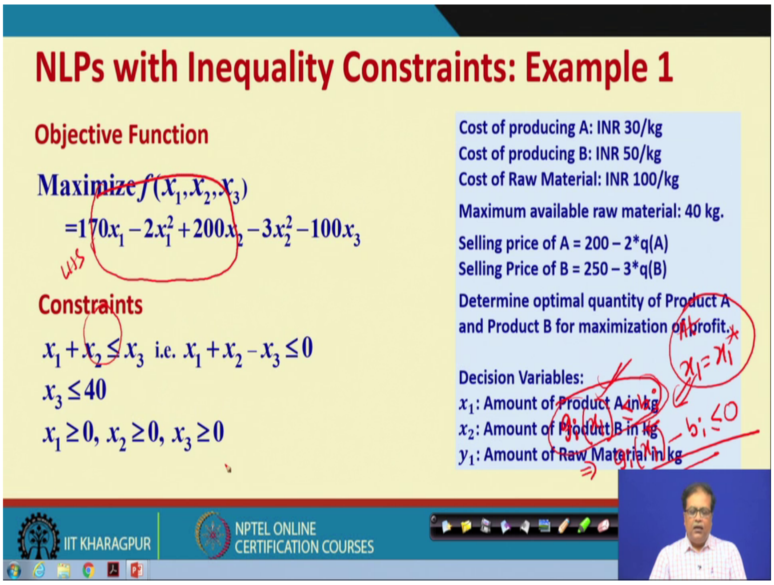

So, in our last lecture, we have formulated the problem and the formulation was like this. Maximize function of x where there are 3 unknowns, the 3 unknowns are amount of product A in kg, amount of product B in kg and the amount of raw material in kg is alright. So, x 1, x 2 and not y 1 that should be x 3 right.

So, these are the 3 unknowns and the constraints where x 1 plus x 2 less than equal to x 3 that is x 1 plus x 2 minus x 3 less than equal to 0, x 3 less than equal to 40 and all the variables greater than equal to 0. So, in our last class we have seen that this particular objective function is we found the Hessian matrix and we found the all the eigenvalues are either negative or 0.

So; that means, you know they were negative semidifinite and we can therefore, conclude that the function is concave and really speaking we can actually find out the optimal point for maximization; so, that much you have already seen in our next class.

Now, the in order to solve that we should know at the optimal point that is x equal to x star, this particular problem I mean the variables should satisfy the KKT conditions. So, that is the necessary condition for optimality. So, let us see how to obtain the KKT conditions for this particular problem.

Now, let us see that. So, the KKT conditions should be like this.

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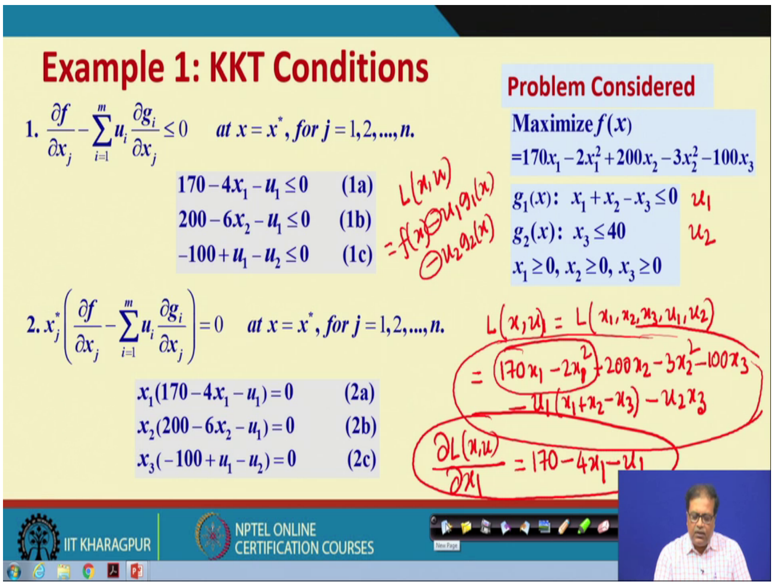

Now, as I have shown you in the previous lecture, a simple way to interpret the first constraint is right out the Lagrangian function the equivalents equivalent Lagrangian function for these, that is we have the f x which is equal to this and we have g 1 x and g 2 x. Let us say we take two multipliers the multiplier u 1 for the first constraint, and multiplier u 2 for the second constraint that is the g 1 x and g 2 x, then the overall function lambda x L of x u 1, u 2 and x n you can write.

So, which is nothing, but L of x 1, x 2, x 3 u 1, u 2. So, that is an total function that will be 170 x 1 minus 2 x 1 square plus 200 x 2 minus 3 x 2 whole square minus 100 x 3 minus u 1, x 1 plus x 2 minus x 3 minus u 2 x 3 right.

So, what we have done? We have combined the objective function and all the constraint into a single function. Now, we differentiate these, we take the partial differentiation with respect to x 1 with respect to x 2, with respect to x 3. So, what will be the del L x u del x 1 that is the first differentiation. So, that should be first of all you see 170 that is the first one, then 4 x 1 that will get from these term.

So, this term these two terms are covered 170 minus 4 x 1 for the rest of the terms we will get nothing, and here we have minus u 1 x 1. So, if you differentiate we get minus u 1 right. So, that will be the partial derivative of these total function with respect to x 1, 170 minus 4 x 1 minus u 1. So, just look that is precisely written 170 minus 4 x 1 minus u 1 less than equal to 0. So, that is the first condition that we obtain by differentiating these function with respect to the x 1.

Now, if you differentiate with respect to x 2, then you see what are some x 2 terms, 200 x 2 so, 200 if you take partial differentiation minus 3 x 2 whole square. So, minus 6 x 2 and any other term here that is minus u 1; so, 200 minus 6 x 2 minus u 1 less than equal to 0 that is the second term. So, what will be the third term differentiate with respect to x 3. So, that is minus 100 and this is plus u 1 x 3 can you see that so, if you differentiate, it will be plus u 1 and if you differentiate minus u 2; So, minus 100 plus u 1 minus u 2 less than equal to 0.

So, I hope you are understood that how we really obtain this, otherwise what you can do you can differentiate f x we can separately differentiate g 1 x and g 2 x with respect to x 1, x 2 or x 3 and then combine them by taking the appropriate multiplier. But anyway if you feel this method is simpler, then you can use and obviously, how to combine the f x and g x whether to use plus sign or minus sign. For maximization problem we usually use minus sign and for minimization problem usually we use plus sign, but then various books may have other kind of conventions.

So, if you look in our one our previous lecture, we have taken plus sign when in a minimization problem. But since the ui's are all scalar values, even if you take another sign ultimately it will be done by wrong I mean the negative value or use that we obtain. But it would not be possible in KKT conditions, because we have strict conditions that all ui's should be greater than equal to 0. So, we have to take the negative sign by making the problem as one of maximization right.

So, this must be remembered that here you cannot substitute these you know. So, when you take the total function L x u equal to f x minus u 1 g 1 x minus u 2 g 2 x. So, here these minus sign has to be always taken right you cannot do something else fine.

So, far we have seen how exactly we take the first condition converted to a set of KKT conditions. Now let us take the second one. The second one is very simple, it is the first condition part only that is 170 minus 4 x 1 minus u 1, multiplied by corresponding variable. So, x 1 into these equal to 0, x 2 into these equal to 0 and x 3 into this equal to 0. So, these are the first 2 KKT conditions right.

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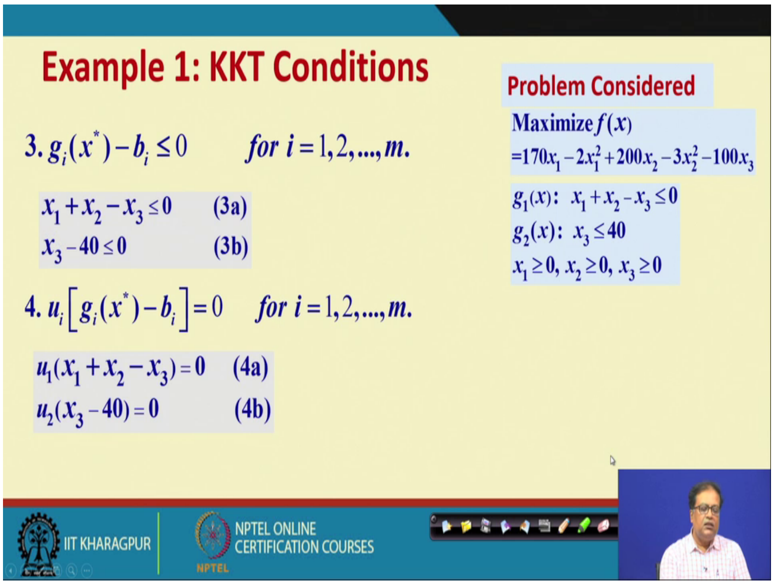

Now, the third one is very very simple really, it is the constraint itself. So, what was the constraint? X 1 plus x 2 minus x 3 less than equal to 0 and this written in this way x 3 minus 40 less than equal to 0. So, this is the third constraint. The fourth constraint is that the corresponding multiplier and this term when you multiply, that should be equal to 0 right. I am not telling this here, but this has the similarity with the duality constraints in linear programming right. Now in duality, they have this complementary slackness conditions which essentially says that either the primal equation will turn to equality or the dual variable should become 0.

So, anyhow if the primal equation terns to equality, then the dual variable exists or if the primal slack exists then the dual variable will not exist, but anyhow which is a similar condition like this, but t hen we that is not exactly what we are discussing here so, same thing here. So, either these slack will vanish from these equation; that means, this will became 0 in other words x 1 plus x 2 should be equal to x 3 and x 3 will be equal to 40 or the corresponding multiplier should be equal to 0. So, these are the third and fourth constraints.

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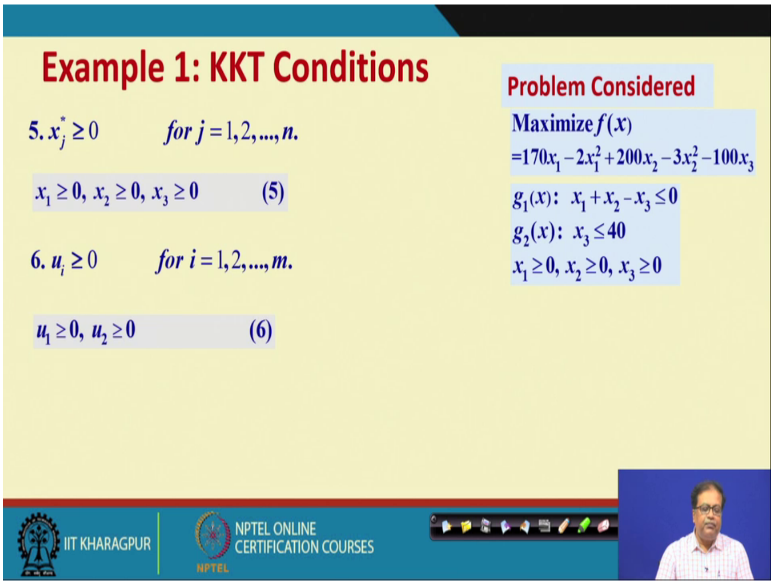

The fifth and sixth constraints are very straight forward, that is these variables x 1, x 2, x 3 and the multipliers u 1 and u 2 should be all equal to 0's.

So, that is all our KKT conditions.

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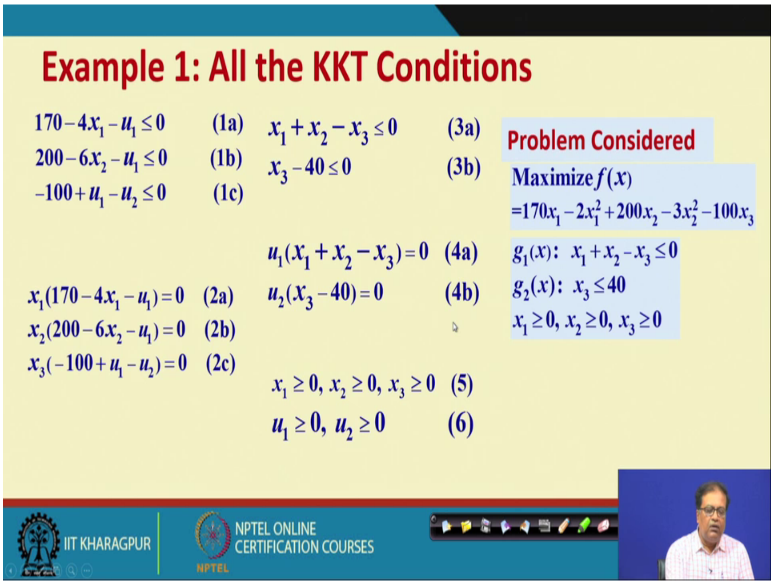

So, all the KKT conditions are put here once again if you if you see. So, these are from partial derivatives, these are that either the you know these equation will turn to equality or the corresponding decision variable should be equal to 0. These are the constraints themselves not the objective function and this is either the constraints will become equality or corresponding multiplier should be 0.

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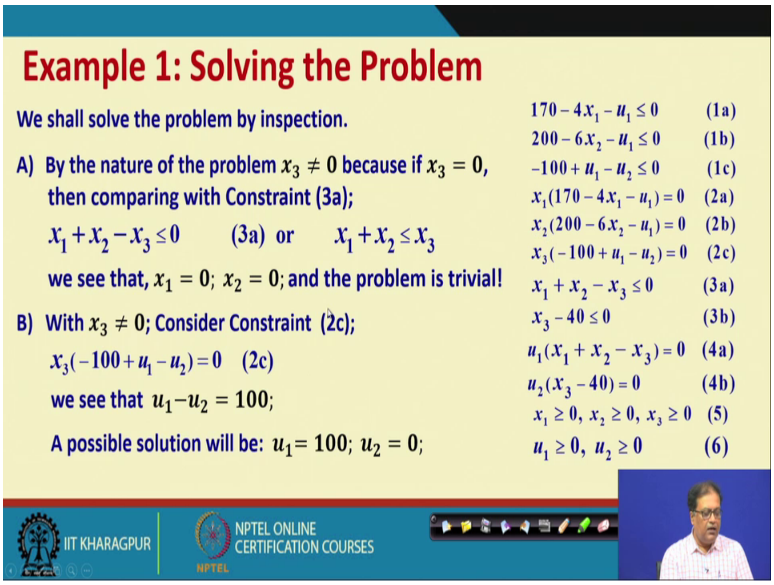

So, those kind of constraints and finally, these are the non negativity constraints.

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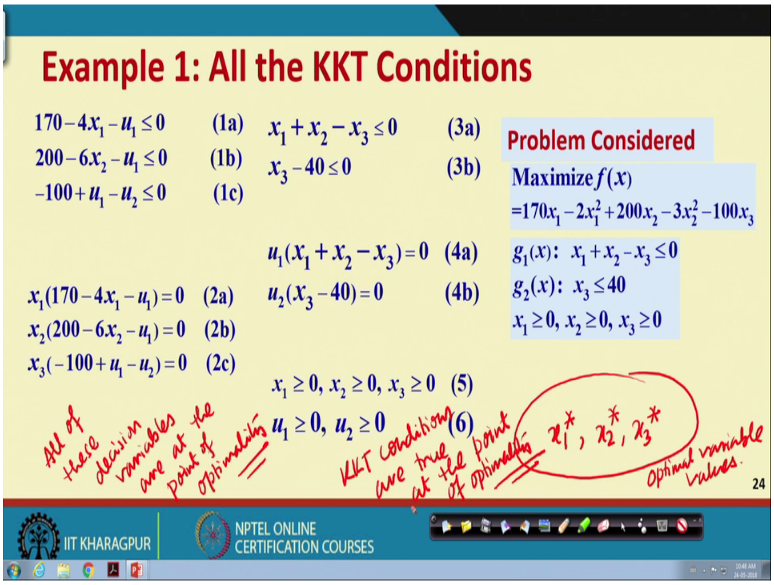

So, these are all the KKT conditions. Now, the question is that as I have told earlier that, really speaking here all the x variables like x 1, x 2, x 3 are really see very important point let me write that all of these constraints sorry all of these decision variables are at the point of optimality right this point has to be very importantly remembered. That means, that really speaking we are real talking about x 1 star, x 2 star and x 3 star right that is the optimal variable values; that means, the KKT conditions are true at the point of optimality, this point has to be remembered.

So, as I am not written, but really they are all actually all these x 1, x 2, x 3 are nothing, but x 1 star, x 2 star and x 3 star right the question is how do I solve this? I have all these conditions so, many of them question if how to solve. See these itself is say challenge in itself really we are not going to go very much into it, because today most of these problems after you obtain the conditions are to be solved by computers.

So, probably you can make use of computers and software packages to really solve this problem, but for simpler problem such as the this we can solve by a method of inspection right. So, I will I will present on inspection method of really solving this. So, let us see how you can solve these by a method of inspection.

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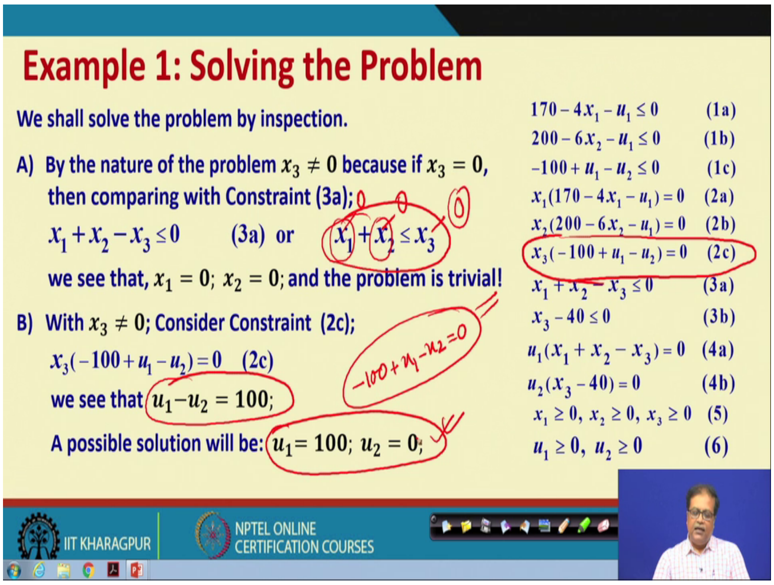

So, first of all by the nature of the problem x 3 cannot be 0, because look here we have a constraint x 1 plus x 2 minus x 3 here less than equal to 0. So, if x 3 is equal to 0 automatically you will see we have a constraint that x 1 plus x 2 less than equal to x 3 if you recall x 3 is the raw material. So, if the raw material is 0, we have no raw material available what does it mean? It means x 1 and x 2 cannot be produced because there is no raw material. So, they also will become 0.

Now, if everything becomes 0, the problem is trivial; that means, the problem vanishes that cannot be the optimal point. So, these therefore, we can say that x 3 will not be equal to 0 is alright now that we know x 3 is not equal to 0. So, look at these third constraint sorry 2 c constraint 2 c, x 3 is not equal to 0.

So, if x 3 not equal to 0 which means what? That means, this term should be 0; that means, minus 100 plus u 1 minus u 2 should be equal to 0 right. So, these equal to 0 that will give u 1 minus u 2 equal to 100 right. So, u 1 minus u 2 should be equal to 100. So, although it is not really just we cannot jump like this. So, we can say a possible solution for these would be u 1 equal to 100 and u 2 equal to 0 right we let us go ahead with this and see if I take these u 1 and u 2 can we solve all of the constraints.

So, precisely that is what we are going to try that.

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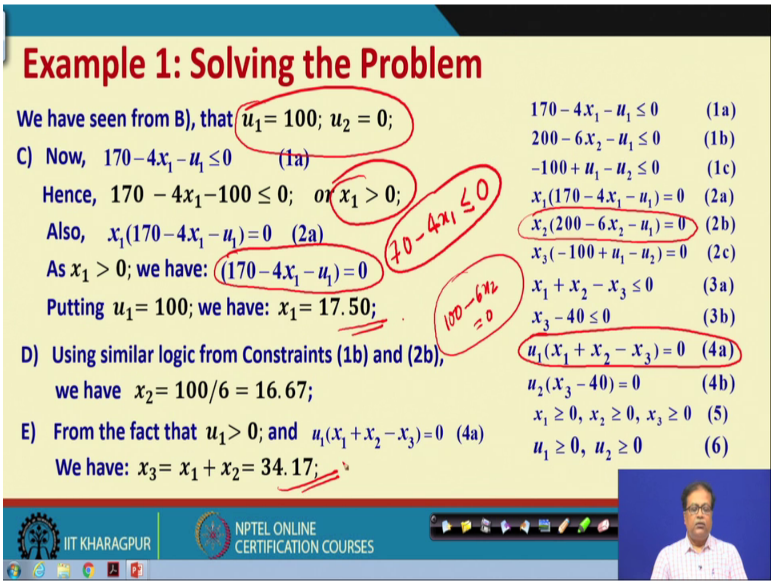

We take u 1 equal to 100 and u 2 equal to 0 right. So, these 2 we take. Now, if we take these if you see that 170 minus 4 x 1 minus u 1 is less than equal to 0 if you take u 1 equal to 100 so; that means, 170 minus 4 x 1 minus 100. So, this is a question constraint 1 a. So, what does it mean? Is means that seventy minus 4 x 1 should be less than equal to 0. So, in other words see let us write it 70 minus 4 x 1 less than equal to 0.

Now, just assume x 1 equal to 0. If x 1 equal to 0 then 70 should be less than equal to 0 which is cannot be. So, therefore, you know x 1 greater than equal to 0. So, because look here x 1 is greater than equal to 0, but x 1 cannot be 0. So, x 1 is greater than 0.

So, really speaking we have proven x 1 equal to 0. So, x 1 not equal to greater than 0. So, now, look at constraint 2 a. The 2 a says x 1 into 170 minus 4 x 1 minus u 1 equal to 0. So, x 1 is greater than 0 right let us say non-negative or non-zero. So, automatically; that means, this second term should be equal to 0. So, you see therefore, 170 minus 4 x 1 minus u 1 should be equal to 0, but u 1 is 100. So, x 1 is 17.5 right. So, this is how we can solve x 1 for these values.

Now, similar logic can be given for 1 b and 2 b right. So, if you look at 1 b and you put u 1 equal to 100 so, 100 minus 6 x 2 less than equal to 0. So, x 2 cannot be 0. If x 2 equal to 0 then 100 should be less than equal to 0, that is impossibility. So, therefore, x 2 is greater than 0 now look at constant 2 b. So, this is constant 2 be x 2 greater than equal to greater than 0 so; that means, if these 2 terms should be multiplied as 0. So, 200 minus 6 x 2 minus u 1 should be equal to 0.

Now, 200 u 1 is 100. So, 100 minus 6 x 2 equal to 0. So, what will be the value of x 2 the value of x 2 will be 100 by 16.67 is alright. Now similar logic can be given in 4 a also, look at 4 a u 1 into x 1 plus x 2 minus 3 equal to 0 here u 1 is greater than 0. So, you know we get x 1 plus x 2 minus x 3 equal to 0 or x 3 equal to x 1 plus x 2 equal to 34.17 right.

So, we have got all of them that is x 1, x 2 and x 3 values.

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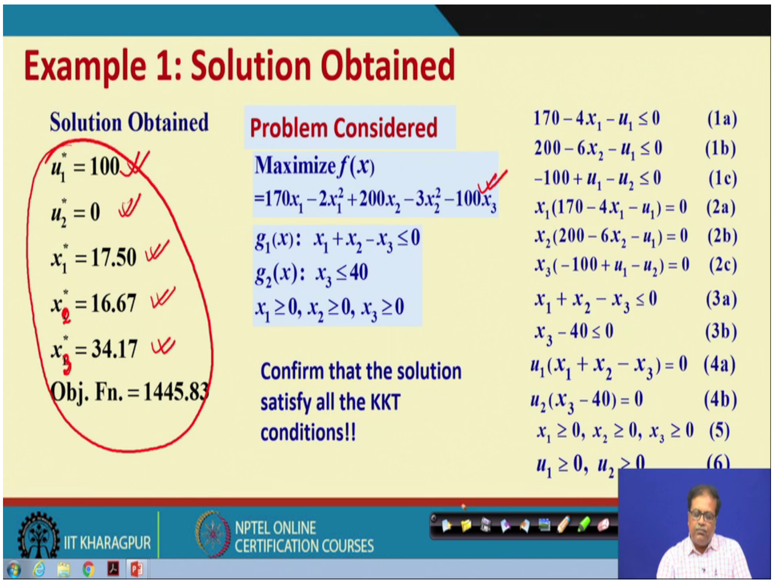

So, we found that these setup solutions that is x 1, u 1 star equal to 100, u 2 star 0, x 1 star 17.50, x 2 star sorry equal to 16.67 and x 3 star equal to 34.17; you know they are actually solutions, which incidentally you know satisfy all the KKT conditions. That see look here we have really not solve the problem, we have obtain them by inspection. So, since we have a obtained by them by inspection, we should ensure it further.

So, like if you try to ensure if you put x 1 star 17.5 and u 1 equal to 100, you see t hat these become 0. So, it is satisfy second one is also satisfied third one yes next 3 yes, this one yes, this one yes, this one u 1 is 100 and this one is 0 yes u 2, x 2, x 2 equal to 40 yes that is also yes so; that means, all the conditions are satisfied; that means, a these trial solution really satisfies all the KKT conditions right.

So, we also seen in your previous lecture that the function is concave we obtained Hessian matrix and we saw that theirs negative semi definite. So, combining these results we can say that this is our optimal solution and the objective function comes out to be 14 45.83 right. So, we have solve this problem. So, in the remaining time, let us quickly check 1 more problem just for confirmation our ideas and we shall see how to quickly really solve such problems.

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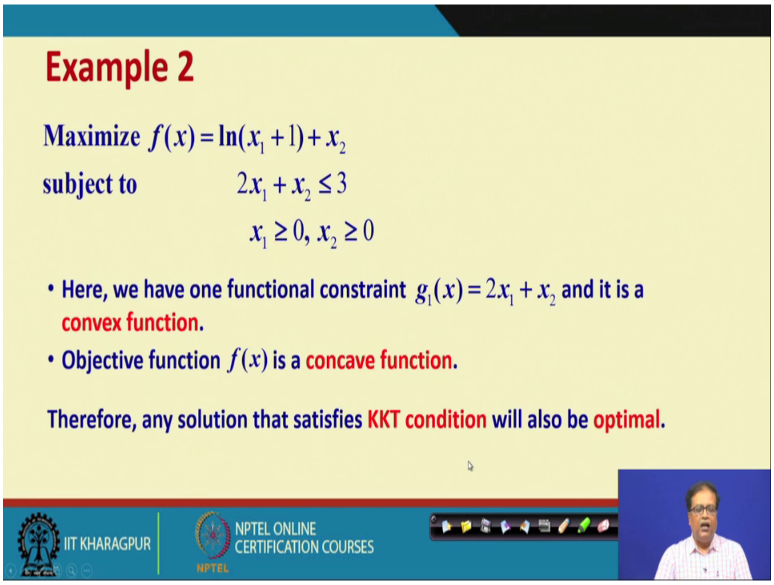

Supposing I have a problem taken from Hillier and Lieberman book, f x equal to l n x 1 plus 1 plus x 2 subject to 2 x 1 plus x 3 less than equal to 3, x 1 less than equal to 0 and x 2 less than equal to 0 right. So, we have one functional constraint and it is a convex function why because it is linear.

So, necessary conditions requires that f x should be concave for a maximization problem right. So, it is so, that this is a concave function we are not proving it, but we know that yes it is. So, therefore, the necessary the sufficient condition is taken care of with concave objective function and convex functional constraint. So, if we find a solution that satisfies the KKT conditions that will be optimal is it alright. So, that is the starting point.

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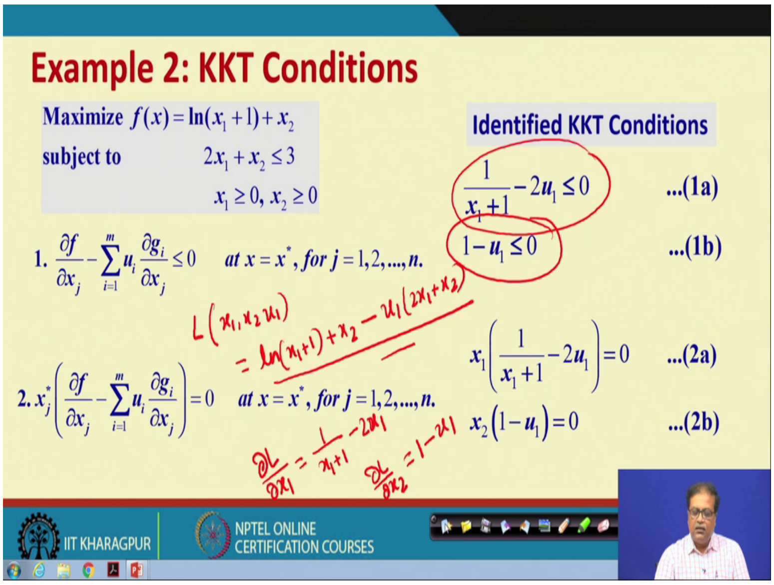

So, let us see how we obtain the KKT conditions. So, these is if I combine the 2 then again we have, we have L x 1, x 2, u 1 equal to ln x 1 plus 1 plus x 2 minus u 1, 2 x 1 plus x 2. So, this is the combined function. So, if you if you differentiate with respect to x 1. So, that is del L del x 1 equal to ln of this function.

So, 1 by x 1 plus 1 that is the partial differentiation, this term will give nothing and this term will give to u one. So, that is that is the first condition. So, this is the del L del x 1 what is del L del x 2? Del l del x 2 will give you know this term will give nothing this term will give 1 and this term will give minus u 1 right.

So, that is the second condition. So, these are the second 2 conditions are identify from these we find x 1 into these term, this is and x 2 into these term they are equal to 0 is it alright. So, this is how the second condition can be obtained also that is x 1 into these and x 2 into these. So, that is the second conditions the third condition can be obtained by the constraint itself right.

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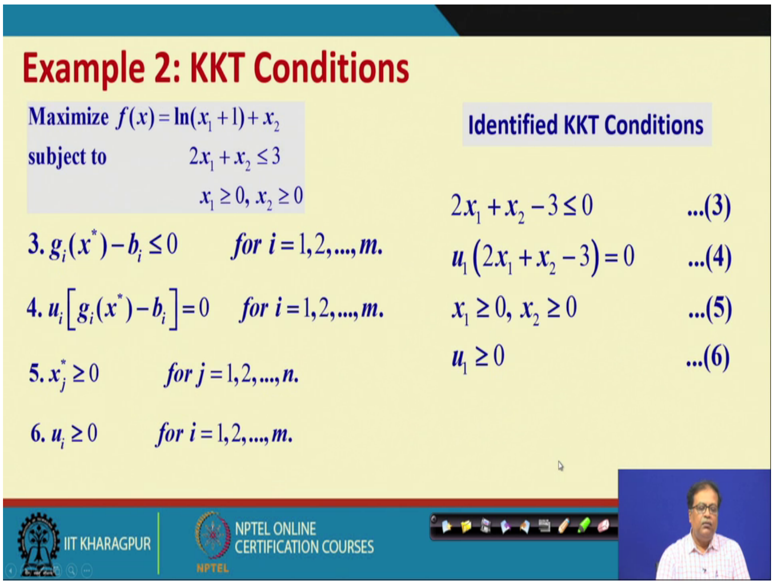

So, what is the constraint itself? That is 2 x 1 plus x 2 minus 3 less than equal to 0 that is the third constrain and what will be the fourth? Fourth is multiplied by the multiplier should be equal to 0 and then fifth and sixth conditions will be that all these term should be non zero so, non negativity constraint right.

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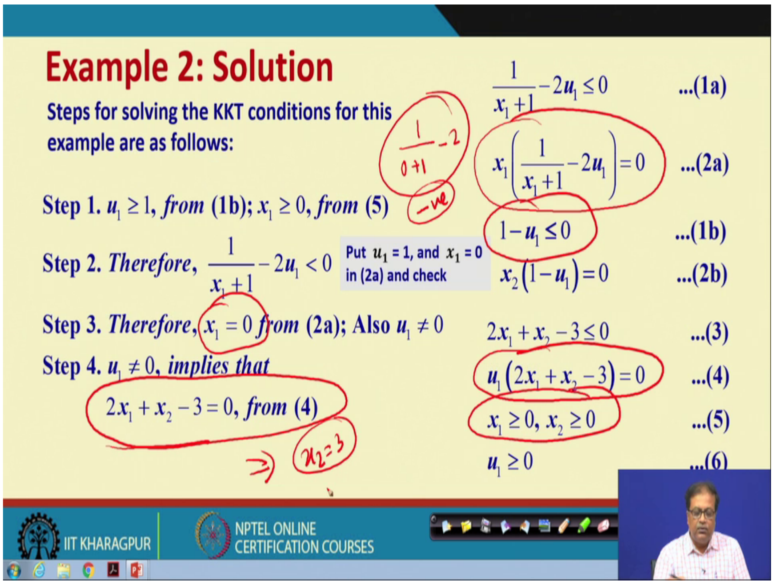

So, we have obtain all the six KKT conditions and those six KKT conditions can be written here. So, I have all the KKT conditions.

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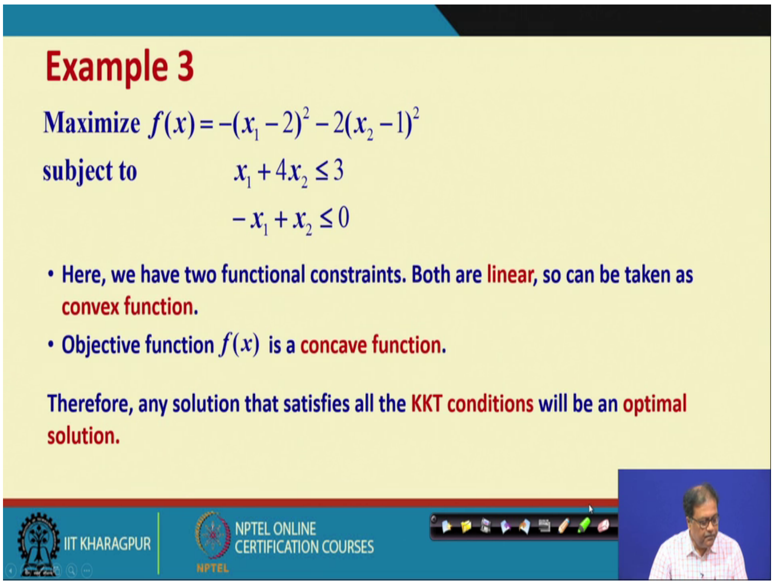

So, these are all the KKT conditions all the KKT conditions.

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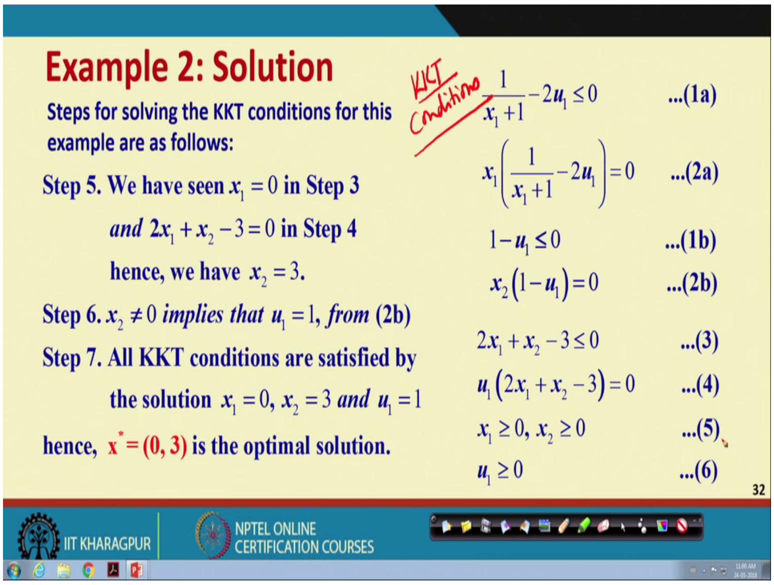

So, look here 1 plus x 1 plus 1 minus 2 u 1 less than equal to 0, x 1 into this equal to 0 1 minus u 1 you know less than equal to 0, that is x 2 into 1 minus u 1 less than equal to 0 yeah the and then. So, that is the second part that is del 1 del x 2 and 2 x 1 that is the constant itself, this is the you know multiplied by the multiplier and these are the non-linear negativity constraints. So, all the KKT conditions will be written on 1 side.

So, now, we can see them then how to solve these problem. So, first of all we have to really find out as set of solutions, then see now again we are solving basically by inspection. So, these logic you have to find out really speaking here some logic is given, but it does not mean that these logical is you know universal and all problems similar logic we will hold right. So, really solving these problems from these point onwards, basically is a task non trivial task we have these constrain we have to really obtain a set of values x 1, x 2 and u 1 which will really give a solution.

So, a quick solution we are just trying out nothing else. So, step 1 if you look at 1 b, then you see u 1 should be greater than equal to 1 right. So, this is what you find and. So, see if I see this constraint and if I see number 5, that is x 1 should be greater than equal to 0. So, we see that u 1 is greater than equal to 1 and x 1 is greater than equal to 0.

So, suppose I put u 1 equal to 1 and x 1 equal to 0 in 2 a. So, this is 2 a. So, put x 1 equal to 1. So, you know x 1 equal to 0 and you 1 equal to 1 what you get? You get 1 by 0 plus 1 minus 2 right. So, you see these are like floor values. So, if I put floor values can you see that this is negative right. So, you see in 2 a by putting u 1 equal to 1 and x 1 equal to 0, I find that this value which within the bracket is actually negative right.

So, it actually is negative. So, now, any x 1 value higher than 0 right and any u 1 value more than 1 will be even words. So, basically as long as these 2 conditions are satisfied that is u 1 is greater than equal to 1 and x 1 is greater than equal to 0, this term will be always negative right this term will be always negative because the lowest values itself it is negative.

So, any higher value of x 1 will become this will be even lower, at any higher value of u 1 these value will be even higher. So, these will be always negative. So, that is what we found out at step 2 that this is negative. So, if this is negative; that means non zero. So, 2 a says x 1 should be 0 then right. So, x 1 should be equal to 0 also u 1 is greater than equal to 1 so; that means, u 1 should not be equal to 0.

So, now look at the fourth condition, u 1 is not 0; that means, 2 x 1 plus x 2 minus 3 equal to 0 right. So, we get these facts that x 1 equal to 0 and 2 x 1 plus x 2 minus 3 equal to 0. So, what does it mean? It means x 2 3 why because x 1 equal to 0. So, x 2 should be equal to 3 right. So, that is how we have been able to obtain the trial solution. So, we have seen x 1 equal to 0, we have seen 2 x 1 plus x 2 minus 3 equal to 0. So, we have x 2 equal to 3 and we already seen u 1 equal to 1 right. So, since x 2 not equal to 0 so, we can take these condition, these condition we can take x 2 not equal to 0.

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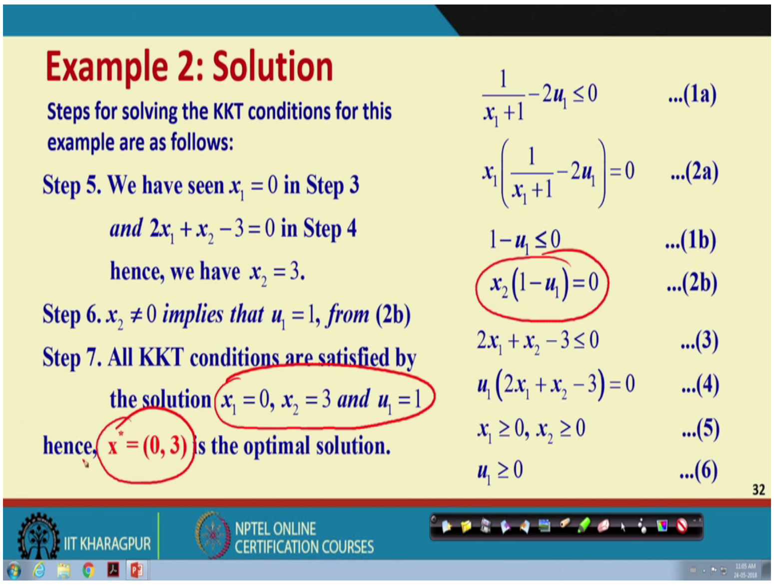

So, if x 2 not equal to 0, then u 1 should be 1. So, all the KKT conditions are now satisfied with x 1 equal to 0, x 2 equal to 3 and u 1 equal to 1 is it alright. So, the optimal solution will be then x star equal to 0 3. So, anyhow in these lecture we have further seen how to solve problems by making use of KKT conditions right.

So, in our subsequent lectures, we shall see more examples on non-linear programming problems right so.

Thank you very much.