Selected Topics in Decision Modeling
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Lecture - 14
Integer Linear Programming

So, in our subject selected topics in decision modeling, we are now at lecture 14 on Integer Linear Programming. So, on integer linear programming we have so, far discussed how to formulate integer programming problems, a number of examples we have seen now in this lecture let us go to the solution methodology.

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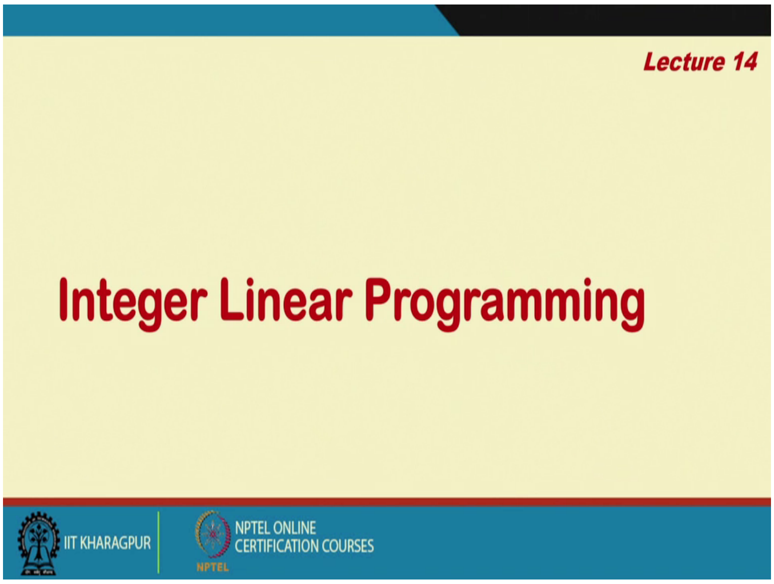

So, there are different methods for solving integer programming problems, mainly integer linear programming problems right.

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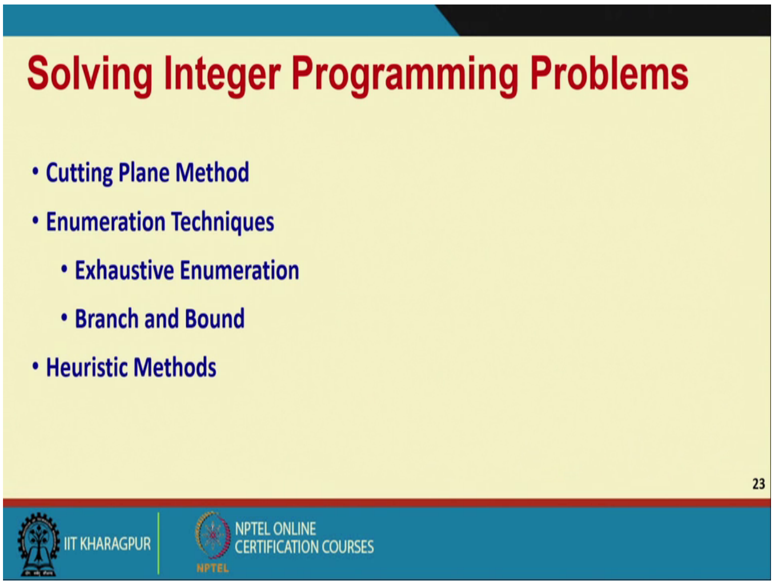

So, with regard to integer linear programming problems the ILP problems, there are several methods. The first method that we shall discuss can be called as cutting plane method, is it alright. The second method is a kind of enumeration technique and there are two very important enumeration techniques we shall discuss, one is called exhaustive enumeration and the branch and bound technique is alright and finally, the heuristic methods.

So, these are the methods that we shall discuss about solving integer programming problem.

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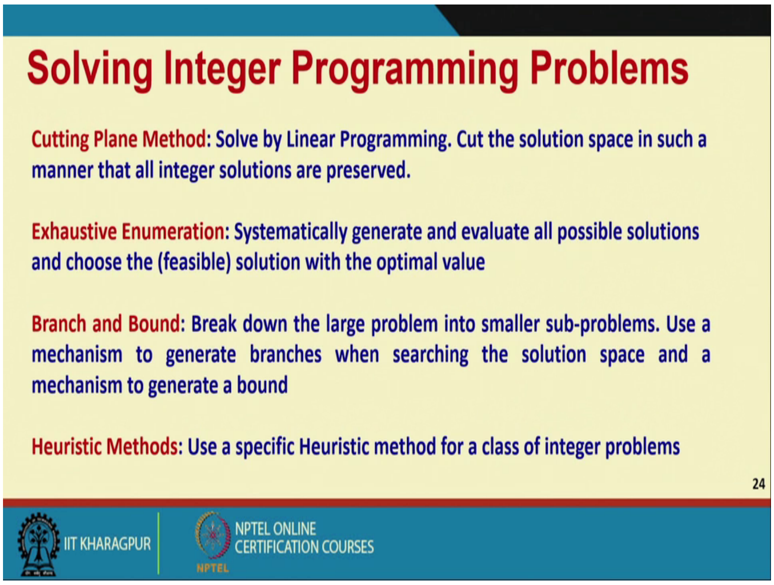

The first method that is the cutting plane method is like, this that first of all solve the problem by linear programming and cut the solution space in such a manner that all the integer solutions are preserved.

So, what is this cutting plane method is that you know we have to solve an integer linear programming problem. So, first solved by linear programming method in the usual way get the solution and then find out in that solution all the set all sets of integer solutions. And find a clever method by which all those integer solutions are preserved is it alright while you cut the solution space you should cut minimally, you know your integer solutions are all intact and your solution space your cutting minimally.

So, that that the current optimal solution is outside your solution space; that means, you have already cut a certain portion of the solution space the minimum portion, where the current optimal solution is excluded. Then again solve the problem by linear programming and you know find the solution, if it is integer the final optimal solution is obtained. And if it is not integer then continue the process. So, this is a cutting plane method.

The second method the exhaustive enumeration is systematically generate and evaluate all possible solutions and choose the solution with the optimum value; obviously, the feasible ones. The problem here on the exhaustive enumeration is that you know if you try to find out all possible solutions of a given problem, you know the number could be very very high. First of all you have to generate them you have to identify the solutions for them by some method.

And finally, you know see which one is the best out of all possible solutions. For a small problem yes exhaustive enumeration is a viable technique, but for large problems definitely not it is alright. So, the use of exhaustive enumeration as a method is highly limited we shall not spend too much time on exhaustive enumeration technique. We shall the other move over to the other very important technique called the branch and bound method. You see the branch and bound technique is a technique, which breaks down the very large problem to smaller sub problems, but then there is a difference.

See we have solved dynamic programming problems in our initial lectures, we also divided a large problem into a series of sub problems, but you know those sub problems you know where different different in the sense that the a particular sub problem in dynamic programming, you know was encompassing, the entire problem, the next sub problem is taking out certain portion, next sub problem is taking out certain problems certain portions and they were all connected by a recursive relationship.

This is not so, in branch and bound in branch and bound the sub systems will be mutually exclusive, but collectively exhaustive is it alright. And, once we are able to divide our total problem into such sub problems and then we basically we generate branches right the each branch holds a sub problem. So, suppose original problem you dividing into 3 sub problems, then initial branching will be 3. And, then after then you know you should be having a method to generate a bound. Bound means the best possible solution no it is a minimization problem say lower bound and a maximization problem a upper bound.

So, those bound should be possible to obtain and then out of every n nodes, you know you have to find what are the bound values and from there it should be possible to obtain the optimal solution is alright.

So, it is an involved technique, but very powerful technique which guarantees an optimal solution in due course of time we shall see the branch and bound technique. Lastly the Heuristic Methods use a specific Heuristic Methods for a class of integer problems. Mostly we shall discuss the travelling salesman problems for heuristic methods please also understand the meta heuristics such as genetic algorithm they are all heuristic methods definitely, but we shall not discuss those problems here, because we have a completely separate set of lectures on meta heuristics right.

So, apart from those evolutionary computing things other heuristic methods we shall discuss in due course of time right. So, these are the methods that we shall discuss for solving integer programming problem. Let us now go ahead with the first one, that is the cutting plane method, but before even cutting plane method let us do a little revision on the linear programming itself right.

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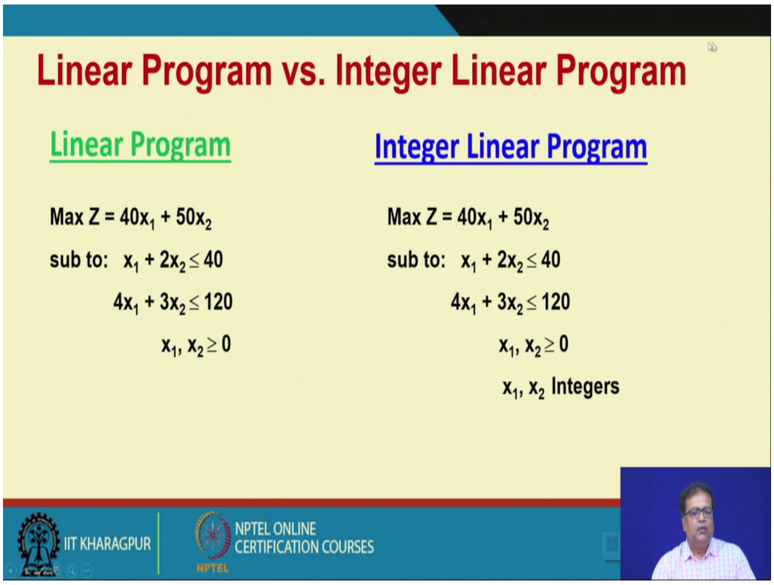

Let us take a very simple linear programming problem; you know the linear programming problem takes a form as you can see here the maximize Z let us say with 2 unknowns X 1 and X 2 is a function let say 40 X 1 plus 50 X 2 subject to let say 2 constraints X 1 plus 2 X 2 less than equal to 40, 4 X 1 plus 3 X 2 less than equal to 120 and X 1 and X 2 greater than equal to 0. The integer linear programming for the same problem is an additional constraint, that X 1 and X 2 are integers.

Now, point to be made here look here the linear programming problems are mostly solved by the simplex algorithm. Before even understanding simplex solution method a quick review, you see most of these problems that we shall discuss should be possible to solve graphically as well, but you must remember that you cannot solve all linear programming problems graphically. You know 2 by 2 S 2 variable problems, 3 variable problems, we can still think of, but more than 3 variables it becomes almost impossible, you know really impossible to solve through graphical techniques right.

So, in such situations the simplex remains the only way possible. So, in the future courses I mean future course of action where, I am going to show you a number of graphical solution methods, please remember I am also going to show you the simplest solution of the same. The graphical part is for understanding for larger problems the graphical solution is excluded; reason is very simple more than 3 by 3 that is more than 3 unknowns the graphical solution is not possible it is alright.

So, in that case simplex remains the only solution method apart from other techniques like branch and bound and so on.

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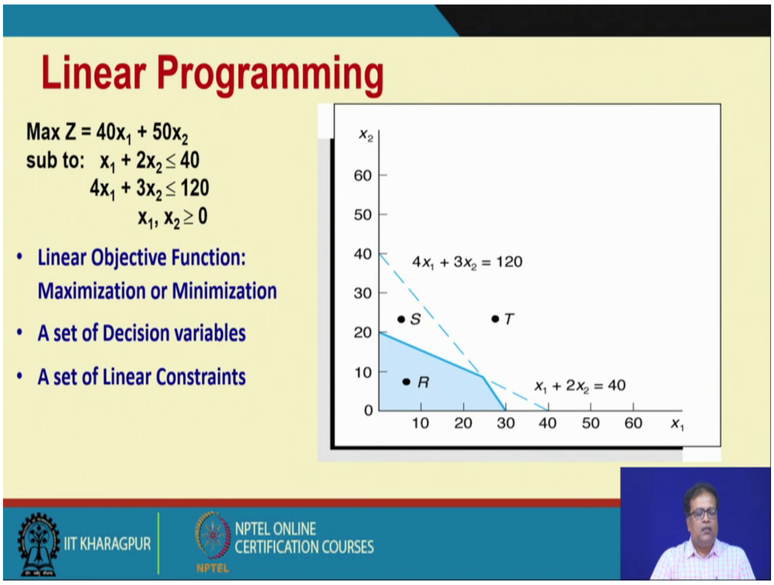

So, the graphical method if we have to solve this problem, then what we do really you know we find out the solution space of this problem, but by drawing the constraints right. By drawing the constraints, how do we draw the constraints are in this manner? That is you know the first one; that is the X 1 plus 2 X 2 less than equal to 40.

So, what we do you see this constraint line which is from 20 to 40, we draw a line. And as you draw this line these line is drawn from let us say you know if X 2 equal to 0 then X 1 is 40 and X 2 equal to 0, then X 1 equal to 20. So, a constraint line is drawn. Now, if I put 0 0 it is satisfied. So, 0; that means, the origin is included; that means, all the portions you know below these line within the X 1 and X 2 axis is covered by this constraint.

Similarly, the other constraints that is 4 X 1 plus 3 X 2 less than equal to 120 is also drawn and all these area you know R plus S, that is covered in the this particular constraint 4 X 1 plus 3 X 2 less than equal to 120 and this line represents 4 X 1 plus 3 x 2 equal to 120. So, the shaded area is then becomes our solution space is alright, that is our solution space.

Now, you know a beautiful understanding that we have about linear programming is that the optimal solution can be found out from one of the basic feasible solutions. What are the basic feasible solutions? Those are the corner points of this solution space right.

So, these are the feasible corner point solutions that is 0 0 0 20 30 and these point where the 2 lines cut each other. Then what we do we find out the objective function value at these 4 points and then the point that returns the highest possible value is our optimal solution is it alright. So, that is how we can solve optimal such linear programming problems graphically.

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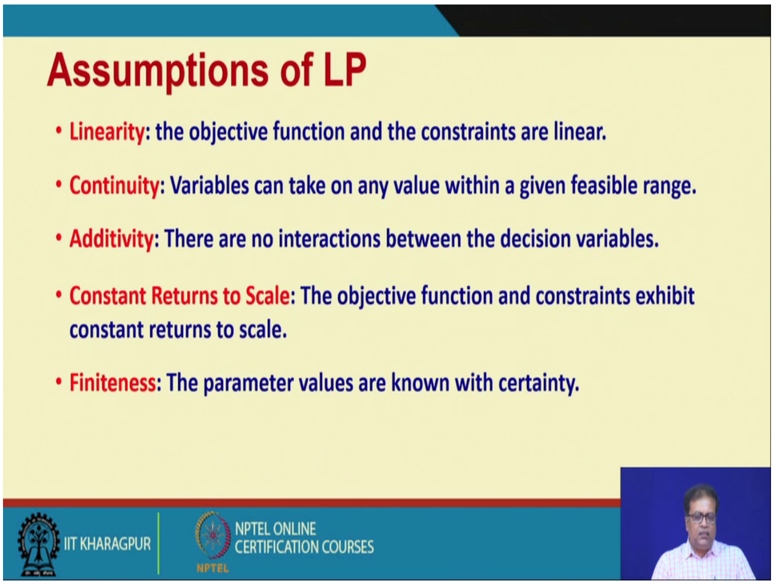

So, these, but then you know understanding of linear programming problem requires the understanding of the assumptions. What are the assumptions the first assumptions the objective function and the constraints are linear let see first. That means, you know all these constraints they are should not be higher order terms.

For example, it should not be X 1 square plus 2 X 2 less than equal to 40 you know. If you have higher order terms, then the they are not linear, you cannot represent them by straight lines, and if you cannot represent the constraint lines by straight lines then the fundamental thing that one of the corner points the feasible corner points will be optimal, that will not be valid. See, even if let us say the individual constraints are all linear and you get a solution space, but if the objective functions is not linear then also this will not happen.

So; that means, the objective function should be linear the constraint should also be linear that is the basic requirements of linear programming problems. The all the constraint should be linear and all the objective function the objective function should also be linear continuity assumption the variables, can take on any value within a given feasible range right. So, supposing what is the feasible range of X 1 and X 2 values. So, here you see the X 1 value could be anywhere between 0 to 40, but feasible range is 0 to 30 right.

So; that means, while X 1 feasible range is 0 to 30 X 2 feasible range is 0 to 20 and any value within that it should be possible say 10.2 should be possible right 0.9 should be possible, but then; obviously, they are not corner points. So, we shall not consider them, but theoretically yes the continuity assumption said that the solution has no restriction for X 2 have any value between, you know 0 to 30 in this particular case you know that could be a possible solution. So, that is the continuity assumption.

The third one Additivity; there are no interactions between the decision variables. You know; that means, that you know we can add the X 1 plus X 2 sometimes what happens you cannot add apples and oranges is not, but in this case the constraints are written in such and way that you can let say if a resource is 40, then we can add certain portion of X 1 and certain portion from X 2 right.

So, this should be possible that is the essential idea of additivity. And, then constant returns to scale the objective function and the constraint exhibit constants returns to scale, what does it mean is that you know you see supposing I have a particular profit for

X 1 let say 40 is it ok. For 1 item it is 40, for 2 items it becomes 80, for 3 items 120, 4 items 140 that is a constant returns to scale.

If, it is not sometimes you see in realistically 1 item did not give you a profit of 40 maybe a profit of 20, but if you have 2 item or 3 item or more item maybe you can make more profit or in a different place maybe for 1 item you can have more profit, but 2 items you give a discount and then you reduce your profit. So, the constant returns to scale may not be true in certain situations, but if that is so, then linear programming assumptions will not hold it is right.

Lastly the finiteness the parameter values must be known with certainty right. So, that is finiteness. Now, tell me if I translate a linear programming problem to an integer linear programming problem, which of these assumptions will not hold, think over it should be the continuity assumption.

The variables can take any value within a given feasible range this is not going to hold right. The continuity assumptions will not hold because the variable will not be able to take any value within the given feasible range it will take only the integer values it is alright.

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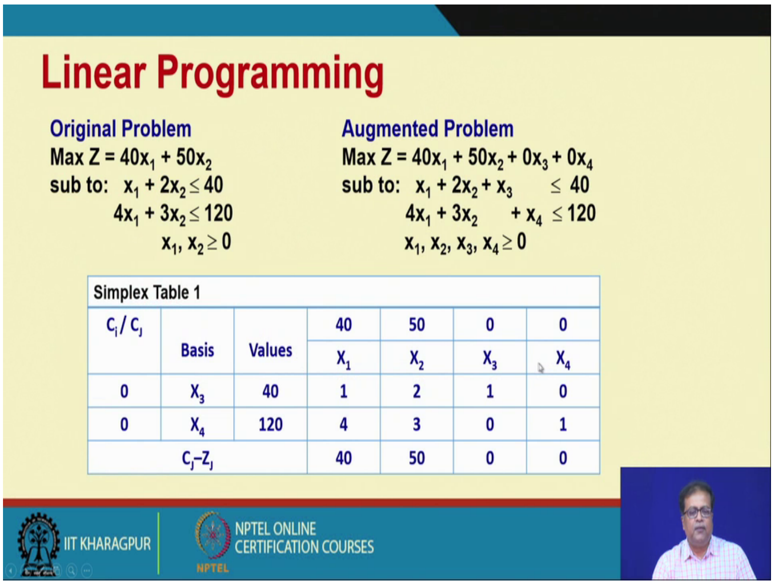

So, that is the essential idea of an you know the integer linear programming problem the continuity assumption will not hold.

Now, let us see how to solve linear programming problem by simplex algorithm. So, we took a very simple problem the same one. So, what we do you know from the original problem we write an augmented problem, what is that augmented problem? We take two more variables, because they are all less than equal to.

So, you know we take that actually this sign should be equal to equal to they are not less than equal to this should be equal to. So, you know they augmented problem than 40 X 1 plus 50 X 2 plus 0 X 3 plus 0 X 4 subject to X 1 plus 2 X 2 plus X 3 equal to 40, 4 X plus 1 plus 3 X 2 plus X 4 equal to 1 0, and X 1 X 2 X 3 X 4 less than equal to 0.

And, then you know we put X 3 and X 4 the you know the variables, which forms an unity matrix you see X 3 X 4 they form an unity matrix. They become the basis and the values are 40 and 120 what we have there? And then we write look here this unity matrix is here with the slag variables and the coefficients we write and we write C J minus Z J values, you see that C i values; that means, the 0 and 0 their profits associated with X 3 and X 4 and 40 and 50 profit associated with X 1 and X 2.

So, the total cost making 1 unit of X 1 will be you give a 1 unit of X 3 and 4 unit of X 4 at a cost of 1 into 0 plus 4 into 0. So, 0 cost forty is your per unit profit. So, C J minus Z J is the net profit per unit of X 1 is 40 and 50 is a net profit per unit of X 2 right. So, this is how we write the first simplex table.

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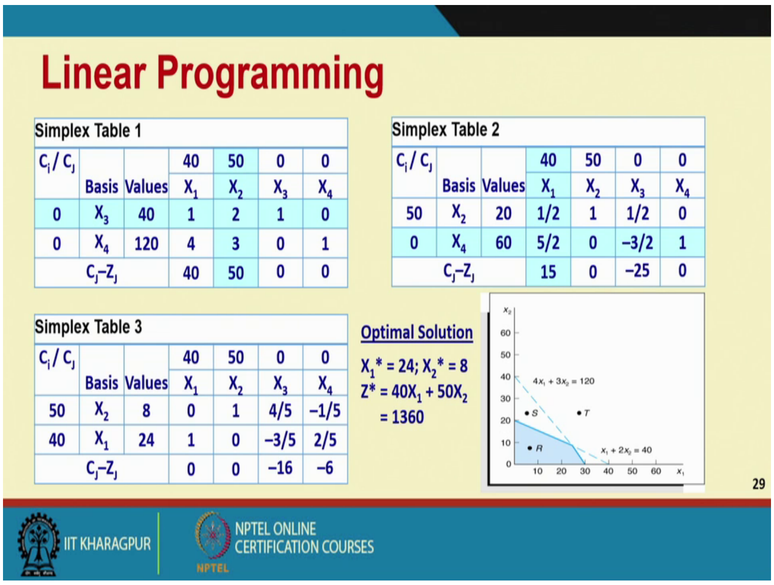

So, once we write the first simplex table then from the first simplex table we find that out of X 3 and X 4, which one becomes a bottleneck resource.

Look here you know we have more profit from X 2. So, X 2 becomes our entering variable and which one is the living variable with X 2 request 2 units of X 3 and 3 units of X 4 is alright. So, with 40 units of X 3 available we can make 20 units of X 2, but with 120 units of X 4 available we can make 40 units of X 2, but can we make 40 because we need both X 3 and X 4. So, which 1 is a bottleneck resource the bottleneck resource will be X 3 here, why because if we make 20 units of X 2 I require 40 units of X 3 and 60 units of X 4 which I have.

But, if I make 40 units of X 2 then I will need 120 units of X 4 which I have, but 80 units of X 3, which I do not have. So, this means we can only make 20 units of X 2. So, that is how we fill this table that this should be 20 that is 40 divided by 2 is 20 right. So, it should be 202 divided by you know 2 that is the key element. So, it become half 1 and then you know these becomes 1 these 1 will become half this 2 will become 1 this 1 will become half this 0 will become 0.

For the remaining units there is a simple formula the what is that formula this 120 will become 120 minus 40 into 3 divided by 2 is it alright. So, you just look at these highlighted areas so, 120 minus forty into 3 by 2 that is 60. So, that is how the 60 is filled up you know like this we fill up all these variables. So, this 4 will become 4 minus 1 into 3 by 2. So, 5 by 2 and since now X 2 has come in the basis these when remaining values will be 0 is alright.

So, like this we fulfill this table and we get the simplex table 2 right. And, then from there we obtain the simplex table 3, you know by suitably transforming these variables now again X 4 becomes the bottleneck resource. So, you know X 4 is now the living variable and X 1 will be now the entering variable.

So, what will be the value X 1 will be sixty by 5 by 2 24 and X 120 minus 60 into half by 5 by 2 equal to 8. So, X 1 equal to 24 and X 2 equal to 8, that is our final solution because look here it is a maximization problem X 1 and X 2 are in the basis; obviously, they are C J minus Z J value should be 0 and X 3 and X 4 they are not in basis. So, they are all minus. So, once we have all negative C J minus Z J; that means, optimal solution is obtained and the optimal solution is X 1 star equal to 24 X 2 star equal to 8.

So, what will be our z value z value will be 1360 and this is indicated here also. So, this point is our optimal value right. So, I am sure that I have gone hurriedly about explaining the linear programming, but that is not our real purpose here, I am sure you know linear programming, but in case you have some difficulty you know you can see maybe the beginners lecture, where we have discussed a little bit on how to solve linear programming in more detail right?.

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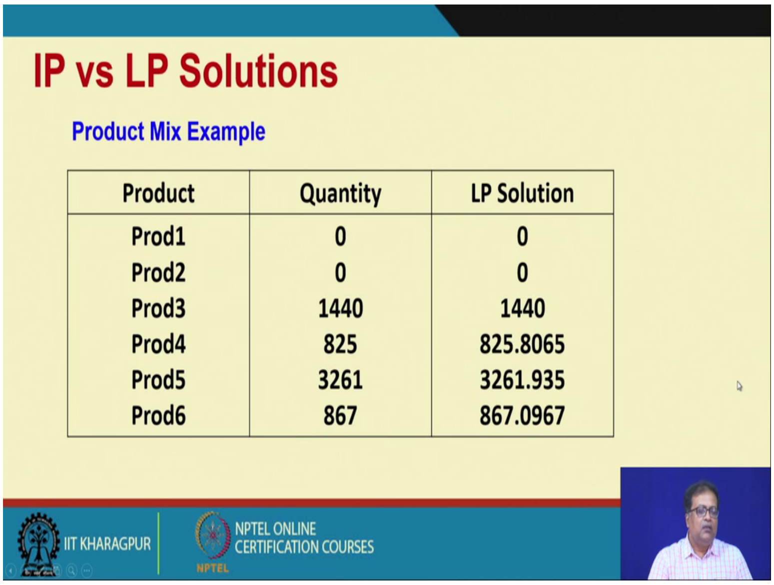

Product	Quantity	LP Solution
Prod1	0	0
Prod2	0	0
Prod3	1440	1440
Prod4	825	825.8065
Prod5	3261	3261.935
Prod6	867	867.0967

So, once we know how to solve linear programming problem one thing should be understood IP versus LP solutions how do they compare?

Suppose in a product mix problem the LP solution is 0, then we can take 0 as the product 1. If LP solutions in integers there is no problem, but if LP solutions is something like 825.8065 or 3261.935 or 867.0967 look these are very high numbers. So, a simple rounding off to 825 326 71 and 867 should not post much of a problem is alight.

So, the very simple thing that is rounding off should be possible in any situations. So, where the number involved are large enough for a rounding off to be meaningful we can definitely use the LP solution as our IP solution right.

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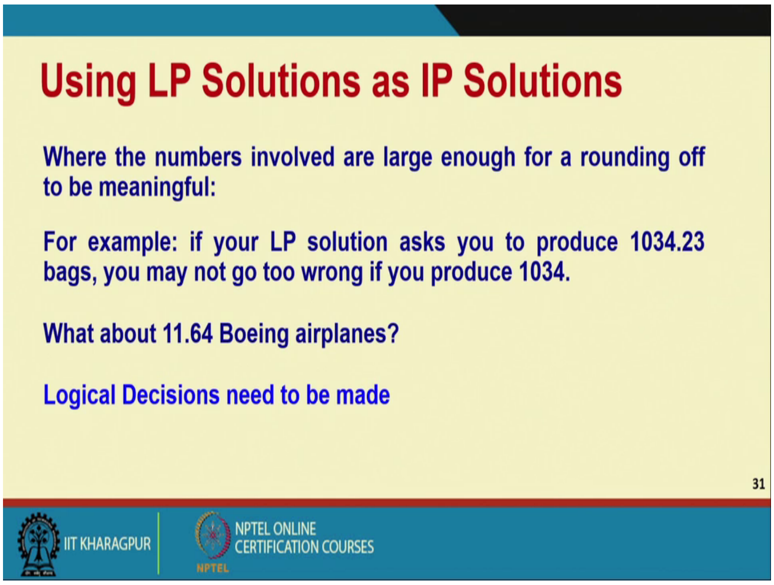

So, if your LP solutions asks you to produce 1034.23 bags you may not go to wrong if you produce 1034, what about 11.64 Boeing airplanes? Look here you know it is very difficult then, because you know 11 Boeing Airplane and 12 Boeing airplanes; obviously, makes a lot of difference is alright. So, every situation is not same. So, we have to be careful, which is the solution what we get here?

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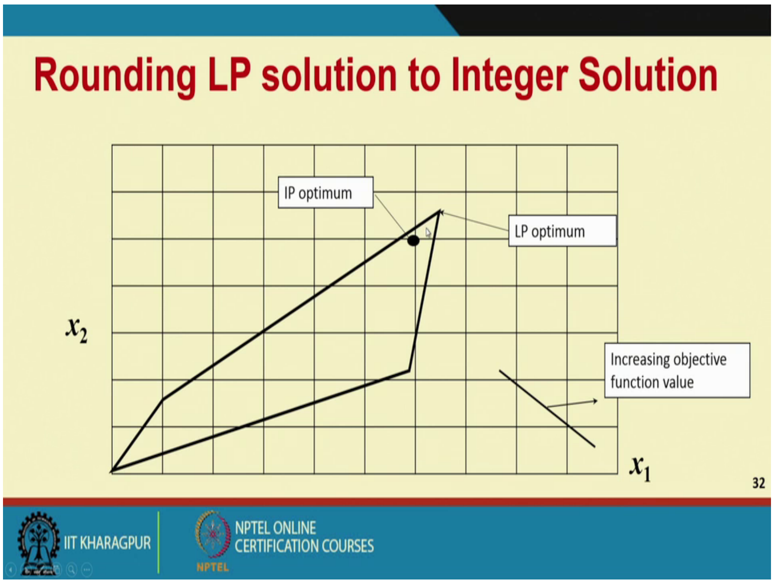

Look here rounding LP solution to integer problem suppose this is our solution space and this is our LP optimum and this is our IP optimum, now this is the integer solution. The

LP and IP they are very near and you know as you move this side objective function value increases. So, it is clear that since there is no other integer solution inside and this side is our profit is increasing all nearby integer solutions are worse than this it is alright. So, we can say yes this is IP optimum we accept it they are should not much of a difficulty.

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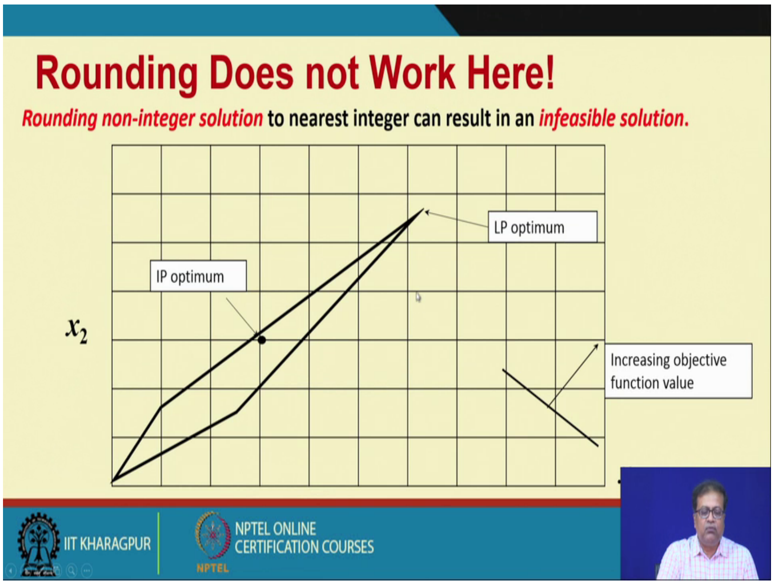

But, look here look at this particular solution space see the LP optimum is here and IP optimum is somewhere here there are so, many IP solutions integer solutions nearby, but non-feasible.

See if the solution space is visible if you draw could draw a graph ah; obviously, your understanding is ok, but supposes you really cannot draw the graph you are solving by simplex how do you know exactly which one is feasible, which one is not feasible you know difficult to know also. So, under such situations rounding off if you round of these value you get either this or this you know or these are these are anything nearby they are not at all feasible right.

So, you cannot round off and get from LP optimum to IP optimum as you have done here look if you round of these you get this there is no problem, but here if you round of these you do not get this right. So, it is not that rounding of works all the time is alright.

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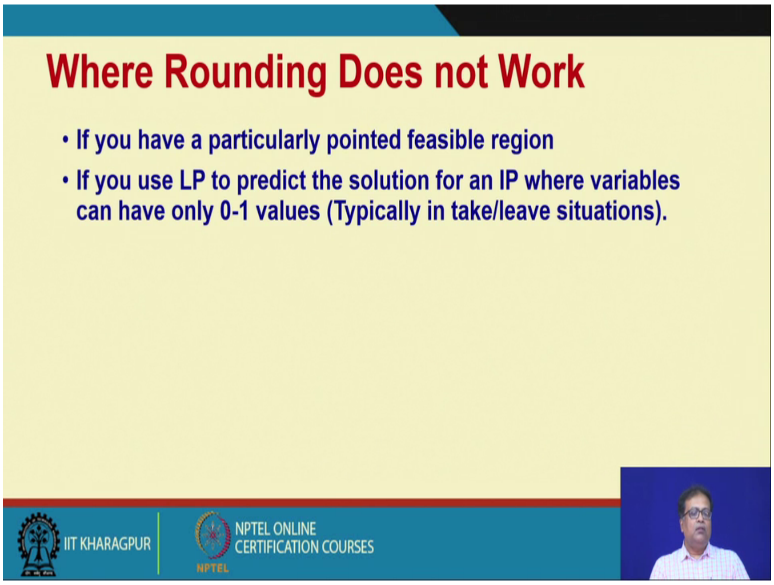

So, it is not rounding off does not you know all the time, if you have particularly pointed feasible region, if you use LP to predict the solution for an IP, where variables can have only 0 1 value, typically in take leave situations you cannot really use rounding off.

So, it is not that you really can use rounding off in every situation you have to use your discretion and see what can work where is alright.

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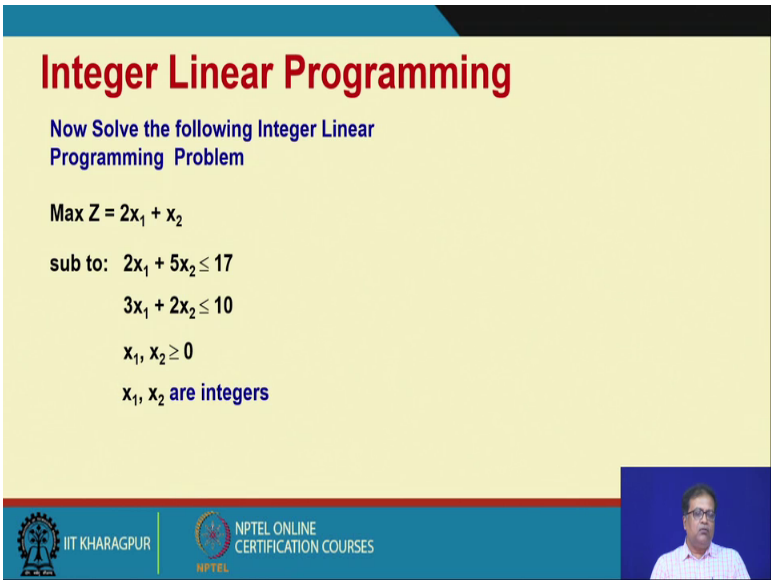

Let us now look at this particular problem, now solve the following integer linear programming problem right. Maximize Z equal to 2 X 1 plus X 2 subject to 2 X plus 5 X

2 less than equal to 17, 3 X 1 plus 2 X 2 less than equal to 10, X 1 and X 2 are 0 let greater than equal to 0 and they are all integers.

So, do you really go ahead with this problem? So, as you have guessed it right first we solve a linear programming solution, the linear programming solution will act as our initial solution. If you are lucky we get an optimum solution there itself then we stop there, but if we do not then we have to really see how exactly we can transform these LP solutions and slowly we can obtain by cutting of the solution space an integer solution right.

So, we will take up these example in detail and we shall solve in our next lecture right.

Thank you very much.