Introduction to Stochastic Processes
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Lecture 36
Communicating Classes & Class Properties

So, what we so far discussed in this just like we know in some applications we will be interested in knowing apriori whether this state is going to be positive recurrent, null recurrent or it is going to be transient. Like in the insurance kind of applications I which just discussed last time. We said that it is important to know whether this is a recurrent state like if I am going to go bankrupt. That the state that I am going to become bankrupt is going to be recurrent I do not want to be in that business.

So, apriori if you know if you could model all the system you want to quickly identify, whether this state is going to be positive recurrent, null recurrent or whatever it is. So, we are just now stating, what are this results that you can what you can say about that, just by looking your Markov chain, say like all these characterization was only simply based on your fjj values. fjj values which you can compute from basically your transition probability matrix.

So, transition probability matrix is basically characterizing your Markov chain. So, if I tell you my Mark chain that you can quickly cover with your transition probability matrix, from that you can compute this fjjs and try to identify of what are the states? How they behave like.

Now, based on this you can try to understand how your things are. Like for example, your transition probabilities. You may start with on transition probability matrix and for that you observed it okay this, this states are transient and these are recurrent and all. But, you notice that I am going to be in recurrent state, most of the time, so this is a bad case, I do not want to get into this.

So, let us change my Markov chain, changing Markov chain is equivalent to saying that, let us consider these parameters. You want to look at a different set of parameters that is like you changing your design or like you changing. Let us instead of starting with this much capital may be let us start with another set of capital or something like that and then you analyse based on that, what are the states which are going to be positive recurrent and transient.

Then you will see that in this I found that, my recurrent states are less may be this is looks like a bit safer bet and then I want to maybe go with this. So, what we are just trying to do is given this set of inputs given one description of Markov chain. See Markov chain as long as somebody tells you these are the transitions happening, you have a Markov chain you are identifying.

But, that Markov chain it can change, if you change the parameters, your transition probabilities. Once you have a new set you can do the same things, so you can do this analysis for any given transition probability matrix and based on that you can say maybe you like you say that, this transition probability looks like if you lead to a situation where things are less risky and maybe as I design or the guy who is actually going to implement them maybe he will see how he can get those parameters.

But, for this computation of all this at null transient and null recurrent we are really do not need to worry about my initial trans probability. I only need to worry about transition probability matrix. Because as you seen when I was trying to do all these calculations, I take given this state. I kinda fix my state I will from there I am trying to analyse. So, the initial distribution is not affecting much, because you already fixing a starting state and after that you are analysing. Initial distribution only affects in which state you are going to start from.

But, that you already fixed now what is only going to decide how your feature is going to evolve is your transition probability matrix. So, that is why I am saying to analyse this you all need to know only transition probability matrix.

Now, we know that we have this different states as of now you have classified them into two broad categories, transient recurrent and further, positive recurrent and null recurrent. Now, is it possible that all my possible states I can group into these classes or like let say I can group into some classes and what is the property of each of this class?

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So, let us try to understand that through what we call as communicating classes. Let us define something, well let us take a Markov chain where the state space is some countable set. As we always denote it by S, as I mean I am again denoting it by S which is countable. Now, we are going to say that.

So, let us take DTMC with countable set of states. Now, take any state j and take another state i, I am going to say that j is reachable from i. That means I am going to hit state j starting from state i. If there exist n positive such that, this n step transition is the probability of n step transition is strictly positive.

And in that case I am going to denote that i goes to j that is from i I can reach state j. Say as long as all I need is to reach j from i is at some point, at some time I should be able to reach it with some positive probability. Write now this n could be very large but that is fine as long as I can do it in some time that is fine.

Now, we are going to say that you take two states i and j, if I can reach j from i and vice versa, like also if I start from j at some point I am go back to state i. If that happens then I am going to say that state i and j communicate. So, if i is reachable from j and j is reachable from i then I am going to call i and j communicate.

And that communication symbol I am going to use it as this double arrowed like. When I say j is reachable from i, I am going to use single arrow taking i to j and when i and j are communicate I will use this one.

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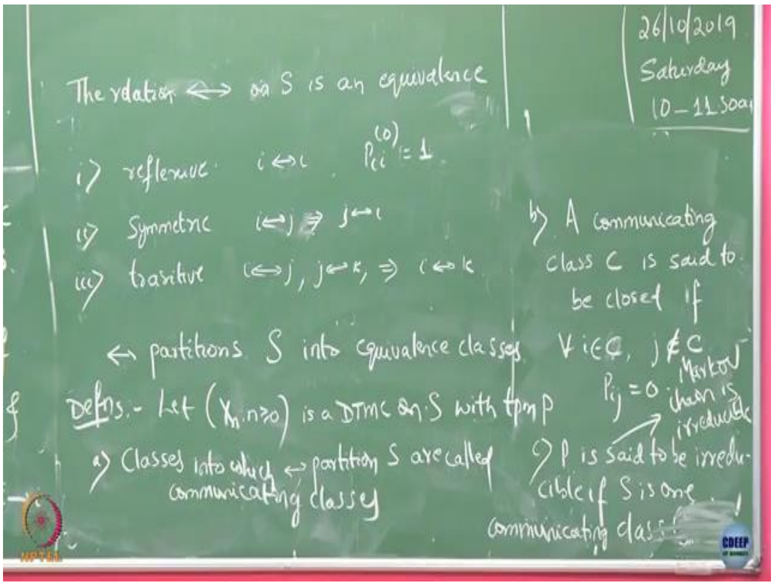

Now, this is the relation, what is this relation this is a communicate relation. Now, we can show that this relation is equivalence relation. So, how many of you know equivalence relation? What are the properties? Reflexive, symmetric, transitive property. What is reflexive property?

Student: (())(10:51)

Professor: Self, so reflexive property. Does this relation satisfy reflexive property? So, then for that what I need to check, i going to itself. So, if this is the case what I need to show, i is reachable from i and again i is reachable from i. To show that what I need to show there exist some n positive, n greater than or equals to 0. So, that I can go from i to i with positive probability. What is that n? 1 or 0? 0 because Pii equals to 1 by your definition, if you are in I, you go to there in 0 time this, that is the meaning of Pii 0 that is what we have defined like.

So, that means if probability 1, if you are in the state in 0 state you will be there in 0th round, so this trivially holds. Now, second is symmetric, what does this mean? Then this implies, is this true? So, what is i mean this is. So, what is i let us take i communicates with j means, i is reachable from j and j is reachable from i. So, that means there exist some positive finite time in which I should be able to reach one from the other. That is same as the so here you start from, i and go to j and then when you look at the other direction start from j to i and just do one and the same thing here.

Student: (())(13:04)

Professor: This is, this is also other direction if you do this this is also. Where in this case we just need to argue that, this implies this. If you start with this then you can say that, this implies this. Now, what is the next? So what is how to show that, is this true? So, how we are going to show this?

So, let us one direction like i communicates with K. So, i communicates with j means in some finite steps i go from i to j and again j communicates with K means in another finite step with positive probability I go from j and K. So, if you take this product I know that in this many rounds at least with positive probability I go from i to K and similarly the opposite direction.

So, this relation is equivalence relation. So, what is the property of an equivalence relation? This relation partitions, but the thing it partitions is your class. So, you have this state S is relation if you apply it is going to partition your S into what you are going to call is equivalence classes.

So, that means all the states within when you have partitions and if you take one particular partition in a way all the states in that partitions are going to be equivalent. That is what we call as equivalent class and here all this we have so many classes, so we are going to call them as equivalent classes.

We have some more definitions, once you have a equivalence relation this is clear that, it partitions. Now, equivalence class is just like the classes we have in this partition we are going to just called them equivalence classes. So, proof for this workout yourself just ensure that you will not get any overlapping sets.

Now, let Xn is greater than DTMC on S. So, let say I have a DTMC with transition probability matrix P. Then, so we are going to as the said S is going to be partitioned by this equivalence class and the different classes we are going to get, we are going to call that as communicating classes.

Because we know that each pair in that particular class or a particular partition is going to communicate with each other. So, we call them as communicating classes. Now, you take a communicating class, so if you are going to take one communicating class and you are going to take one element in that communication class and take state which is outside this communication class and if let say i belongs to that communication class C and j is outside this communicating class. If Pij is equals to 0, then we are going to call this communicating class C as closed.

That means the probability of you are going from any state of this to outside state of this class is going to be 0. That means you are not going basically outside of this class to any state. So, you are that is the kind of closed class you have and if this does not hold this property then we are going to call it as open communicating class.

Now, we further say that, the transition probability P is going to be irreducible, if there is only one communicating class. That mean entire state space is just one communicating class and in this case, so and this case we say the Markov chain itself is irreducible.

Again to define this communicating classes, all what I need to know? Is this enough if I know my transition probability matrix, yes because to define this I need to know all this P transition probability matrix. But, we are in the case of time homogeneous Markov chains for this, this Pijs n step transition probability can be obtained from my one step transition probability matrix.

So, all these properties that I have defined that this relation, whether my class is going to be communicating and whether my class this communicating class is closed or it is going to be open. All I need to know is just my transition probability matrix from that itself I can define all these things.

And now, what are the transition probability I am going to deal with, I am going to call it as irreducible. If that leads to me only one communicating class. So, that my state space is not going to partition it is just going to be one communicating class. So, once I say one communicating class I have should be reachable from one state to another state within in that communication class. So, that is why I am going to call it as irreducible, you cannot partition into two parts that is the case.So, in that case we are just calling our Markov chain itself is irreducible.

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A quick example on this, let us have some states like this, where we can go from here, go from here, go from here, go from here and also go from here. So, the points where I have put these arrows that means these transitions are possible with positive probability. Where the leak does not exist that means that transition have with 0 probability. So, let us call this state 1,2,3,4. So, how many states are there here? Four states and how many communicating classes are there here?

What are those communicating classes? 1, 2 because I am able to reach 1 to 2 and also 2 to 1 and what is other communicating class 3 to 4. This also I am able to reach one from each other there and now is this communicating class a closed communicating class. So, this is one class. Let us call this C1 and this is called C2. Is this communicating class a closed one? Why is that? We can go with positive probability from a state in this class to a outside state C2. Whereas, this C2 is going to be a closed communicating, it is a closed one.

That is right because we are not going anywhere outside. So, what is if I add another states here and allow this to happen? So, in this case, if I add this will this 5 becomes goes inside C2. But, can I reach from 5 to 1? So I can go here with positive probability from there I can go here, in that case I will include this 5 also in that.

But, suppose if I remove one link here, it say that can become another itself class with just one element in that and in this case 1 and 2 is going to be a closed one, it is not going to be a closed one write. Because it can escape from this two, to a state which is outside this class.

Now, we are saying closed this one, we are going to say closed if it do not have any escape route to go outside, outside a state with some positive probability. So, in this case when I wrote this, do you have any escape route to be outside go outside? To take state other than 1 and 2. You go 1 and 2 and 1 and 2 you may be just be like that, you will not possibility and here there is a possibility for you that you escape from being in C1 you can go to here because there is a positive property.Now, if I just said there is add 1 and then let you go there than you are able to escape from this class. So, that is what is no more closed.

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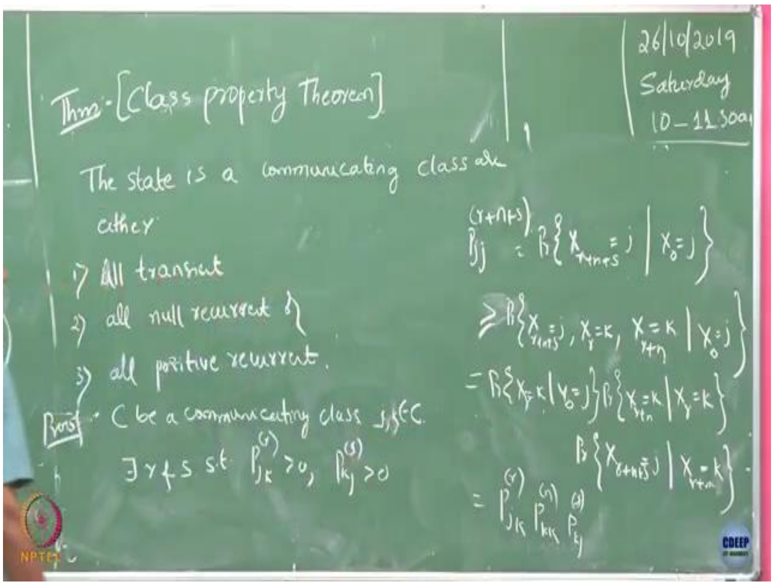

Now, comes is called Class property theorem. If there is only one thing what is there to define closeness it is just going to be close, there is only one state in that. And the definition becomes kind of a vacuous here, it is just one state and you have to go into that state. So, when we say it comes to this state and it is not going anywhere means it remains in this state always. It is just staying in that not moving anywhere.

Now, the states, so this theorem is stating that, the communicating class we have will be such that, all the states in a communicating class are going to be just one type. They are all other going to be all transient, are going to be all null recurrent or going to be all positive recurrent. It is not possible that, you have a communicating class that will have some states to be recurrent and some other states to be positive recurrent or null recurrent, which is going to be either one of them.

So, let us quickly look in to this why this is the case. Suppose let us take on communicating class and let us take lets C be a communicating class and take i, j belongs to C. So, if i, j belongs to C what I know there exist R and S such that, PjK of R is going to be positive and PjK of S is going to be positive.

This is by definition, if i and j belongs to the same communicating class. It must be the case that, I am going to go from j to K with positive probability in some finite steps and similarly I should go from K to j in some finite step with positive probability. Now, suppose now I want to claim that, if assume that j is transient I want to prove that, then K is also transient.

And if I assume j to be positive recurrent, then I want to show that then K is also positive recurrent and if j is null recurrent then K is also null recurrent. So, let see why that is the case. Let first look into the case where, I want to go from j to j in this much steps, r plus n plus s. This r and s I am going to be take this whatever this value, n is variable format any n. Now, what is this? This is probability that X r plus n plus j, S is equals to given we are going to start from X naught equals to j, this is the meaning. Now, what I will do is?

I will instead of going from state j from the beginning to against state j in r plus n plus n steps. I will want to reach this while going to some states in between also. So, let us further condition this by saying that, rn plus s equals to j what is that conditioning amount to add. Xr equals to K then X r plus n is equals to K given X naught equals to j. So, what I want here is instead of directly going here you I also want it to be first reach state K in the first r steps. Subsequently, in the next r n steps that is r plus nth round.

I want you again hit state K and from there in the next n steps that is r plus n plus s round I want to hit state j. So, is this probability is going to be lower bound for this, yes because I added this two extra conditions on this. I basically asking this Markov chain to go through this special state at this steps. Not just looking at the final step here. So, if you just now apply the chain rule here, what you will get is. So, already applied the chain rule and Markov property here and by definition the first term is going to be PjK r rounds.

Then Pkk in n rounds and then P going to be jj, this is k here. This is going to be Kj in s rounds. It is correct? So I have just applied chain rule and applied Markov property and this is just by definition.

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And now what we have basically done is, Pjj r plus n plus s is going to be less than or equals to PjK r PKK n steps and then PKj in s steps. Now, I want to use this property, first thing you can check that, Pjj of n you are going to be Pjj r plus n plus s this I am simply going to take, n greater than or equal to 1 here and this one here.

Now, this is because I am just taking this guy as lower bound, this guy have lower bounded and then summing it over all possible ns and now further I am going to lower bound is by this inequality which I have obtained here. So, here s is the variable, this r and s are fixed and I know that, this PjK r and PKj s both are strictly positive quantities. So, now let us use the property that we have stated in a theorem. Suppose if I assume j is going to be transitive. Let us assume j is transitive, transient.

So, suppose if I assume j is transient, what I know about this quantity as n goes to infinity? We just know it is to be finite. I know that if my state j is transient this is going to be finite and I know that this quantity here is some positive quantity and this quantity here is again some positive quantity.

If that is the case then what I can say about this quantity, this is also finite. Then what will be say in that theorem? If that the theorem when the summation Pij n is finite if and only if j is transient. That theorem was if and only if case. So, now if I assume this quantity to be finite, let that is following making assumption that j is transient.

Then I already concluding that these guy is also finite. What does this imply? What is transient? So, that implies K is also transient. So, what if I take any two states j and K in my class, what we are concluding is. If j is transient, so is K. So, it must be the case that any states in this communicating class they must all be transient. So, now let us look at the case where i is sorry.

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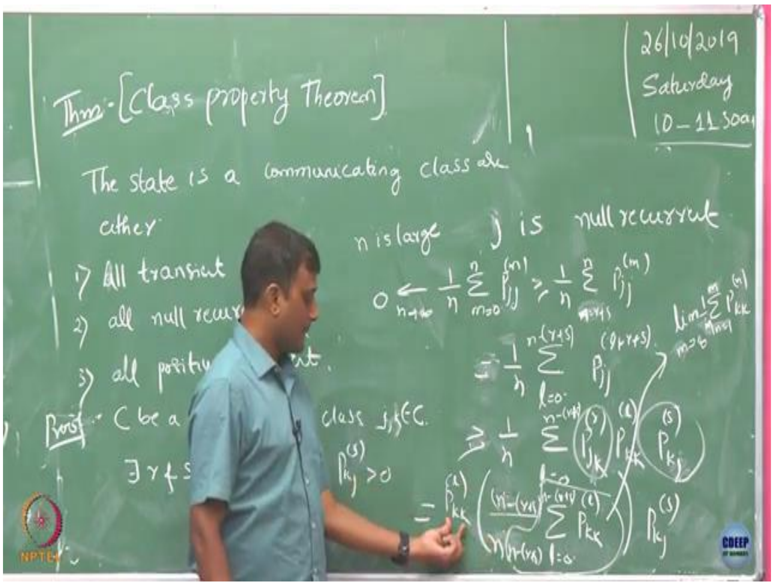

Let us look at, where j is now you want to look at null recurrent it must be what I want to look at, now let say null recurrent. Now, we are going to say that, if j is null recurrent then we will going to show that K is also going to be null recurrent. Suppose let say j is null recurrent and if I assume that, that implies K is transient, but then I am contradicting my first statement. If suppose K is transient then it must be the case that j is must be also transient. But, I am assuming that j is null recurrent. So, by this contra-positive argument, this cannot be the case. So, j could be either null recurrent or positive recurrent. So, let see what it is. So, we have to then use the another result we have used. Where, instead of looking at the sum we look at that averages.

So, let us try to look at the average and then the sum is going to write it as, one by n which has to be have few steps will be done. What I am doing here is basically I am taking this 1by n and summation of n terms here. That is the average of the first n sums here and then in this

I am going to drop out all the points before r plus s and looking at the some beyond that. So, basically here I am assumed that n is large, it is at least r plus s. So, that is why it is starting from this, I am only having few terms here that is why it is a lower quantity.

And then I am just re-indexing it, so instead of starting from m r plus 1, I am going to start from n l equals to zero and then re-indexing this quantity and also when l is 0 this is going to be I am just re-indexing these quantities equals to this quantity and then I am going to use this relation that have already got that is, l equals to 0 to n what I have got. Pjj of r, PKK of n and PKj, what did I do? l here and then Pkj s.

So, this we have already shown that, this can be lower bounded like this. Now, I am going to do little bit manipulation, I will pull out this term and this term outside. So, this it minus r plus s. PKK of l times one by n eps PKK. So, there are how many terms here, l goes from 0 to n minus r plus n, r plus s.

Just we are done, now what I am going to do, so is this clear I have just pulled this outside and thus the reorganised. I have taken 1 by n and this quantity here. But now look at this, this is summation of how many terms here, l equals to 0 to n minus r plus s. So, what I will do is simply, this n I will keep in numerator n minus r plus s and n minus r plus s. I have just multiplied and divided by this term. Now, if you look at this quantity here, this is now average of n minus r plus s terms.

So, there are how many terms, there are n minus r plus s terms and I am also dividing by the same number of amount. Now, let us apply our result what would be say as n goes to infinity. If I let n goes to infinity and this guy j is null recurrent.

What did I say where does this quantity goes to? The limit n tends to infinity, where does this go? So, this goes to 0 as n goes to infinity and now as n goes to infinity, you look at this there is n minus r plus s divided n. What does this ratio go to, that ratio goes one because r plus n this is constant term that does not add much.

And where does this ratio will go? This ratio is going to be same as limit as PKK l as, n as n goes to infinity. Because you are just skipping some finite terms here. If you look into the infinite summation of this divided by that same number here. This is going to be the limit of, so this quantity is going to be the limit of PKK of n.

So, this is going to be let say equals to 1 to m or maybe m equals to infinity, m equals to one to n. So, this quantity is just going to this. So, now we are what we have shown is, as j goes to, as n goes to infinity, this term is going to be 0 and this term is one here but this is going to be positive quantity.

This last term is also going to a positive quantity. In that case, what we say this limit must be 0. Because this quantity is upper bounded by 0. So, then what does this mean? The j is also the j is null recurrent then K is also null recurrent. So, now last thing, suppose now let say if j is my positive recurrent. It is because it cannot be transient or null recurrent. So, only option your left with is positive recurrent.