Course Name: Introduction to Microwave and Optical Metamaterials
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Week-2
Lecture-9

Lec 9: Electromagnetic Waves in Materials

Hello everyone, welcome to lecture 9 of the online course on the introduction to microwave and optical metamaterials.

Lecture Outline

- Maxwell's Equations
  - in Free Space
  - in Source-free Medium
  - In Conductive Medium
- Boundary Conditions
- Power & Intensity
- Energy & Power Density
- Momentum
- Electromagnetic Waves in Dielectric Medium
  - Linear, Nondispersive, Homogeneous, and Isotropic Medium
  - Nonlinear, Dispersive, Inhomogeneous, or Anisotropic Medium

In today's lecture, we will be discussing electromagnetic waves in materials. So, here is the lecture outline: we will briefly discuss Maxwell's equation in free space, then we will see how it changes in a source-free medium and then in a conductive medium. We will also discuss the boundary conditions, power and intensity, energy and power density, momentum, and then we will look into electromagnetic waves in a dielectric medium such as linear, non-dispersive, homogeneous, and isotropic mediums. Also in a non-linear dispersive inhomogeneous or anisotropic medium.

Maxwell's Equations in Free Space

- An electromagnetic field is described by two related vector fields that are functions of position and time: the Electric field $\mathbf{E}(r,t)$ and the Magnetic field $\mathbf{H}(r,t)$.
- In general, therefore, six scalar functions of position and time are required to describe light in free space.
- Fortunately, these six functions are interrelated since they must satisfy the celebrated set of coupled partial differential equations known as Maxwell's equations.
- The electric- and magnetic-field vectors in free space satisfy Maxwell's equations:

Maxwell's Equations (Free Space)	
Divergence equations	Curl equations
$\nabla . \mathbf{E} = 0$	$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$
$\nabla . \mathbf{H} = 0$	$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, as discussed in previous lectures, an electromagnetic wave can be described by two related vector fields that are basically functions of position and time.

So, what are those? The electric field E is a function of r and t. Similarly, you have the magnetic field H, which is also a function of r and t. So, r basically represents the x, y, z coordinates. So, there are basically six scalar functions of position and time that are required to describe light in free space.

So, 3 from here, and 3 from here. Fortunately, these six functions are basically interrelated since they must satisfy the celebrated coupled partial differential equations that we all know as Maxwell's equations. So, this is how the electric and magnetic field vectors in free space look; they will be interrelated with Maxwell's equations. So, you all know that there are two divergence equations and two curl equations, okay? So, in free space, the divergence of E is 0, the divergence of H is 0, and this is the curl equation. So, it shows that the curl of E equals minus mu naught dou H by dou t and the curl of H equals epsilon naught dou E by dou t.

$$\nabla . \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla . \mathbf{H} = \mathbf{0} \qquad \nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

So epsilon naught and mu naught, you know that vacuum permittivity and permeability, and you know the standard values, right.

Maxwell's Equations in Source-free Medium

- In a medium devoid of free electric charges and currents, two additional vector fields are required — the electric flux density (also called the electric displacement) $\mathbf{D}(r,t)$ and the magnetic flux density $\mathbf{B}(r,t)$.
- The four fields, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$, are related by Maxwell's equations in a source-free medium:

Maxwell's Equations (Source-free Medium)	
Divergence equations	Curl equations
$\nabla.\mathbf{D} = 0$	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
$\nabla.\mathbf{B} = 0$	$\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t}$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad \& \qquad \mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

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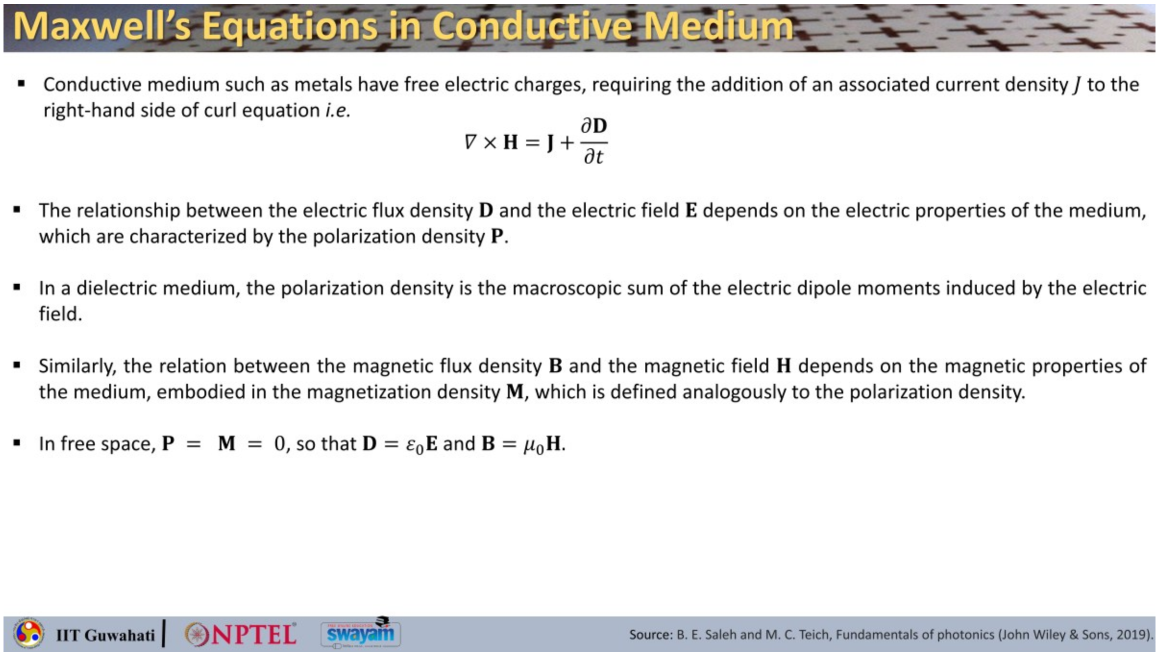
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Now how do things change in a source-free medium? So if you consider a medium that is devoid of free electric charges and current, you basically bring in two additional vector fields. One is called the electric flux density or the electric displacement, denoted by capital D, which is also a function of position and time, and then you have the magnetic flux density B, which is also a function of position and time. So, now we have four fields: E, H, D, and B, right? Electric and magnetic fields, and then you have electric flux density and magnetic flux density, right? So they are related by Maxwell's equations in a source-free medium in this particular form. So now we change those divergence equations to divergence of D equals 0, divergence of B equals 0, and the curl equations look like this: curl of E equals minus dot B dot t. Curl of H equals dot D dot E.

Now, what is D? This is basically where your medium comes into the picture. So, you have epsilon naught E that is coming from the vacuum part, okay? This was there in the earlier case as well, but P tells you about the polarization in a particular medium, okay? So, this is where the medium gets into the equation. Similarly, in the case of magnetic flux density, this applies to free space, but then you have an additional term, magnetization, coming into the picture for the medium.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad \& \qquad \mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$



Now, if you want to see how the equations work for a conductive medium such as metal, which has a lot of free electric charges, okay, so they are free to roam, the charges are free to roam; that means there will be an associated current density J, okay. So, in that case, you need to add that current density term to the right-hand side of the curl equation.

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

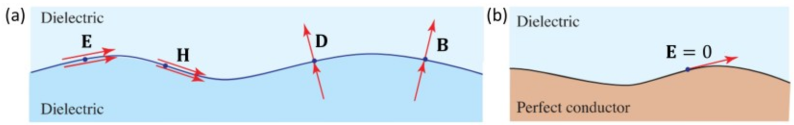
So, your fourth equation changes like this: the curl of H equals J plus dot D dot E, right? The relationship between the electric flux density D and the electric field, as I mentioned, depends on the medium properties, which are characterized by the polarization density P. Similarly, in a dielectric medium, the polarization density is nothing but a macroscopic sum of all the electric dipole moments that are induced by this external electric field in the medium. And if you think of the other equation, the magnetic flux density equation, you will see that the magnetic flux density B is dependent on the magnetic field H, and also on the magnetization density M, which is very similar to the polarization density.

So, this happens in a medium in the presence of an external magnetic field. So, your equations are basically this, right? So, this is where the magnetic properties of the medium and the electric properties of the medium get into Maxwell's equations.

And if you want to retrieve the equation for free space, you can simply set the polarization and magnetization to 0. In that case, your, you know, d simply becomes epsilon naught E, and B becomes mu naught H. So, you retrieve the same equation for free space. So, this kind of equation, when you are relating D with P and then B with M, is where the connection between the medium and the electromagnetic waves is being established.

Boundary Conditions

- In a homogeneous medium, all components of the fields **E**, **H**, **D**, and **B** are continuous functions of position.
- At the boundary between two dielectric medium, in the absence of free electric charges and currents, the tangential components of the electric and magnetic fields **E** and **H**, and the normal components of the electric and magnetic flux densities **D** and **B**, must be continuous.



Boundary conditions at: (a) the interface between two dielectric media; (b) the interface between a perfect conductor and a dielectric material.

- At the boundary between a dielectric medium and a perfectly conductive medium, the tangential components of the electric-field vector must vanish.
- Since a perfect mirror is made of a perfectly conductive material (a metal), the component of the electric field parallel to the surface of the mirror must be zero.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Now, let us look at the boundary conditions.

So, in the case of a homogeneous medium, we understood that all the components of the fields, like E, H, D, and B, are continuous functions of position. Now, at the boundary between two dielectric mediums, we have discussed this briefly in the previous lecture as well. Say you have medium 1 and medium 2, and we are considering that in the absence of any free charges or current, what we will see is that the tangential components of the electric and magnetic fields are basically continuous. Whereas the normal components of the flux densities, electric flux density, or you can say magnetic flux density, must be continuous. Now, if you consider a boundary between a dielectric medium and a perfect conductor, in that case, the tangential component of the electric field vector must vanish.

Since this one behaves like a perfect mirror, or you can say a perfect mirror is basically made of perfectly conducting material, something like a metal, the component of the electric field parallel to the surface of the mirror must be 0. So, this requires that at normal incidence the electric fields of the reflected and incident waves. must have equal magnitudes and a phase shift of π so that you know their sum will add up to 0. So these boundary conditions are an integral part of Maxwell's equations. They are basically used to determine the reflectance and transmittance of waves at various boundaries and to find the propagation of waves in periodic dielectric structures and waveguides.

So it is very important to understand the boundary conditions.

Power & Intensity

- The flow of electromagnetic <u>power</u> is governed by:

$$\text{Poynting Vector} \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}$$

- The direction of power flow is along the direction of the Poynting vector, *i.e.*, orthogonal to both $\mathbf{E}$ and $\mathbf{H}$.

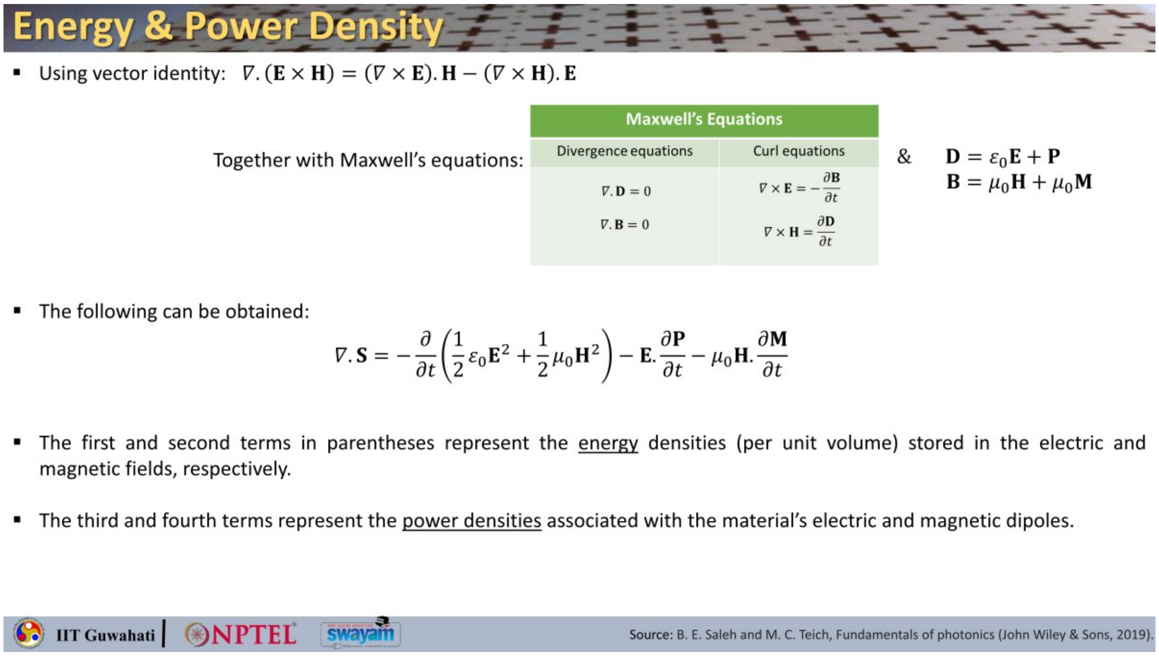
- The optical <u>intensity</u> (power flow across a unit area normal to the vector $\mathbf{S}$) is the magnitude of the time-averaged Poynting vector $\langle \mathbf{S} \rangle$.

- The average is taken over times that are long in comparison with an optical cycle.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Next, we will look into some of the properties of electromagnetic waves, such as power and intensity. So when you say the flow of electromagnetic power, it is basically governed by or given by the Poynting vector S, which is obtained as a cross product of the electric and magnetic fields and indicates the direction of the power flow. Will be orthogonal to both E and H. So, it is something like, you know, you take the cross product, okay.

So, if you use your right thumb to point, you know, if you take your fingers towards the electric field and then bend them around to the orthogonal direction. You will get the magnetic field, and your thumb will be pointing in the direction of the power vector. So, the next important parameter is optical intensity, which is nothing but the power flow across a unit area that is normal to the vector s, and that is nothing but the magnitude of the time-averaged Poynting vector. So, that is given by the average of the pointing vector at this time, and this average is taken over times that are long in comparison with an optical cycle. You can further utilize this vector identity, which is, you know, del dot E cross H.



So, you are basically taking the divergence of the Poynting vector, and you can write it like this: curl of E dot H minus curl of H dot E. Okay, when you combine this with the Maxwell's equations that you have seen for any material, you can obtain del dot S, which is this term. Okay, it turns out to be minus dot d dot E half epsilon naught E squared plus half mu naught H squared minus E dot dot P dot E minus mu naught H dot M dot E. So, as you can see here, the first two terms, this one basically gives you the electric energy density per unit volume, and this one gives you the magnetic energy density of the wave. Okay, and the third and fourth terms basically give you the power densities.

So, this gives you energy density, this gives you power density, this is the electric power density, and this is the magnetic power density, okay for the electric and the magnetic dipoles. So, this equation, also known as the Poynting theorem, therefore represents the conservation of energy. So, the power flow escaping from the surface of an incremental volume equals the time rate of change of the energy stored inside the volume.

$$\nabla.(\mathbf{E}\times\mathbf{H}) = (\nabla\times\mathbf{E}).\mathbf{H} - (\nabla\times\mathbf{H}).\mathbf{E}$$

$$\nabla.\mathbf{S} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0\mathbf{E}^2 + \frac{1}{2}\mu_0\mathbf{H}^2\right) - \mathbf{E}.\frac{\partial\mathbf{P}}{\partial t} - \mu_0\mathbf{H}.\frac{\partial\mathbf{M}}{\partial t}$$

Momentum

- An electromagnetic wave carries linear momentum, which results in radiation pressure on objects from which the wave reflects or scatters.
- In free space, the linear momentum density (per unit volume) is a vector:

$$\varepsilon_0 \mathbf{E} \times \mathbf{B} = \frac{1}{c^2}\mathbf{S}$$

- The average momentum in a cylinder of length c and unit area is $(\langle \mathbf{S} \rangle)/c^2) \cdot c = \langle \mathbf{S} \rangle / c$.
- This momentum crosses the unit area in a unit time, so that the average rate (per unit time) of momentum flow across a unit area oriented perpendicular to the direction of **S** is $\langle \mathbf{S} \rangle / c$.
- An electromagnetic wave may also carry angular momentum and may therefore exert torque on an object.
- The average rate of angular momentum transported by an electromagnetic field is $r \times \langle \mathbf{S} \rangle / c$.

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Next, we will learn about momentum. So, an electromagnetic wave carries linear momentum, which results in radiation pressure on objects that the wave is moving over.

Falling or the wave reflects, or you can say from which the wave reflects or scatters. So, in free space, the linear momentum density calculated per unit volume is a vector, and it looks like this. So, epsilon naught E cross B would be equal to 1 over C squared S, where S is the point vector. So, if you so, this is your linear momentum. Now, if you consider the average momentum in a cylinder of length c and unit area, it is this.

So, it has a unit area; in that case, the momentum will be, you know, s by c squared, and then you multiply by length because the area is 1. So, you get the volume you get as over c. So, this momentum basically crosses a unit area in a unit time. So, that is how you can interpret it. So, you can say this is the average rate per unit time of momentum flow across a unit area oriented perpendicular to the direction of s, and that value is basically s by c.

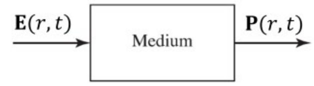
$$\varepsilon_0 \mathbf{E} \times \mathbf{B} = \frac{1}{c^2}\mathbf{S}$$

$$(\langle \mathbf{S} \rangle)/c^2) \cdot c = \langle \mathbf{S} \rangle / c$$

Now, interestingly, your electromagnetic wave may also carry angular momentum. and that may exert, you know, torque on an object. And the average rate of the angular momentum transported by an electromagnetic field will be r cross the time average of the Poynting vector by c. So, these are some important properties of electromagnetic waves, the parameters that are used.

Electromagnetic Waves In Dielectric Medium

- It is useful to regard the $\mathbf{P}-\mathbf{E}$ constitutive relation as arising from a system in which ε is the input and $\mathbf{P}$ is the output or response.
- $\mathbf{E}=\mathbf{E}(r,t)$ and $\mathbf{P}=\mathbf{P}(r,t)$ are functions of both position and time.



- In response to an applied electric field $\mathbf{E}$, the dielectric medium creates a polarization density $\mathbf{P}$.
- A dielectric medium is said to be linear if the vector field $\mathbf{P}(r,t)$ is linearly related to the vector field $\mathbf{E}(r,t)$. The principle of superposition then applies.
- The medium is said to be nondispersive if its response is instantaneous, *i.e.*, if $\mathbf{P}$ at time t is determined by $\mathbf{E}$ at the same time t and not by prior values of $\mathbf{E}$.
- The medium is said to be homogeneous if the relation between $\mathbf{P}$ and $\mathbf{E}$ is independent of the position r.
- The medium is said to be isotropic if the relation between the vectors $\mathbf{P}$ and $\mathbf{E}$ is independent of the direction of the vector $\mathbf{E}$, so that the medium exhibits the same behavior from all directions.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

With that, we will see how electromagnetic waves behave in a dielectric medium, right? So, it is useful to consider the P-E constitutive relation.

So, P is the polarization density and the electric field. So, that arises from a system in which epsilon is the input and P is the output, or you can say the response, okay. So, E is basically a function of both position and time, similarly to P. Okay, so you can think of it like this. So, there is a medium called the dielectric permittivity of epsilon.

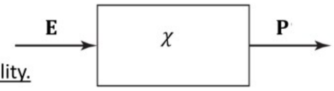
When an electric field falls on that medium, it generates a response or output. In the form of P, right? So, a dielectric medium is said to be linear if the vector field P is linearly related to the vector field E, right? In that case, the principle of superposition will apply. The medium is also called non-dispersive if its response is instantaneous. That means if P at time t is determined only by the electric field at the same time and not by the, you know, electric field values that were previously there, right? So, non-dispersiveness is clearly an idealization since all physical systems. No matter how rapidly they may vary or respond, they do have a finite response time, don't they? So, for the sake of simplicity, we will consider non-dispersive where the electromagnetic response is instantaneous.

The medium is also said to be homogeneous when the relationship between P and E is basically independent of position R. Okay, you can call a medium isotropic if the relationship between P and E is independent of the direction of the vector E; this means the material or the medium basically exhibits similar behavior in all directions. So in that case, P and E, we will now look into a linear, non-dispersive, homogeneous, and isotropic medium.

Linear, Nondispersive, Homogeneous, and Isotropic Medium

- Let us first consider the simplest case of linear, nondispersive, homogeneous, and isotropic dielectric medium. The vectors **P** and **E** at every position and time are then parallel and proportional, so that:

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$



where the scalar constant χ is called the Electric Susceptibility.

- A linear, nondispersive, homogeneous, and isotropic medium is fully characterized by a single constant, the electric susceptibility χ.
- The electric flux density can be written as:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi \mathbf{E} = \varepsilon_0 \mathbf{E}(1+\chi) = \varepsilon \mathbf{E}$$

where the scalar quantity $\varepsilon = \varepsilon_0(1+\chi)$ i.e. electric permittivity of the medium.

- The relative permittivity $\varepsilon/\varepsilon_0 = 1 + \chi$ is also called the dielectric constant of the medium.
- Under similar conditions, the magnetic relation can be written in the form: $\mathbf{B} = \mu \mathbf{H}$

where μ is the magnetic permeability of the medium.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, this is the first case we will consider, and this is the simplest one. So, our medium is linear, as I mentioned, non-dispersive, homogeneous, and isotropic.

So, the vector fields P and E are parallel and proportional at every position and time. So that you can simply write P equals epsilon naught chi E. Right. So, chi is a scalar constant known as electric susceptibility, okay. So, a linear non-dispersive homogeneous and isotropic medium, as you can see here, can be fully characterized by a single constant, which is called chi.

So, this one constant is good enough for mapping all the electric properties in this kind of case. So, you can write the electric flux density D as epsilon naught E plus P; P can be expressed from this equation as epsilon naught chi E. By taking epsilon naught E common, you get 1 plus chi, which is nothing but epsilon r, and when you multiply epsilon naught by epsilon r, you get epsilon. So, this is the permittivity of the medium, the electric permittivity of the medium. So, as I mentioned, this term is basically epsilon r, which is nothing but epsilon over epsilon naught; that is the relative permittivity, and that is given by 1 plus chi.

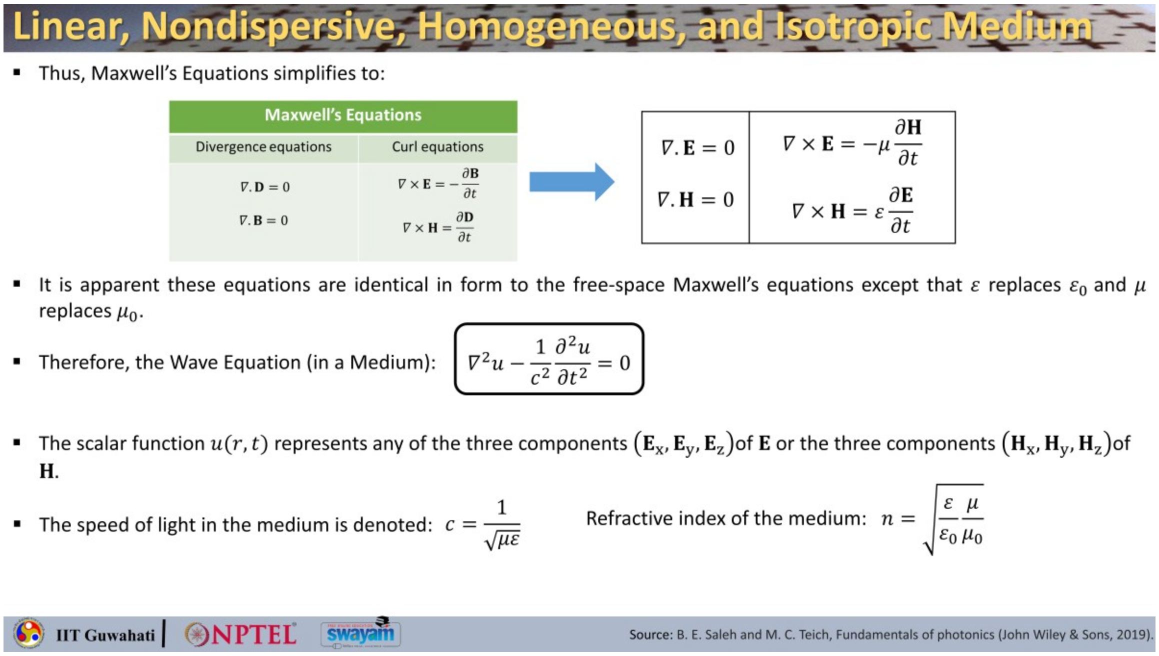
$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi \mathbf{E} = \varepsilon_0 \mathbf{E}(1+\chi) = \varepsilon \mathbf{E}$$

$$\varepsilon = \varepsilon_0(1+\chi)$$

$$\varepsilon/\varepsilon_0 = 1 + \chi$$

Under similar conditions, the magnetic relation can also be written in the form of B equals μH.



Now, in such cases, you can simplify Maxwell's equations to this form. So, we are just, you know, keeping things simple here. So, it is apparent that these equations are identical to the form of the free space Maxwell's equations; the only difference here is that epsilon is replacing epsilon naught. and mu is upside down replacing mu naught. So, from this, you can also write down the wave equation in a medium that can be written as $\nabla^2 u - (1/c^2)\, \partial^2 u/\partial t^2 = 0$, okay. So, here the scalar function urt represents three components, okay: E_x, E_y, and E_z of the electric field, or the three components H_x, H_y, and H_z of the magnetic field H, okay.

$$\boldsymbol{\nabla}.\mathbf{E} = \mathbf{0} \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\boldsymbol{\nabla}.\mathbf{H} = \mathbf{0} \qquad \boldsymbol{\nabla} \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

These are the three components. So, either these three components are correct or these three components are right. So you can get the wave equation. So, if you see this equation, it has got c, which is basically the speed of light, and you can correlate the speed of light with the permittivity and permeability of the medium using this equation: C equals 1 divided by the square root of mu epsilon. Right. So, for a non-magnetic medium, if you consider mu and mu naught to be equal, okay.

$$c = \frac{1}{\sqrt{\mu\varepsilon}}$$

So, this goes off. So, whatever you have here is nothing but epsilon r, and you can write this as 1 plus chi, okay? So, the refractive index n can be written as the square root of epsilon r or the square root of 1 plus chi.

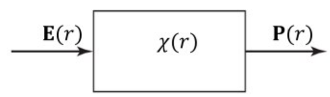
$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}$$

Inhomogeneous Medium

- We now consider nonmagnetic dielectric medium for which one or more of the properties of linearity, nondispersiveness, homogeneity, and isotropy are not satisfied.

<u>For inhomogeneous medium:</u>

- Consider an inhomogeneous dielectric (such as a graded-index medium) that is linear, nondispersive, and isotropic.
- The simple proportionalities, $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$ and $\mathbf{D} = \varepsilon \mathbf{E}$, remain intact, but the coefficients χ and ε become functions of position: $\chi = \chi(r)$ and $\varepsilon = \varepsilon(r)$.
- The refractive index, therefore, also becomes position dependent so that $n = n(r)$.



- An inhomogeneous (but linear, nondispersive, and isotropic) medium is characterized by a position dependent susceptibility $\chi(r)$.
- For media with gradually varying dielectric properties, $i.e.$, when $\varepsilon(r)$ varies sufficiently slowly so that it can be assumed constant within distances of the order of a wavelength, the wave equation:

$$\nabla^2 u - \frac{1}{c^2(r)}\frac{\partial^2 u}{\partial t^2} = 0$$

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Now, let us see how things change in the case of an inhomogeneous medium. So, if you consider a non-magnetic dielectric medium for which one of the properties of linearity, non-dispersiveness, homogeneity, and isotropy is not satisfied. So, let us first start with this case, as I mentioned, in a homogeneous medium.

So, we are considering a graded-index medium. So, it is not homogeneous; the graded index means the refractive index varies along the direction and position. So, otherwise, it is linear, dispersive, and isotropic, okay. So it is just a graded index, so it is inhomogeneous. So in that case, the simple proportionalities, like P equals epsilon naught chi E and D equals epsilon E, will remain intact; just that, you know, the coefficients chi and epsilon will now become functions of position. So, you have to write chi equals chi r and epsilon equals epsilon r because it is a graded index.

So, along the position, the refractive index changes, and this tells you that the refractive index is also position-dependent. So, you should represent it as a function of r. So, this is how things look in this particular scenario where you have an inhomogeneous but otherwise linear, non-dispersive, and isotropic medium, right? So, this is basically characterized by a position-dependent susceptibility χ_r. For media with gradually varying dielectric properties, that means when your ε_r varies sufficiently slowly, it can be assumed constant within distances of the order of the wavelength, and in that case, the wave equation can be written as $\nabla^2 u$ - c^2, which is position-dependent; you can put this as the speed of light in that particular medium, which is position-dependent, and then you have $\partial^2 u/\partial t^2 = 0$, right.

$$\nabla^2 u - \frac{1}{c^2(r)}\frac{\partial^2 u}{\partial t^2} = 0$$

So, what is important to note here is that because your refractive index n is n_r, that means your c_r is also nothing but the speed of light in vacuum c_0 divided by n_r; that means you have a specially varying speed, and that is all happening because your refractive index is specially varying.

Anisotropic Medium

For anisotropic medium:

- The relation between the vectors **P** and **E** in an anisotropic dielectric medium depends on the direction of the vector **E**; the requirement that the two vectors remain parallel is not maintained.
- If the medium is linear, nondispersive, and homogeneous, each component of **P** is a linear combination of the three components of **E**:

$$\mathbf{P}_i = \sum_j \varepsilon_0 \chi_{ij} \mathbf{E}_j$$

where the indices $i, j = 1, 2, 3$ denote the x, y, and z components, respectively.

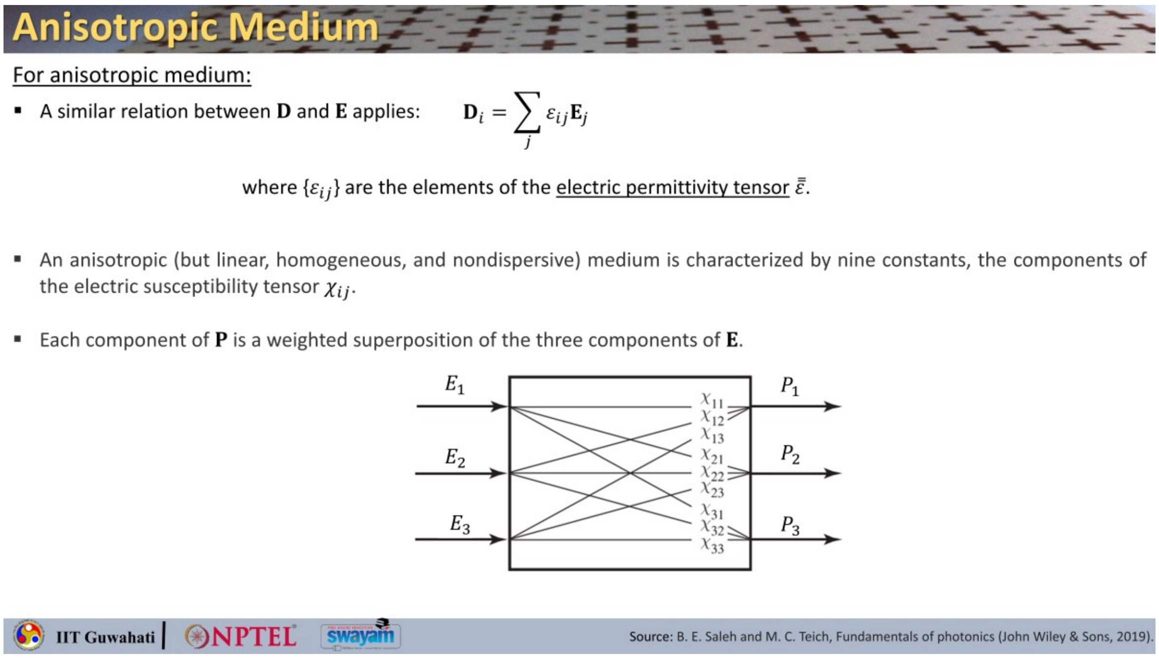
- The dielectric properties of the medium are then described by a 3×3 array of constants $\{\chi_{ij}\}$, which are elements of what is called the electric susceptibility tensor χ.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Next, for an anisotropic medium. The relationship between the vectors P and E depends on the direction of the electric field. And the requirement that the two vectors are isotropic is that the two vectors are parallel, but here they are not parallel, right? So, if the medium is linear, non-dispersive, and homogeneous, okay. In that case, it is anisotropic. So, each component of P is basically a linear combination of the three components of the electric field. So, you can write Pi equals sigma over j epsilon naught χ_{ij} E_j.

$$\mathbf{P}_i = \sum_j \varepsilon_0 \chi_{ij} \mathbf{E}_j$$

So, each component of P that is Pi will depend on the three components of j. So, i and j are basically here 1, 2, 3. that indicates your x, y and z. So, the dielectric property of the medium will in this case be you know because i and j both are running from 1 to 3. So, it is a 3 by 3 area of constants, okay, as you can see here.



So, you are basically getting a tensor susceptibility tensor that is determining the material property. So, once you understand how this works, you can apply the same relation for d and e as well in the case of an anisotropic medium. So, D_i becomes sigma over j epsilon ij E_{ij}. That means j is running from 1 to 3, and i is also running from 1 to 3, okay. So, your epsilon ij is basically giving you the elements of the electric permittivity tensor, right? So, in anisotropic media, it is a linear homogeneous and dispersive medium; in that case, the permittivity will also have 9 components, and these components are basically coming from the electric susceptibility tensor, right? So, if you go back to the previous equation, you will see that E_i, okay, is you can see from this one that P_i equals sigma j, okay, epsilon naught χ_{ij} E_j.

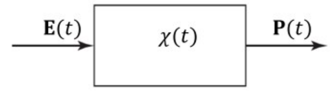
$$\mathbf{D}_i = \sum_j \varepsilon_{ij}\mathbf{E}_j$$

So, all three components of the electric field will give you each component of polarization. So, P_1 will be dependent on E_1, E_2, and E_3 via the, you know, constant susceptibility constants of χ_{11}, χ_{12}, and χ_{13}. Similarly, P_2 will be dependent on E_1, E_2, and E_3, as these are the susceptibility elements χ_{21}, χ_{22}, and χ_{23}. Similarly, P_3 will have χ_{31}, χ_{32}, and χ_{33}.

Dispersive Medium

For dispersive medium:

- The relation between the vectors **P** and **E** in a dispersive dielectric medium is dynamic rather than instantaneous.



- The vector $\mathbf{E}(t)$ may be thought of as an input that induces the bound electrons in the atoms of the medium to oscillate, which then collectively give rise to the polarization-density vector $\mathbf{P}(t)$ as the output.
- The presence of a time delay between the output and the input indicates that the system possesses memory.
- Only when this time is short in comparison with other times of interest can the response be regarded as instantaneous, in which case the medium is approximately nondispersive.
- For dispersive medium that are linear, homogeneous, and isotropic, the dynamic relation between $\mathbf{P}(t)$ and $\mathbf{E}(t)$ may be described by a linear differential equation such as that associated with a driven harmonic oscillator:

$$a_1 \frac{d^2\mathbf{P}}{dt^2} + a_2 \frac{d\mathbf{P}}{dt} + a_3\mathbf{P} = \mathbf{E}$$ where a_1, a_2, and a_3 are constants.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Now, we will see how things change in the case of a dispersive medium.

For a dispersive medium, the relation between P and E is dynamic rather than instantaneous. So, you now have to think of time dependence in the case of the electric field and the polarization density, as well as in the material parameter. This is coming from the material parameter because things do not change instantaneously, right? So, the vector E(t) may be thought of as an input that basically induces some bound electrons in the atoms of the medium to oscillate, which then collectively give rise to some polarization density factor P(t) as the output. So, that takes some time, and there is a presence of time delay between the output and the input, which indicates that the system possesses some memory. So, only when this time is short in comparison to the other times of interest.

Okay, this response can be regarded as instantaneous. Then, you do not care. Okay, in such cases, you can consider the medium as non-dispersive, right? But if you say that it takes some finite amount of time, then it is a dispersive medium. So, for a dispersive medium that is linear, homogeneous, and isotropic, the dynamic relationship between P(t) and E(t) may be described by a linear differential equation, such as that associated with a driven harmonic oscillator, something like this:

$$a_1 \frac{d^2\mathbf{P}}{dt^2} + a_2 \frac{d\mathbf{P}}{dt} + a_3\mathbf{P} = \mathbf{E}$$

So, a_1, a_2, and a_3 are constants.

Dispersive Medium

For dispersive medium:

- The linear-systems approach may be used to investigate an arbitrary linear system, which is characterized by its response to an impulse (impulse response function).
- An electric-field impulse of magnitude $\delta(t)$ applied at time $t = 0$ induces a time-dispersed polarization density of magnitude $\varepsilon_0\chi(t)$, where $\chi(t)$ is a scalar function of time with finite duration that begins at $t = 0$.
- Since the medium is linear, an arbitrary electric field $\mathbf{E}(t)$ then induces a polarization density that is a superposition of the effects of $\mathbf{E}(t')$ for all $t' \le t$, so that the polarization density can be expressed as a convolution:

$$\mathbf{P}(t) = \varepsilon_0 \int_{+\infty}^{-\infty} \chi(t - t')\mathbf{E}(t')dt'$$

- This dielectric medium is completely described by its impulse response function $\varepsilon_0\chi(t)$.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

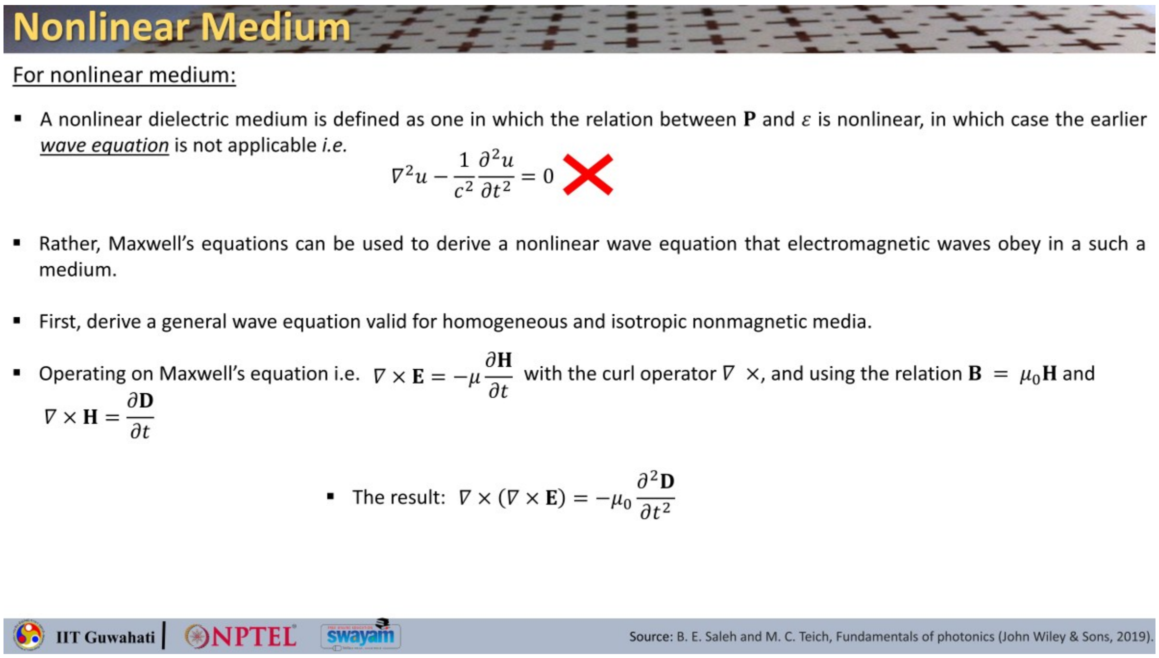
So, what we have seen is that in a dispersive medium, it is linear, homogeneous, and isotropic. The relationship between P(t) and E(t) is governed by a dynamic linear system. Described by an impulse response function epsilon naught chi t, which basically corresponds to a frequency-dependent susceptibility, chi f, you can also think of it in frequency domain terms. So, a linear system approach may be used to investigate any arbitrary linear system that is characterized by its response to an impulse. So, you are basically thinking about the impulse response function, right? So, an electric impulse electric field impulse, if you consider a magnitude delta t, that is applied at time t equals 0, will introduce, or you can say induce, a time-dependent polarization density of magnitude chi naught epsilon naught, and the magnitude will be around epsilon naught chi t. So, this chi t is basically a scalar function of time with some finite duration that begins at t equals 0.

Now, since the medium is linear, any arbitrary electric field E(t) induces a polarization density that is a superposition of the effects of E(t'). For all t prime less than or equal to t, it covers everything before that time t, so that the polarization density can be expressed as a convolution. So, you can write P(t) equals epsilon naught integral from minus infinity to plus infinity x, or you can say chi(t - t prime) E(t) prime dt prime, right. So, this is how the dielectric medium is completely described by its impulse response function, epsilon naught chi t.

$$\mathbf{P}(t) = \varepsilon_0 \int_{+\infty}^{-\infty} \chi(t - t')\mathbf{E}(t')dt'$$

Alternatively, a dynamic linear system may be described by its transfer function, which governs the response to harmonic inputs.

The response function is a Fourier transform. of the impulse response function. In this particular example, the transfer function at frequency f is nothing but epsilon naught chi f, where chi f will be the Fourier transform of this chi t. So, you know that is basically becoming a frequency-dependent susceptibility. For magnetic media under similar assumptions, the relationship between the magnetization mt and magnetic field H(t) will be very similar to the convolutional equation that you have shown here.



Now, we move to a non-linear medium. For the case of a non-linear dielectric medium, it is defined as one in which the relation between P and epsilon is non-linear, in which case the earlier wave equation is not applicable; that means this is not applicable. Rather, you need to take Maxwell's equations to derive a non-linear wave equation that electromagnetic waves obey in such a medium, okay. So, first we have to derive a general wave equation that is valid for a homogeneous and isotropic non-magnetic medium.

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

So, operating on Maxwell's equations, the curl of E equals minus mu dou H dot E with a curl operator, and using the relation P equals mu naught H and curl of H equals dou D dot E, you can write the curl of the curl of E equals minus mu naught dou D dou squared D dot E squared. Right. So, you can take, like, basically what you are doing: you are taking the curl of the curl of this.

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial \mathrm{t}}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}$$

Nonlinear Medium

For nonlinear medium:

- The result: $\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}$
- Making use of the vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and the relation $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ yields:

$$\nabla(\nabla . \mathbf{E}) - \nabla^2 \mathbf{E} = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- For homogeneous and isotropic medium $\mathbf{D} = \varepsilon \mathbf{E}$; thus $\nabla . \mathbf{D} = 0$ is equivalent to $\nabla . \mathbf{E} = 0$.
- Substituting this, along with $\varepsilon_0 \mu_0 = 1/c_0^2$ provides:

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

- This wave equation applicable for all homogeneous and isotropic dielectric medium: nonlinear or linear, nondispersive or dispersive.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, here you are also introducing the curl of H, and from the curl of H, you are putting this D, right? So, you have got this one; then you use this particular vector identity that the curl of the curl of E is nothing but the gradient of the divergence of E minus the Laplacian of E. And then you have this relation that you already know: D equals epsilon naught E plus P. When you put this together, you get this one. So, you get that this is the left-hand side, okay? That is this part, and this becomes this minus epsilon naught mu naught dou square E dot square minus mu naught dou square P dot square.

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla 2\mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla(\nabla . \mathbf{E}) - \nabla^2 \mathbf{E} = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

So, if you consider a homogeneous and isotropic medium, D equals εE. In that case, the divergence of D is 0. That is equivalent to the divergence of E becoming 0. So, this part blows off. So, you are just left with this one. So, you write, remove the minus from all sides, and then bring this term to the left side; you will get this.

Okay, and one replacement is done. Here, you can see that epsilon naught mu naught is nothing but 1 over c naught squared; c naught is the speed of light in vacuum. So, you can use this equation, and this is the equation in a non-linear format. medium right. So, ideally, this is particularly the equation that is applicable for all homogeneous and isotropic dielectric media, whether they are linear or non-linear, non-dispersive or dispersive. So, this is the master equation, you could say, right?

$$\nabla^2 \mathbf{E} - \frac{1}{c_0^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

Nonlinear Medium

For nonlinear medium:

- Now, if the medium is nonlinear, nondispersive, and nonmagnetic, the polarization density $\mathbf{P}$ can be written as a memoryless nonlinear function of $\mathbf{E}$, say $\mathbf{P} = \Psi(\mathbf{E})$, valid at every position and time.
- The simplest example of such a function is: $\mathbf{P} = a_1\mathbf{E} + a_2\mathbf{E}^2$ where a_1 and a_2 are constants
- Under these conditions becomes a nonlinear partial differential equation for the electric-field vector $\mathbf{E}(r,t)$:

$$\nabla^2 \mathbf{E} - \frac{1}{c_0^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \Psi(\mathbf{E})}{\partial t^2}$$

- The principle of superposition is no longer applicable by virtue of the nonlinear nature of this wave equation.
- Nonlinear magnetic materials may be similarly described.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Now, let us focus on the special case that we want this to be non-linear, okay.

So, if the medium is non-linear, non-dispersive, and non-magnetic. The polarization density P can be written as a memory-loss non-linear function of E. So, you can write P equals psi of E okay which is valid at every position and time because it is non-dispersive. So, there is no time dependence, okay, and it is happening everywhere. So, you can take this dependency, which can be represented like this: this is the simplest case; say the polarization is dependent up to the second power of E.

So, you have a_1E plus a_2E^2, okay. So, under this kind of condition, you have to take the partial differential equation, okay. So, you get a non-linear partial differential equation for the electric field vector E, ok. So, if you just refer to the previous case, the generalized one, it's okay.

There you put this, and this is how it looks. So, P is now psi E. This dependency can be anything that you put here. So, now because it is non-linear, the principle of superposition will no longer be applicable due to the non-linear nature of this wave equation, and non-linear magnetic materials can also be described using a similar method. So, most dielectric media are approximately linear unless the optical intensity is substantial, as in the case of focused laser beams, right

$$\mathbf{P} = \Psi(\mathbf{E})$$

$$\mathbf{P} = a_1\mathbf{E} + a_2\mathbf{E}^2$$

$$\nabla^2\mathbf{E} - \frac{1}{c_0^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial^2\Psi(\mathbf{E})}{\partial t^2}$$

So, with that, we will thank you for this lecture. If you have any queries, you can drop an email at this email address.