Course Name: Introduction to Microwave and Optical Metamaterials
Professor Name: Dr. Debabrata Sikdar
Department Name: Electronics and Electrical Department
Institute Name: Indian Institute of Technology, Guwahati
Week-2
Lecture-7

Lec 7: Basic Electromagnetic Theory-II

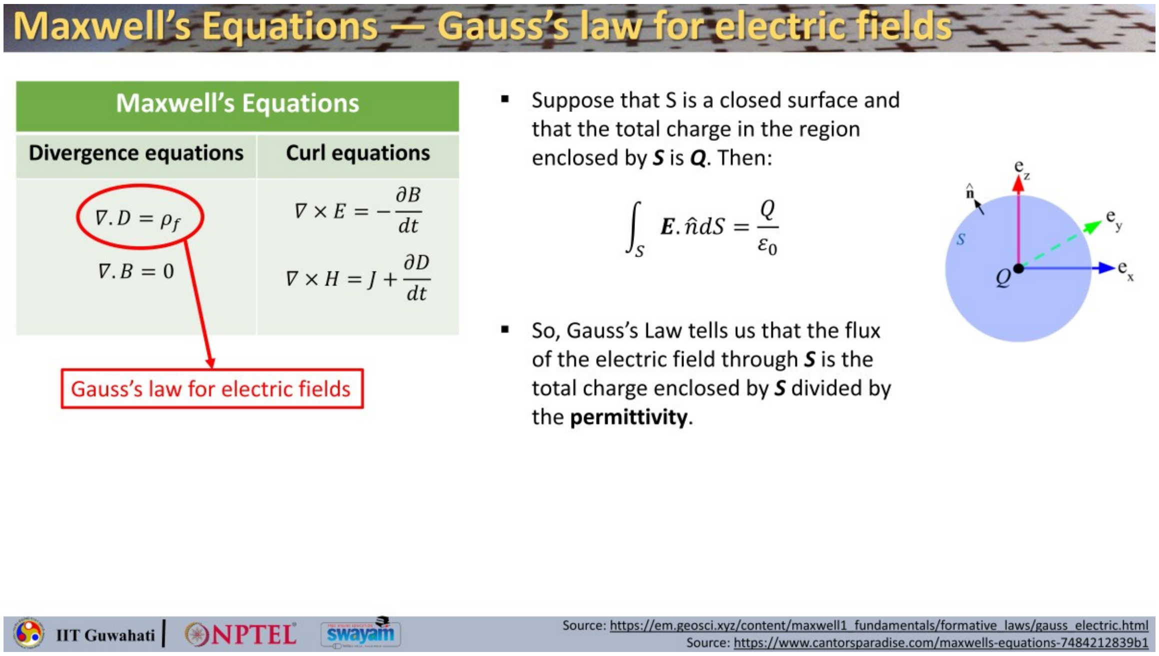
Hello students, welcome to Lecture 7 of the online course on Introduction to Microwave and Optical Metamaterials.

Lecture Outline

- Maxwell's Equations
  - Gauss's law for electric fields
  - Gauss's law for magnetic fields
  - Faraday's law
  - Ampere-Maxwell equation

- The Wave Equation
  - from Maxwell's Equations
  - speed of EM waves

In this lecture, we will continue with the basic electromagnetic theory. So, here is the outline of the lecture. We will discuss the four Maxwell's equations in more detail; the first one we will cover is Gauss's law for electric fields, the second one, as all of you know, is Gauss's law for magnetic fields, followed by Faraday's law, and then we will look into Ampère's equation and what Maxwell contributed to that, which has finally become the fourth equation in that set. Maxwell's equation, which is also known as Ampère-Maxwell's equation. From that, we will see how we can obtain the wave equation from Maxwell's equations and also how we can obtain the speed of electromagnetic waves.



So, students, let us discuss Gauss's law for the electric field first. So, just a background about that, this law was published posthumously in 1867 as part of a collection of works by the famous German mathematician Gauss. So, if you look into Maxwell's equations, it is basically a set of two divergence equations and two curl equations. So, let us focus on the first one here, and this particular law is called the Gauss Law for electric fields.

Now, how do you understand or interpret this law? It says del dot D equals rho f. So, let us assume that you have a closed surface called S, and the total charge that is inside this closed surface is basically Q. Therefore, you can say that the charge Q is enclosed by this closed surface S. So, N cap shows the normal from the surface, and these are the 3 coordinate factors E x, E y, and E z. Now, this Gauss's law also states that the flux of the electric field passing through this closed surface S is basically equal to the total charge that is enclosed.

$$\boldsymbol{\nabla}. D = \rho_f \qquad \boldsymbol{\nabla} \times E = -\frac{\partial B}{dt}$$

$$\boldsymbol{\nabla}. B = 0 \qquad \boldsymbol{\nabla} \times H = J + \frac{\partial D}{dt}$$

$$\int_S \boldsymbol{E}.\hat{n}dS = \frac{Q}{\varepsilon_0}$$

In this surface, divided by the permittivity. So, if you assume that the material within the surface is vacuum or air, then the permittivity is only epsilon naught. So, the right side basically becomes a total charge that is Q divided by ε_0. And how do you calculate the total flux of the electric field through this surface? So, you take the electric field vector and calculate what is coming out normally. So, you take the dot product with this surface normal n hat and then you integrate it over the entire surface.

So, this gives you the total flux that comes out from this closed surface S, and that is equal to the charge Q enclosed by this surface divided by the vacuum permittivity. So, one of the great features of this particular law is that S can be any surface; you do not need to think of a sphere, a cube, or any other regular shape; it can be any surface that can completely enclose the charge distribution. Okay, and then the flux through this surface can be calculated in a similar way, and then you can find the total amount of charge that is enclosed divided by epsilon naught.

Maxwell's Equations — Gauss's law for electric fields

Maxwell's Equations	
Divergence equations	**Curl equations**
$\nabla . D = \rho_f$	$\nabla \times E = -\frac{\partial B}{dt}$
$\nabla . B = 0$	$\nabla \times H = J + \frac{\partial D}{dt}$

Gauss's law for electric fields

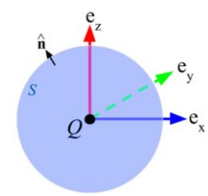
- The differential form is obtained with the **divergence theorem**:

$$\int_V (\nabla . \mathbf{E}) dV = \int_S (\mathbf{E} . \hat{n}) dS$$

and $\frac{Q}{\varepsilon_0} = \int_V \frac{\rho}{\varepsilon_0} dV$

$$\therefore \int_V (\nabla . \mathbf{E}) dV = \int_V \frac{\rho}{\varepsilon_0} dV$$

$$\therefore \nabla . \mathbf{E} = \frac{\rho}{\varepsilon_0}$$



Source: https://em.geosci.xyz/content/maxwell1_fundamentals/formative_laws/gauss_electric.html
Source: https://www.cantorsparadise.com/maxwells-equations-7484212839b1

Now, moving ahead, when you take that equation, the calculation of the flux, okay, in this form, you can also take the help of the divergence theorem, and that will help you obtain the differential form that you are seeing here. So, this particular equation is basically in differential form, and this is the one shown in the integral form of the Maxwell equation, and how you can go back and forth from one form to the other.

So, you can see here that this particular one can also be written as the divergence of E over this surface, okay, over this volume, okay. So, you can have the volume integral of the divergence of E, which is the same as the surface integral of the electric flux, and q over epsilon naught can be simply written as q, which is nothing but the total amount of charge. So, if you know the surface charge density, volume charge density rho, okay. If you know the volume charge density rho and integrate it over the volume. So, the integral of rho dv is basically giving you the total amount of charge enclosed.

So, q divided by epsilon naught can be replaced by this. So, what we are doing in this equation is that the left-hand side can now be written as this. So, it is basically a volume integral of the divergence of the electric field, and the right side, which was basically Q by epsilon naught, can now be written as a volume integral of charge density over epsilon naught. So, you can just put this on the left side and the right side, and you will see both are basically taking the volume integral. So, what is happening? Quantities within the integral are basically now the same; that is how their integration is also yielding the same result.

So, you can say del dot E equals rho over epsilon naught, okay. Now, if you multiply this E vector by epsilon naught, what you get is D, the electric flux, and this is particularly this equation.

$$\int_V (\nabla.\mathbf{E})dV = \int_S (\mathbf{E}.\hat{n})dS$$

and $$\frac{Q}{\varepsilon_0} = \int_V \frac{\rho}{\varepsilon_0} dV$$

$$\therefore \int_V (\nabla.\mathbf{E})dV = \int_V \frac{\rho}{\varepsilon_0} dV$$

$$\therefore \nabla.\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Maxwell's Equations — Gauss's law for magnetic fields

Maxwell's Equations	
Divergence equations	**Curl equations**
$\nabla . D = \rho_f$	$\nabla \times E = -\frac{\partial B}{dt}$
$\nabla . B = 0$	$\nabla \times H = J + \frac{\partial D}{dt}$

Gauss's law for magnetic fields

- Gauss's law for magnetism states that **no magnetic monopoles exists** and that the total flux through a closed surface must be zero.
- ∇ . **B** = 0 is derived from:

$$\int B.dS = 0$$

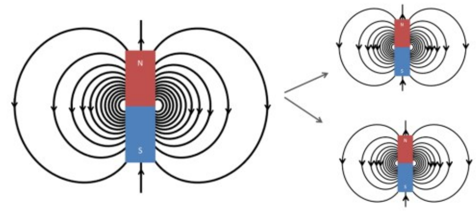


Figure. When a bar magnet is cut in two, you get two bar magnets.

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So, now let us move on to the second equation, which is basically Gauss's law for the magnetic field. So, this particular equation we are talking about, del dot B equals 0. What is B? B is basically the magnetic flux.

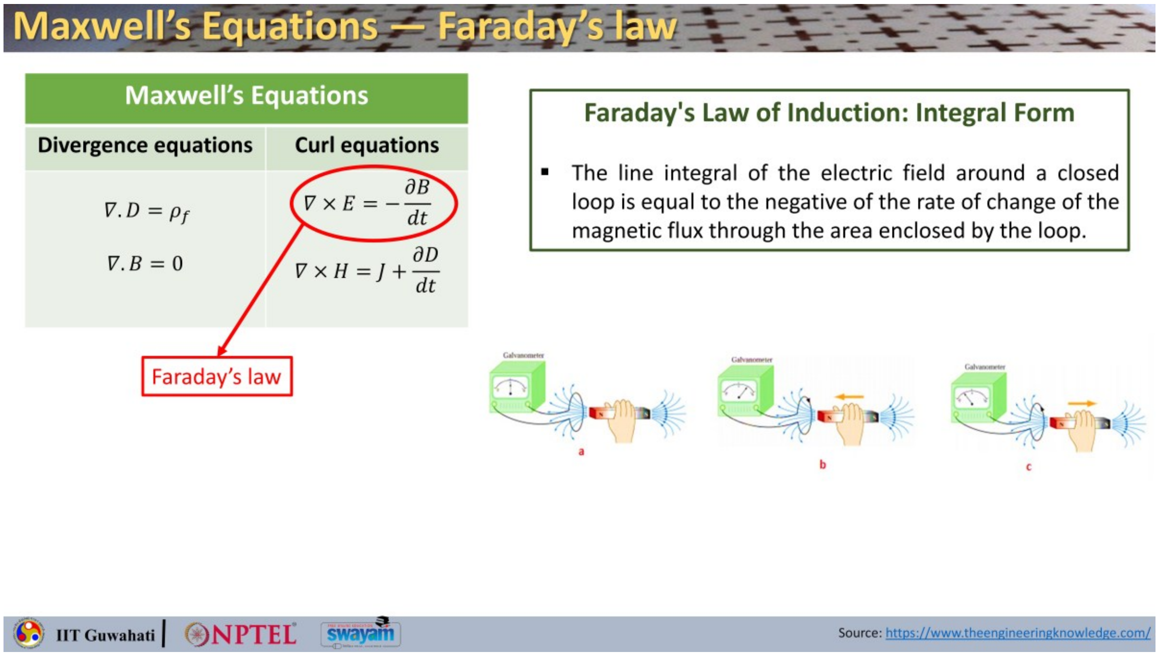
$$\int B.dS = 0$$

Now, Gauss's law for magnetism basically states that no magnetic monopole exists and that the total flux through a closed surface must be 0. This is also true that you know because the total flux through a closed surface is always 0, which means you cannot separate out the poles and only keep a positive pole or a north pole without having a south pole along with it; that is not possible. So, you can say that the number of magnetic field lines that enter and exit through an arbitrarily closed surface will always remain the same. So, this particular thing, here you can see a bar magnet. So, this is how the magnetic field lines exit from the North Pole and enter the South Pole.

But now if you think that you can simply, you know, split this magnet into two and make only a north pole and only a south pole, that is not going to happen. When you break them into two smaller magnets, they themselves also have a north pole and a south pole, and the field lines will behave exactly in the same way as the earlier larger magnets.

So, how do you derive that del dot B equals 0? It basically comes from this particular integral form. So, you can integrate the magnetic flux over the closed surface, and that will come out to be 0. And that is because, if you think of any surface like this, from the positive pole the lines are basically exiting, but because of the north pole, it is exiting.

For the South Pole, that many lines are only entering. So if you think of enclosing this particular magnet in a closed surface, you will see that the number of magnetic field lines exiting that surface from the north pole will be equal to the number of field lines entering the surface again because of the south pole. So the net flux will be zero.

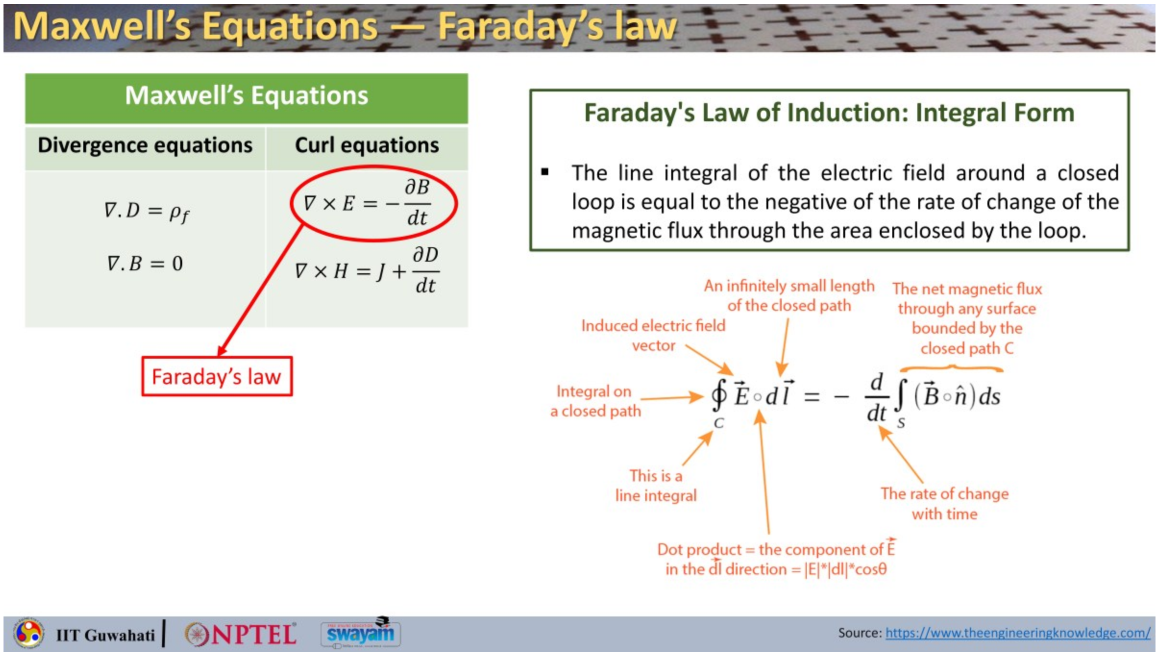


So now let us look at the third one. which is called Faraday's law.

So, this is one of the first two equations that basically connect the electric and magnetic fields, okay? So, you can see that here E is basically now dependent on or connected with B as well. So, it tells you that E is a conservative field in the absence of a magnetic field or if the magnetic field is basically constant in time. It means that if the magnetic field is not present at all or if the magnetic field is constant, then this term becomes 0. In that case, you know that the electric field is a conservative field. Now, this electromagnetic induction was basically discovered independently by two scientists, Michael Faraday in 1831 and Joseph Henry in 1832.

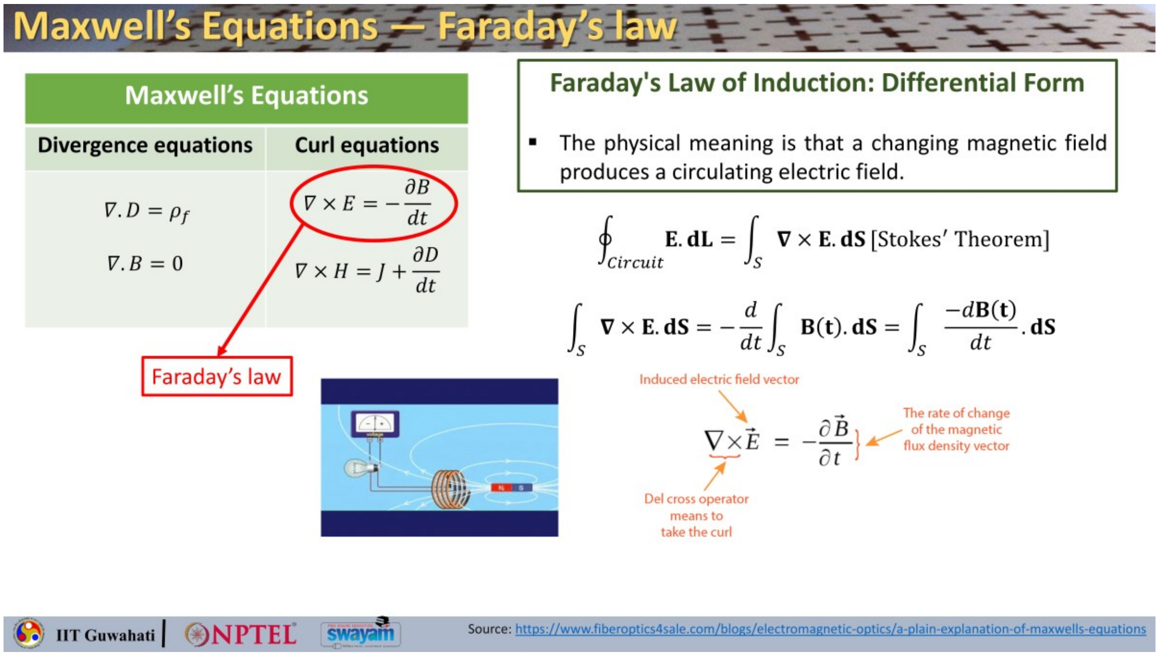
So, Faraday was the first to publish the results of his experiments; thus, this particular equation got his name. So, we call this Faraday's law, and Faraday's law of induction in the integral form looks like this. So, these are all in differential form; as you can see in integral form, this is how it looks, and it tells you the story in a much more comprehensive manner. So, here you can see the line integral of the electric field around a closed loop. So, this tells you about a closed-loop line integral.

That is how the symbols are displayed. So, the line integral of the electric field around a closed loop is basically equal to the negative of the rate of change of the magnetic flux; this is the magnetic flux enclosed by the loop.



So, there are a couple of things here; as you can see, d/dt tells you the rate of change with respect to time. This particular integration is giving you the net magnetic flux through any surface that is bounded by this closed path C, okay? And what are these you know you can see here dL is basically telling you an infinitesimal small length of this closed path. And you are basically doing a dot product.

So, E is also a vector; dl is a vector. So, the component there is the dot product. So, you are basically computing the modulus of E, the modulus of dl, and then cos theta, right? So, this is basically the induced electric field vector because of the change in the magnetic flux, right? And this is how you obtain it, fine. So, the direction of this electromotive force, okay, is given by Lenz's law, and you all know that Faraday's law is the basis for electric generators, and it also forms the basis for inductors and transformers. So, it is a very important law in electromagnetism.



So, you can also see this is how it is written in the differential form, and you can always go from the integral form to the differential form by using some of the theorems that you have already studied. So, I will quickly tell you what it shows. So, from the curl of E equals minus dou B dot E, that also tells you that the physical meaning behind this is that a changing magnetic field will basically produce a circulating electric field, ok. So, here you can also see that when the magnetic field lines are, or you can see the magnetic field line is kind of at the center of the loop. Okay, so the current induced is 0, but as it moves away or goes in, you get a negative current; when it comes out, you get a positive current, or vice versa, depending on the magnetic poles, whether the north pole is this side or the other way.

So, this basically tells you how the current is introduced because of the time-varying magnetic field, and that is basically recorded in the galvanometer. Now, if we go back to this particular form and try to use Stokes' theorem. So, on the left-hand side, we have seen that you have a closed-loop line integral, okay. So, E · dl over a closed loop, and you can use Stokes' theorem. So, you can convert this line integral into a, you know, surface integral of the curl of that vector, okay.

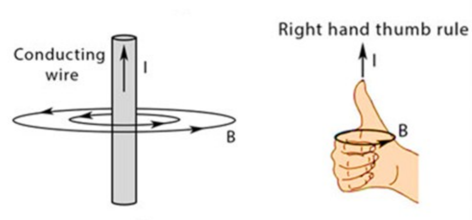
Now, this is the same as, so on the right side, you can see from here that you already have a surface integral. So, it is easy for you to write down these two equations side by side, and you see you are calculating both surface integrals. So, what is happening is that whatever is getting integrated, that quantity is now basically matching on both sides. So, you can see that the curl of E is basically minus d/dt of ... So, that is how you can get this differential form equation. So, this is very simple: E represents the induced electric field vector; this is basically del cross, which means you are essentially taking the curl. So, you are getting a circulating electric field that is basically coming from a time-varying magnetic flux, or you can say the rate of change of the magnetic flux density factor gives you the circulating electric field, which is also seen here. When the magnet is basically moving, the flux that is going through these lines is changing. So, that induces a different amount of current, and the direction of the current also reverses based on whether the magnet is going in or coming out because the magnetic field lines come out from the north pole and go into the south pole.

$$\oint_{Circuit} \boldsymbol{E}.\boldsymbol{dL} = \int_S \boldsymbol{\nabla} \times \boldsymbol{E}.\boldsymbol{dS} \text{ [Stokes' Theorem]}$$

$$\int_S \boldsymbol{\nabla} \times \boldsymbol{E}.\boldsymbol{dS} = -\frac{d}{dt}\int_S \mathbf{B}(\mathbf{t}).\boldsymbol{dS} = \int_S \frac{-d\boldsymbol{B}(\boldsymbol{t})}{dt}.\boldsymbol{dS}$$

Ampere's Law - no time dependence (Incomplete)

- Suppose you have a conductor (wire) carrying a current, **I**. Then this current produces a Magnetic Field which circles the wire.
- Ampère had shown how to make magnetism from electricity.
- **Right Hand Thumb Rule:**
  - Thumb points in the direction of the electric current and fingers curl around the current indicating the direction of the magnetic field.



Source: https://www.sciencefacts.net/amperes-law.html

Now we move on to the last law in Maxwell's equations. So before Maxwell actually put them together, all these laws were known independently. So the world actually knew the first half of this particular equation. Okay, it is known as Ampere's law, and it is very simple: it says that if you have a wire or a conductor carrying current, then this current will produce a magnetic field that will basically circle the wire.

Okay. So, Ampere basically showed how to make magnetism from electricity. So, a current-carrying wire can give you a circulating magnetic field. Now, what are the directions? How can you do it? You can actually get the magnetic field direction from the right-hand thumb rule. Put your thumb pointing in the direction of the current. Okay, your fingers will curl; your fingers of your right hand will curl, indicating the direction of the magnetic field.

Ampere's Law - no time dependence (Incomplete)

Ampere's Law – Integral Form

- From **Biot-Savart's Law**, the magnetic field due to a long straight wire is:

Since, $\vec{B}$ and $d\vec{l}$ are in the same direction,

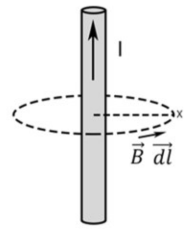
$$\vec{B}.d\vec{l} = B\,dl\cos 0 = B\,dl$$

Therefore,

$$\oint \vec{B}.d\vec{l} = \frac{\mu_0 I}{2\pi r}\oint dl$$

$$\Rightarrow \oint \vec{B}.d\vec{l} = \frac{\mu_0 I}{2\pi r}\oint (2\pi r)$$

$$\boxed{\Rightarrow \oint \vec{B}.d\vec{l} = \mu_0 I}$$



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Okay? So, that is how it is following the right-hand thumb rule. So, you can also write the amperes law in integral form. So, we will see where it comes from; it basically starts with the magnetic field that is there in the magnetic field. Because of this kind of carrying, what is the magnetic field that you can obtain from the Biot-Savart law? So, you can write that you know the magnetic field due to a long straight wire will be simply B equals mu naught i over 2 pi r. So, what are these? These are basically, you know, mu naught is the vacuum permeability, I is the amount of current, and 2 pi r is basically the circumference of the wire.

Now that you know B and dl in this particular case, dl is basically the distance along which the magnetic field lines are circulating. So you can take B and dl in the same direction, so you can write b dot dl equals b dl cos 0; ideally, theta is 0, but then they are both in the same direction, so you can take theta equals 0, which gives you b dl. So now if you basically integrate it along this closed loop, you write the closed integral, the line integral of b dot dl, and that will be equal to, you know, b, which you already know is basically mu naught i by 2 pi r. So what are you left with? You are basically left with this integral of dl, right? So when you take the integration along the closed loop, you will get nothing but 2 pi. So, this closed line integral will basically give you 2 pi r.

So, this 2 pi r and 2 pi r cancel out. So, what you get is simply the integral line integral of B dot dl equals mu naught i. So, this is the integral form of Ampere's law for static fields. Now, you can also write down that you know Ampere's law in differential form, so what you can do is first put B equals mu naught H in the integral form of Ampere's law, which is the one we have just seen: the loop or line integral of B dot dl equals mu naught I. So, B can now be written as μ_0H. So, what happens is μ_0 will cancel out from both sides; you will simply have, you know, the line integral of H dot dl is basically equal to the current enclosed, okay, by that loop or whatever is the current passing through that loop, right? So, once you have this equation, You can obtain the differential form of Ampère's law by applying Stokes' theorem.

$$\vec{B}.\,d\vec{l} = B\,dl\cos 0 = B\,dl$$

$$\oint \vec{B}.\,d\vec{l} = \frac{\mu_0 I}{2\pi r}\oint dl$$

$$\Rightarrow \oint \vec{B}.\,d\vec{l} = \frac{\mu_0 I}{2\pi r}\oint (2\pi r)$$

$$\Rightarrow \oint \vec{B}.\,d\vec{l} = \mu_0 I$$

Ampere's Law - no time dependence (Incomplete)

Ampere's Law – Differential Form

- We can put $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ in the integral form of Ampere's law $\oint \vec{B}.\,d\vec{l} = \mu_0 I$.
- Then, the differential form of Ampere's law can be determined by applying **Stokes Theorem**:

$$\oint \mathrm{H}.\,\mathbf{dL} = \boldsymbol{I}_{enc} \quad \Rightarrow \quad \oint \mathbf{H}.\,\mathbf{dL} = \int_S (\nabla \times \mathbf{H}).\,d\mathbf{S}$$

$$\boldsymbol{I}_{enc} = \int_S \mathbf{J}.\,\mathbf{dS} \quad \Rightarrow \quad \int_S (\nabla \times \mathbf{H}).\,\mathrm{d}\mathbf{S} = \int_S \mathbf{J}.\,\mathrm{d}\mathbf{S}$$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J}$$ → Incomplete and not valid for electrodynamics

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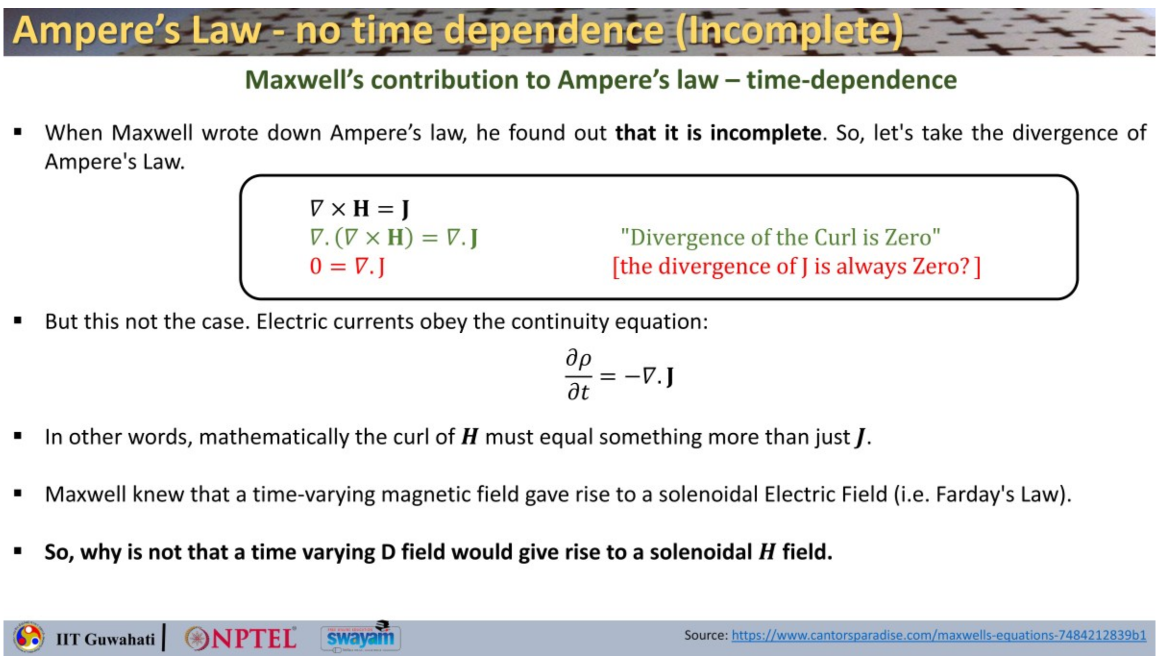
In Stokes' theorem, you can replace this line integral with the curl of the magnetic field, and then it will become a surface integral, right? So, this one can simply be written as the curl of H ds. So, that is now the surface integral, and if you know the surface current density, that is J, and if you integrate it over the entire surface, you will get the total current enclosed. So, this term on the right can now be written like this. So, you can write it as the surface integral J dot ds. So, what are you doing again? So, you are bringing everything to the same kind of surface integral.

So, whatever is getting integrated will be basically the same thing. So, here you see that you are basically equating these two. So, essentially, the curl of H is nothing but J because on both sides you are taking a surface integral. So, this is what Ampere's law looks like in the differential form: the curl of H equals J. Now, this is incomplete and it is not valid for electrodynamics.

$$\oint \boldsymbol{H}.\,\mathbf{dL} = \boldsymbol{I}_{enc} \quad \Rightarrow \quad \oint \mathbf{H}.\,\mathbf{dL} = \int_S (\nabla \times \mathbf{H}).\,d\boldsymbol{S}$$

$$\boldsymbol{I}_{enc} = \int_S \mathbf{J}.\,\mathbf{dS} \quad \Rightarrow \quad \int_S (\nabla \times \boldsymbol{H}).\,\mathrm{d}\boldsymbol{S} = \int_S \mathbf{J}.\,\mathrm{d}\boldsymbol{S}$$

$$\Longrightarrow \nabla \times \mathbf{H} = \mathbf{J}$$



Why? Because you know that when Maxwell basically wrote down this Ampère's law, he found that it is not complete. Because when he took, he started with Ampere's law, okay, and then he took the divergence of this Ampere's law on both sides. You see, you take the divergence, so the divergence of a curl is 0. That means the divergence of J, which is this one, is becoming 0. But is it always the case? The case? The answer is no.

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla.(\nabla \times \mathbf{H}) = \nabla.\mathbf{J} \qquad \text{"Divergence of the Curl is Zero"}$$

$$0 = \nabla.\mathrm{J} \qquad \text{[the divergence of J is always Zero?]}$$

$$\frac{\partial \rho}{\partial t} = -\nabla.\mathbf{J}$$

Ampere's Law - no time dependence (Incomplete)

Maxwell's contribution to Ampere's law – time-dependence

- When Maxwell wrote down Ampere's law, he found out **that it is incomplete**. So, let's take the divergence of Ampere's Law.

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla.(\nabla \times \mathbf{H}) = \nabla.\mathbf{J} \qquad \text{"Divergence of the Curl is Zero"}$$

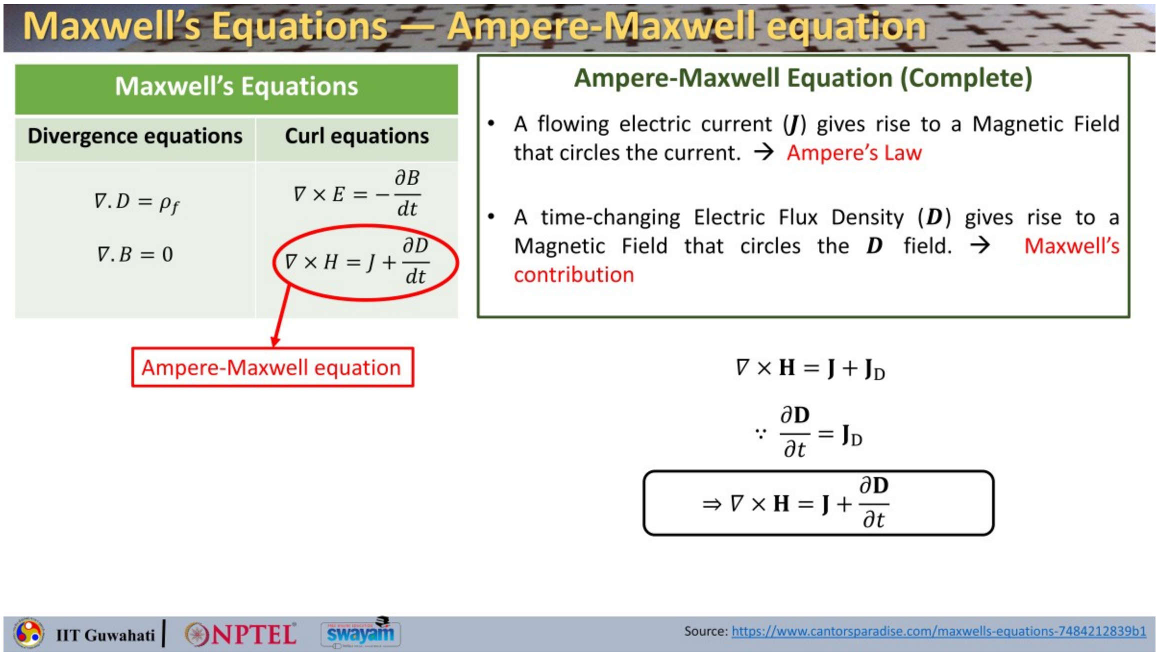
$$0 = \nabla.\mathrm{J} \qquad \text{[the divergence of J is always Zero?]}$$

- The universe loves symmetry, so Maxwell introduced the term named as the *displacement current density*:

$$\frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{J}_{\mathrm{d}} \qquad \text{[Displacement Current Density]}$$

In electromagnetic theory, the continuity equation is an empirical law that expresses local charge conservation, and charge conservation is much more fundamental than Maxwell's equations; it basically states that the divergence of the current density J. In amperes per square meter, it will be basically equal to the negative rate of change of the charge density rho, which is typically represented in coulombs per meter cubed. So, you can say that the divergence of J should ideally be minus dou rho over dou t. So, this is what is happening: as I told you, the electric currents basically obey the continuity equation, which is this, and from Gauss's law we know that the divergence of D basically gives you rho; D is the electric flux, right? So, in other words, it looks like this is basically this one. So, to make it equal to 0, you basically have to add something to it.

$$\frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{J}_{\mathrm{d}}$$



So, you can intuitively see that the curl of H is not only J; there is basically something more than J. So, that is what Maxwell found out, and he knew that a time-varying magnetic field gives rise to a solenoidal electric field, which is Faraday's law. So, why not, you know, a time-varying D field would give rise to a solenoidal H field, right? So, because the universe loves symmetry, Maxwell introduced a new term called displacement current density, denoted as J_d, which is basically dot d dot t, okay. Now, what is this term? So if you look at what Maxwell has done, he has basically added this new term right into Ampere's law, and now this equation becomes Ampère-Maxwell's equation.

Right. So, this is a complete one; it tells you that a flowing electric current gives you a magnetic field that basically circles the current. So, that was coming from Ampere's law, and then Maxwell basically added this new term, which states that a time-changing electric flux density also gives rise to a magnetic field that circles the d field, and that is basically Maxwell's contribution. And the second part, this part, is introduced for electrodynamics. Right. And that is how you can write the curl of H: it is not only J, it is basically J plus J_D.

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_\mathrm{D}$$

$$\because \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_\mathrm{D}$$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

What is J_D? J_D is dot D dot T, and okay, so you can basically get this particular equation. So, what is happening? Once again, if you take the divergence on both sides, you will have the divergence of the curl of H, which is 0; the divergence of J is basically giving you minus dou rho dou t, and then the divergence of D will give you rho. So, this term will give you plus dou rho dou t, this minus dou rho dou t, and this plus dou rho dou t, which will give you 0, and that is how this equation is also satisfied. So, the fourth Maxwell's equation is basically telling you that the magnetic field can be generated in two ways, namely with electric current as well as with a changing electric field.

Maxwell's Equations — Static vs Dynamic

Table: Comparison of Maxwell's equations for static and time-varying electromagnetic fields.

	Electrostatics / Magnetostatics	**Time-Varying (Dynamic)**
Electric & magnetic fields are...	independent	possibly coupled
Maxwell's eqns. (integral)	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{encl}$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ $\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{encl}$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{encl}$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{s}$ $\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{encl} + \int_S \frac{\partial}{\partial t} \mathbf{D} \cdot d\mathbf{s}$
Maxwell's eqns. (differential)	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D}$

Note: Differences in the time-varying case relative to the static case are highlighted in blue.

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Source: https://www.cantorsparadise.com/maxwells-equations-7484212839b1

So finally, here we can summarize Maxwell's equations in terms of the static and dynamic fields, where the E and H fields are basically independent in the case of electrostatics or magnetostatics.

But they are coupled in the case of time-varying or electrodynamics, okay. So the equations here on the left show the equations for electrostatics and electromagnetics because, as soon as, their dynamics in the electrodynamics, these new terms. In blue, they are basically added up. So, these are Maxwell's equations in the integral form, and these are Maxwell's equations in the differential form. Now, we basically know how we can go from one form to the other.

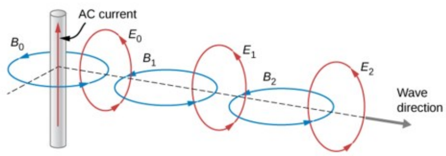
So, Maxwell's equations in integral form give you a much clearer picture and understanding of the entire scenario from which we can derive these differential forms that are easy to compute using some theorems like Stokes' theorem or the divergence theorem.

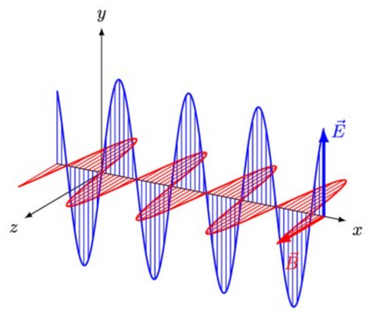
Wave Equation

- The **electromagnetic wave equation** is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum.

- The homogeneous form of the equation, written in terms of either the electric field $\boldsymbol{E}$ or the magnetic field $\boldsymbol{B}$.

$$\nabla^2 \boldsymbol{E}(r,t) = \frac{1}{v^2}\frac{\partial^2 \boldsymbol{E}(r,t)}{\partial \boldsymbol{t}^2}$$





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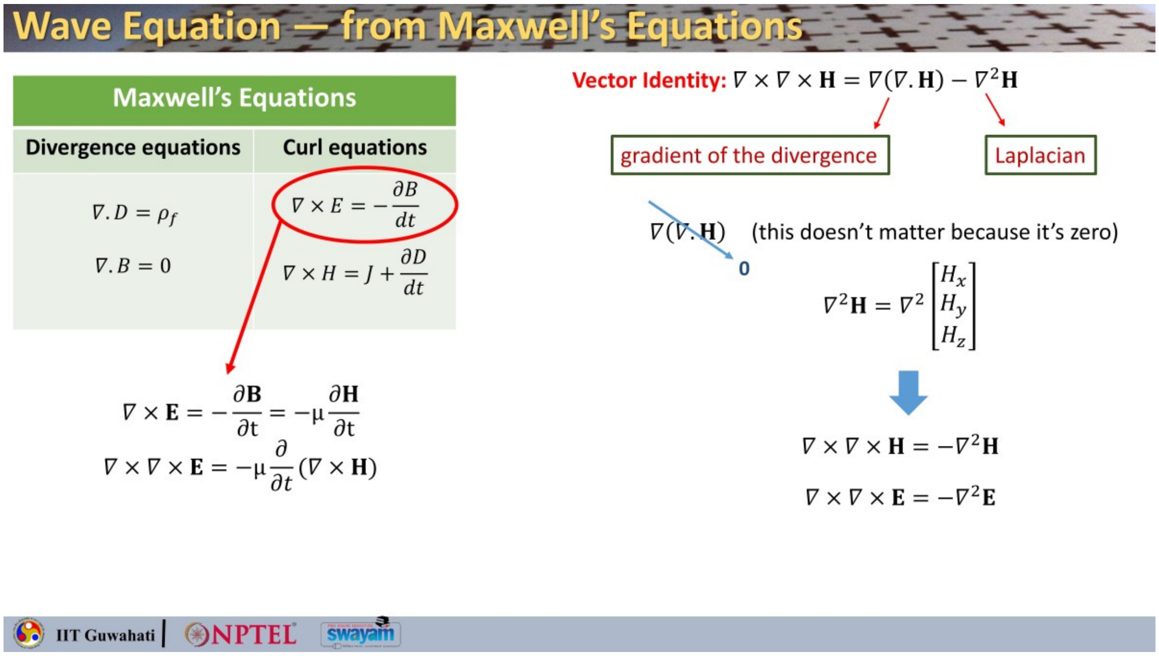
Source: https://phys.libretexts.org/Bookshelves/University_Physics
Source: https://commons.wikimedia.org/wiki/File:EM-Wave.gif

Now that we know how to determine the electric and magnetic fields in electrodynamics, we want to determine how these fields will propagate through a region, and this can be described by the electromagnetic wave equation, which is essentially a second-order partial derivative that describes the propagation of an EM wave or electromagnetic wave. through a medium or in vacuum. So, here you can see how the wave is propagating, and you have an electric field vector that is oscillating along the y direction; you have the magnetic field vector oscillating along the z direction, and the field is propagating along the x direction. So, the homogeneous form of the equation can also be thought of like this: you have electric current, which has a circulating magnetic field, and then you have an electric field between the loops, and so on.

So, this is how they are coupled, and they are propagating forward. So, the homogeneous form of the equation can be written either in terms of the electric or magnetic field. del square E(r,t) is ($1/v^2$) dou^2 E(r,t) by dou t^2. So this is how the wave equation typically looks. So it can be written for both the electric field as well as the magnetic field, okay.

$$\nabla^2 \boldsymbol{E}(r,t) = \frac{1}{v^2}\frac{\partial^2 \boldsymbol{E}(r,t)}{\partial \boldsymbol{t}^2}$$



So our goal here is to determine how the wave equation can be derived from Maxwell's equations. So you can see Maxwell's equations here. So, to start, let me throw out a vector identity, which is basically a mathematical manipulation that is true for all vector fields, and here we will assume that we are in a source-free region. So, there are no charges and no current is flowing. So, we will start with this equation, say curl of E equals minus dou B dot E.

So, B is mu H. You can write it like that, and then you take the curl on both sides. So, you can see it is the curl of the curl of E, which is nothing but minus mu dot dot E curl of H. So, you can use this particular vector identity that the curl of the curl of H equals the gradient of the divergence of H minus the Laplacian of H or del squared H. So, this is basically the gradient of the divergence and this is the Laplacian. Now, for the magnetic field, the divergence is 0, so this equation basically boils down to the curl of the curl of H being minus del squared H, and because of the symmetry, you can also write the same equation for the electric field, and that is also fine because we are talking about a source-free region or a region where no current is flowing.

So, you can also write that the curl of the curl of E is minus del squared E. So, if you go to this equation, the curl of the curl of E equals minus mu dot dot T curl of H. So, what you have seen is that the left side can be written as minus del squared E. What about the right side? The right side is basically the curl of H. So, the curl of H can be represented from the fourth Maxwell's equation, which is J plus dot D dot E, right? So, this equation becomes like this.

So, you have minus del squared E, which is basically mu dot dot E J plus dot D dot E, right? Clear? Now, because there is no current, it is a source-free region; this term also becomes 0. So, what do you have? The left side remains as it is. The right side only becomes minus mu, okay. So, where is the minus coming from? So, you have a relation from the curl of the curl equation; the curl of H can be written as minus del squared E equals minus mu dot J plus dot E dot E. Now, because it is a source-free region, this J term becomes 0, so you have minus del squared E equals minus mu, D can be written as epsilon E, so you have epsilon outside, so you will have this one.

$$\times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}} = -\mu \frac{\partial \mathbf{H}}{\partial \mathrm{t}}$$

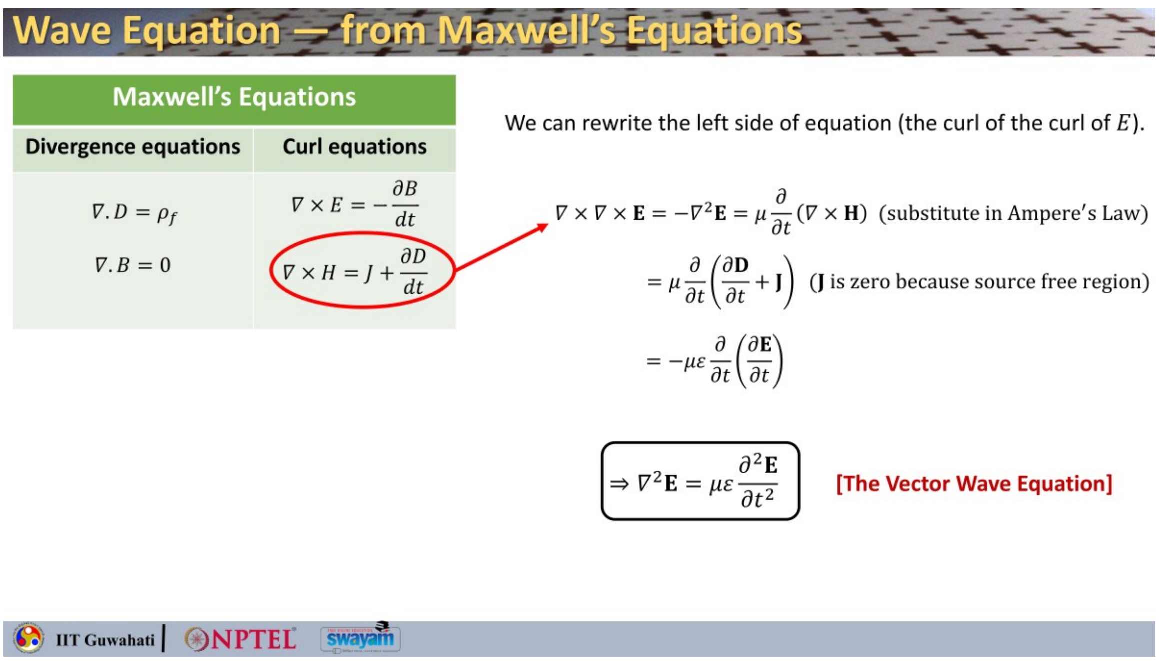
$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$

$$\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla . \mathbf{H}) - \nabla^2 \mathbf{H}$$

$$\nabla^2 \mathbf{H} = \nabla^2 \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

$$\nabla \times \nabla \times \mathbf{H} = -\nabla^2 \mathbf{H}$$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E}$$



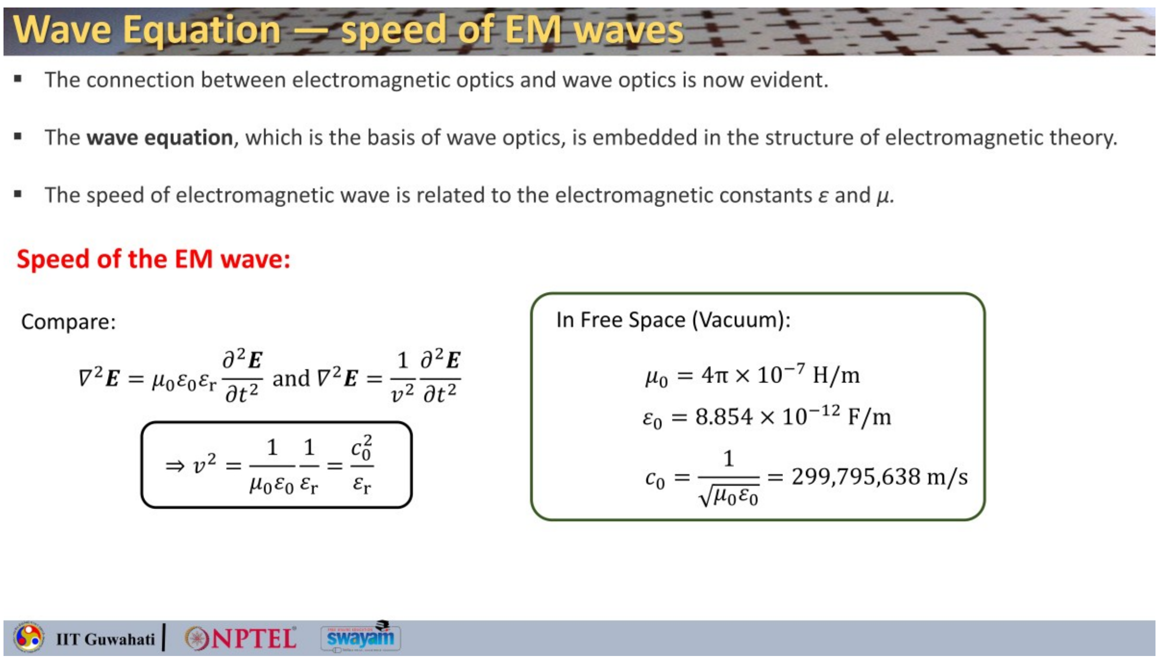
Okay, which is nothing but the second derivative, the partial derivative of E with respect to t, okay. So this is how you get there. So del square E is basically mu epsilon dot square E dot t square, fine. So this is the vector wave equation. So you know it actually is basically three equations because you can think of Ex, Ey, and Ez, okay.

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} = \mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$

$$= \mu \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

$$= -\mu\varepsilon \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\Rightarrow \nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$



These are the three components of the electric field. Now, we will also see how this wave equation can be used to obtain the speed of electromagnetic waves. So, the connection between electromagnetic waves and wave optics is now more evident; you can see that the wave equation, which is basically the basis of wave optics, is embedded within the structure of electromagnetic theory. The speed of the electromagnetic wave is related to the electromagnetic constants epsilon and mu. So, if you want to obtain the speed of an electromagnetic wave, you have to compare this equation, okay, that you have just obtained: del square e equals, here it was, if you see from the previous slide, here it was mu epsilon.

So, mu is basically mu naught mu r. So, relative permeability and vacuum permeability mu r is 1 in vacuum and for any other material or in most materials. So, you take mu naught only; epsilon can be written as epsilon naught, which is the vacuum permittivity and the relative permittivity, and then you have this term. If you compare it with the wave equation, you will see that this term is basically 1 over v squared. So, you can say v squared is basically 1 by mu naught epsilon naught times 1 by epsilon r. So, this first ratio is basically c squared, which is the speed of light in a vacuum.

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \text{ and } \nabla^2 \boldsymbol{E} = \frac{1}{v^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\Rightarrow v^2 = \frac{1}{\mu_0 \varepsilon_0} \frac{1}{\varepsilon_r} = \frac{c_0^2}{\varepsilon_r}$$

Wave Equation — speed of EM waves

- The connection between electromagnetic optics and wave optics is now evident.
- The **wave equation**, which is the basis of wave optics, is embedded in the structure of electromagnetic theory.
- The speed of electromagnetic wave is related to the electromagnetic constants ε and μ.

Speed of the EM wave:

Compare:

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \text{ and } \nabla^2 \boldsymbol{E} = \frac{1}{v^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\Rightarrow v^2 = \frac{1}{\mu_0 \varepsilon_0} \frac{1}{\varepsilon_r} = \frac{c_0^2}{\varepsilon_r}$$

In Free Space (Vacuum):

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299{,}795{,}638 \text{ m/s}$$

So, c squared divided by epsilon r. So, you have these values; these are the constants people use: mu naught, epsilon naught, and from that, you can obtain what C naught is, which comes out to be very close to 3 into 10 to the power 8 meters per second. So, this is how you can obtain the speed of light or the speed of any electromagnetic wave in any medium, which is basically having this epsilon r. So, this is v squared equals c naught squared. So, if you want to find what v is, v equals c divided by the square root of epsilon r. Now, if this material is lossless, then the square root of epsilon r will simply be the refractive index of the material, ok.

So, with that, we will conclude this lecture. So, we will go into the electromagnetic properties of materials in more detail in the next lecture. If you have any queries regarding this lecture, you can always drop me an email at this particular email address, mentioning the lecture number and the course title in the subject.