Course Name: Introduction to Microwave and Optical Metamaterials
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Week-10
Lecture-47

Lec 47: Principles of Transformational Electromagnetics

Hello students, welcome to lecture 47 of the online course on the introduction to microwave and optical metamaterials. In today's lecture, we will learn about the principles of transformational electromagnetics. This is also popularly known as transformation optics because this concept was initially developed for that Optics domain, but it can also be extended to microwave frequencies.

Lecture Outline

- Transcriptional Electromagnetics (TEM)
  - Introduction
  - Fundamentals
  - Principles
- Coordinate Transformation
  - Cylindrical to Cartesian
  - Spherical to Cartesian
  - Cartesian to Cartesian
- Form Invariance of Maxwell's Equations

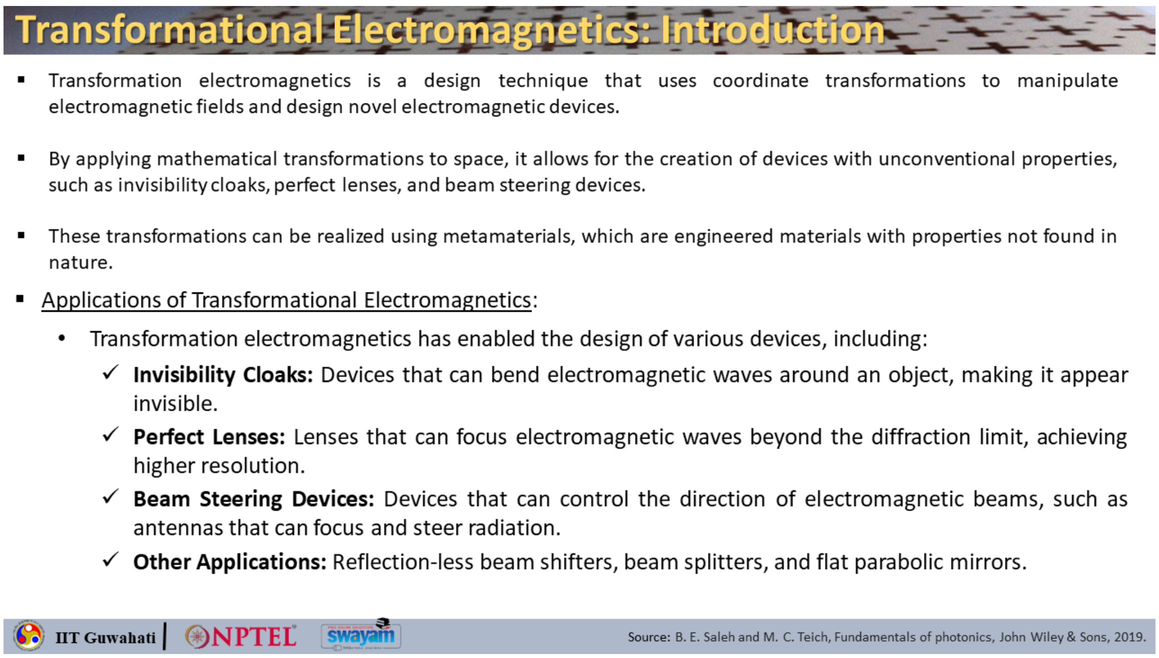
So, we will discuss the fundamental principles of this field, taking transformational electromagnetics into account. As you know, the key term TEM can be referred to in shorthand.

Then we will see how you perform the coordinate transformation from cylindrical to Cartesian and from spherical to Cartesian. And Cartesian to Cartesian, and finally, we will look into how Maxwell's equation shows form invariance. That is very important for this transformational electromagnetics or transformational optics to work.

Transformational Electromagnetics

So, now let us look at what transformational electromagnetics is.

So, this is basically a design technique that uses coordinate transformation to manipulate electromagnetic fields and allow you to design novel electromagnetic devices. So, by applying a mathematical transformation to the space, this technique basically allows the creation of devices with Unconventional properties, something like invisibility cloaks, perfect lenses, and beam steering devices. so how do you realize this kind of devices you can realize this kind of transformations by using metamaterials, which is the theme of this course, you can engineer the properties of the materials. Because you do not need to rely on natural materials and their fixed properties.



So, the applications of transformational electromagnetics can be for invisibility cloaks that is you know You can develop devices that can bend electromagnetic waves around a particular object, making that object appear invisible. Other applications could be like a perfect lens that can focus electromagnetic waves beyond the diffraction limit. So, it can help you achieve a much higher resolution and resolve very, very fine details of a particular image or feature. Then you can also make beam steering devices that allow you to control the direction of the electromagnetic beam. You know, such as antennas that can focus and steer the radiation based on your application.

There are other applications as well that can be like reflectionless beam shifters; you can design beam splitters and also flat parabolic mirrors.

Fundamentals of TEM

- Transformation electromagnetics is a mathematical tool that facilitates the design of optical materials that guide light along desired trajectories.
- The underlying concept relies on a geometrical transformation that converts simple trajectories into desired ones.
- In order that Maxwell's equations remain valid, the optical parameters associated with the transformed equivalent system must also be modified, and this establishes the character of the required optical material.
- As a simple example of such equivalence, a local compression of the coordinate system by a scaling factor is equivalent to a local increase of the refractive index by the same factor, so that the optical path length (product of length and refractive index) remains unchanged.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, if someone asks you what transformation electromagnetics or transformation optics is, the clear definition is that it is basically a mathematical tool that facilitates the design of optical materials that can guide light along any desired trajectory. So, the underlying concept here basically relies on the geometrical transformation that can convert simple trajectories into your desired trajectory. So, in order for you to know that Maxwell's equations remain valid, the optical parameters associated with the transformed. equivalent system must be modified and this establishes the character of the required optical material.

So, you can take one simple example of such equivalence. So, you can think of, you know, a local compression of the coordinate system. A scaling factor is going to be equivalent to the local increase in the refractive index of some medium by the same factor. So that you know, the optical path length, which is basically the product of the length and the refractive index, remains unchanged, Right. So, that way you can think of how the local compression or expansion of a coordinate system can be. Mapped into the material property rights are those materials you need to design using metamaterials.

Fundamentals of TEM

- A four-step design procedure provides a guide:
  1. Begin with a pilot physical system for which the optical trajectories are known, such as a homogeneous and isotropic material.
  2. Find a coordinate transformation that converts these trajectories to the desired ones.
  3. Determine the transformed physical parameters of the equivalent material.
  4. The new material will implement the desired electromagnetic trajectories in the original coordinate system.
- Since geometrical transformations generally involve changes of directions and introduce direction-dependent scaling, the transformed parameters are generally both anisotropic and spatially varying.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, in summary, you can say that there are it is basically a four-step design procedure, okay First, begin with a pilot physical system for which the optical trajectories are known such as you can consider homogeneous and isotropic material. Then you find a coordinate transformation that converts these trajectories into the desired ones, Okay.

To do that, you have to determine the transformed physical parameters of the equivalent material. which can do the same kind of, you know, distortion to the electromagnetic wave or light, okay. So, that should go into the material property, and then finally you need to implement that new material. That will give you the desired electromagnetic trajectories in the original coordinate system.

Now, since geometrical transformations generally involve changes in direction and introduce direction-dependent scaling, the performance parameters are generally both isotropic and spatially varying.

So, the principles of TEM tell you that the mapping you will see between the old and the new system has to go into the material properties. Now, how do you normally consider the light-matter interaction or electromagnetic wave-matter interaction?

Principles of TEM

- Let $\{\varepsilon_{ij}\}$ and $\{\mu_{ij}\}$ be the elements of the permittivity and permeability tensors of the original material in the original coordinate system (x_1, x_2, x_3).

- The elements of the permittivity and permeability tensors of the equivalent material (denoted by the superscript "e") in the transformed coordinate system (u_1, u_2, u_3) are then related to the original elements by the matrix equations:

$$\varepsilon' = \frac{A^T \varepsilon A}{|\det A|} \qquad \mu' = \frac{A^T \mu A}{|\det A|}$$

  - The quantity A^T is the transpose of A, and ε and μ are 3×3 matrices whose elements are $\{\varepsilon_{ij}\}$ and $\{\mu_{ij}\}$, respectively.

- Here A is the 3 × 3 Jacobian transformation matrix, whose elements are the partial derivatives:

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \qquad i, j = 1,2\ 3 \ldots$$

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

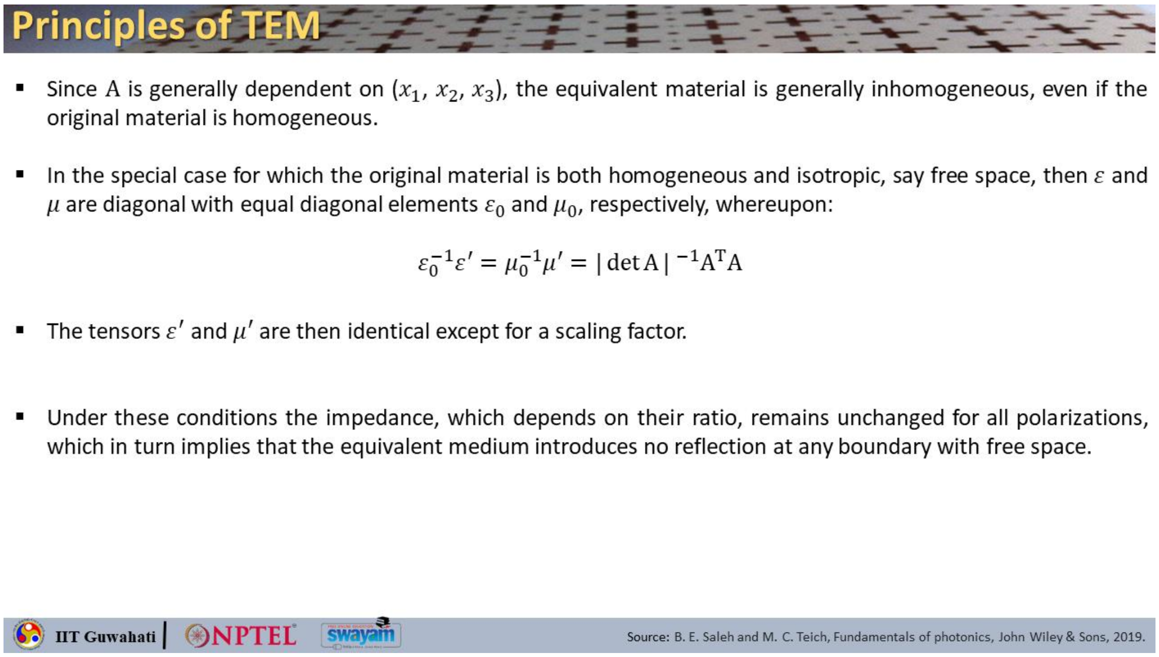
You consider the two basic parameters of the material: its electric permittivity and magnetic permeability. So, let us consider epsilon ij; these are now in tensor form, and mu ij are the elements of the tensor. So, ij will tell you one particular element of the tensor, okay. So, this represents the permeability and the permeability tensor elements of the original material in the original coordinate system.

So, you can say it is x1, x2, x3, or you could have named them x, y, z, whichever you like. Now, the elements of the permittivity and permeability tensors of the equivalent material are denoted by the superscript prime is okay in the transformed coordinate system. So, you can consider the unit vectors to be, or the coordinate system to have, the 3 axes u1, u2, and u3, okay. The unit vectors will obviously be u1 cap, u2 cap, and u3 cap. In this particular coordinate system, you can, you know, correlate the parameters of the new permittivity.

And the permeability parameters of the old system using this formula. So, you can see the new permittivity epsilon prime will be $\varepsilon' = \frac{A^T \varepsilon A}{|\det A|}$. Similarly, μ' will be $\frac{A^T \mu A}{|\det A|}$.

So, now what are A and A transpose? A transpose is this; obviously, it is basically the transpose of A. Now, A is the most critical thing here; we are coming to that, but before that, we have to just understand epsilon.

And mu is basically a 3 by 3 matrix; that is why you can have this epsilon ij as each element mu ij, where i and j both run from 1 to 3. Now, here is how A looks, okay. So, A is basically a 3 by 3 Jacobian transformation matrix. So, the elements are basically the partial derivatives, okay? The partial derivatives of the new coordinate system divided by the old one, okay? So, $A_{ij} = \frac{\partial u_i}{\partial x_j}$; so i and j both are 1, 2 and 3 okays.



So, you can see that A is generally dependent on x_1, x_2, and x_3. So, the equivalent material is generally inhomogeneous even if the original material is homogeneous. So, in a special case, if you consider the original material to be both homogeneous and isotropic, say free space, in that case. You know that epsilon and mu are basically diagonal, and they have equal diagonal elements of epsilon naught and mu naught, right? So, in that case, you can simply write $\varepsilon_0^{-1}\varepsilon'' = \mu_0^{-1}\mu'$ will be nothing but $|\,det\,A\,|^{-1}A^TA$, okay. It is coming from the previous set of equations; nothing new here, okay. So, what I want to tell you here is that the tensors epsilon prime and mu prime are identical except for a scaling factor, right? So, each has its own scaling factor, right? So, this will remain the same.

Under these conditions, you can see the impedance, which is basically the ratio of this permittivity and permeability. that remains unchanged for all polarizations, which in turn basically implies that the equivalent medium will introduce no reflections at the boundary with free space.

So, this is the beauty of this transformation in electromagnetics allows you to define and make the material work according to your desired plan for the wave project trajectory.

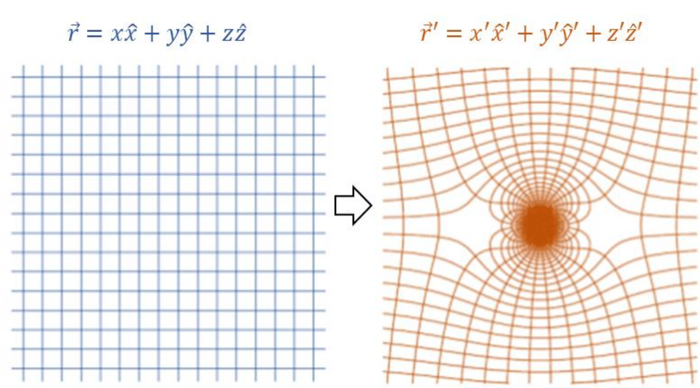
Coordinate Transformation

Now let us move into coordinate transformations and see how they are done.

So, we can basically map one coordinate space into another, okay?

Coordinate Transformation

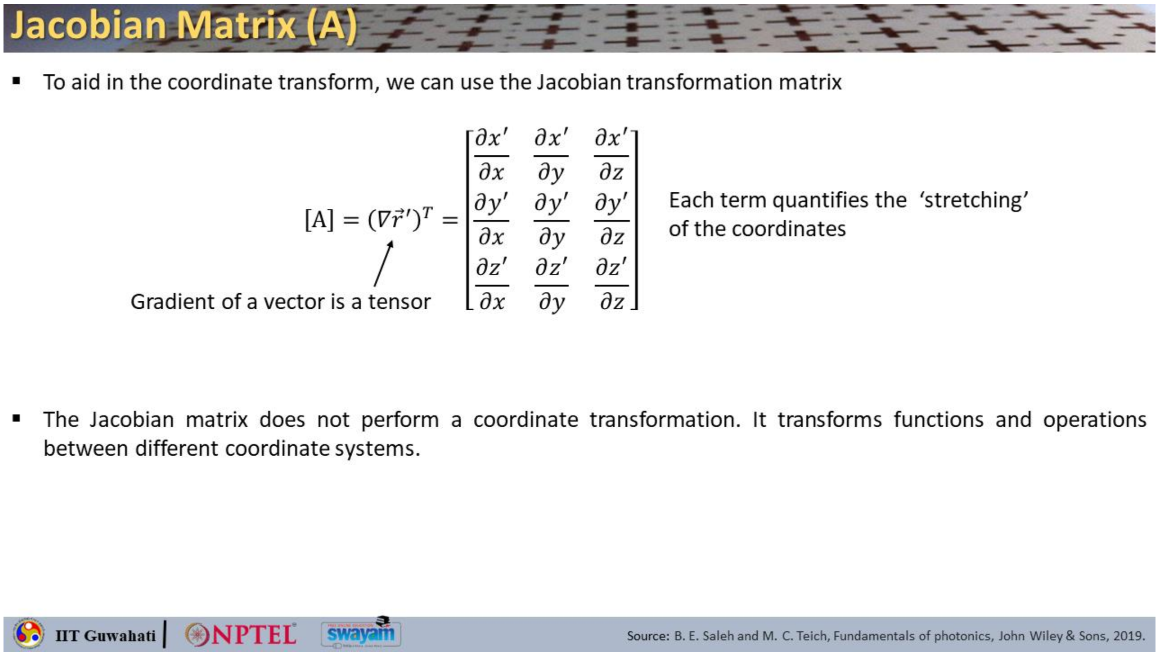
- We can map one coordinate space into another:



Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, you can simply take this kind of coordinate space where a position vector r is given as $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, Okay, that can be transferred into this kind of coordinate space, where r prime, the position vector, is now $\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$, right.

Now, we have seen that in doing this kind of coordinate transformation, the most critical element is the Jacobian transformation matrix A is also called J in a few places. So, A is basically $(\nabla\vec{r}')^T$.



So, this is how you can basically calculate each element as $\begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix}$, right. Fine, so what you see is that the Jacobian matrix does not perform a coordinate transformation on its own. It basically transforms the functions and operations between different coordinate systems. For example, let us convert a cylindrical coordinate system to a Cartesian coordinate system.

Cylindrical to Cartesian

- The Cartesian and cylindrical coordinates are related through:

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

- The Jacobian Matrix is then:

$$[A] = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{bmatrix} \Longleftrightarrow \begin{matrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & \frac{\partial x}{\partial z} = 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & \frac{\partial y}{\partial z} = 0 \\ \frac{\partial z}{\partial \rho} = 0 & \frac{\partial z}{\partial \phi} = 0 & \frac{\partial z}{\partial z} = 1 \end{matrix}$$

$$[A] = \begin{bmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, this is how the Cartesian and cylindrical coordinates relate. You have all known this for a long time. So, x can be written as ρ cos φ, y equals ρ sin φ, and z remains z. So, I am not showing the regular picture of. How a cylindrical coordinate system and a Cartesian coordinate system look like, I believe all of you know that, right? So, my idea here is to tell you how the Jacobian matrix will look.

So, we are going from cylindrical coordinates to Cartesian coordinates. So, you see the denominator is basically having the old coordinate system and the numerator is basically having the new coordinate system. So, this is how you can calculate the Jacobian matrix A: $[A] = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{bmatrix}$.

So, once you compute these, x is basically ρ cos φ, okay. So, you can calculate: $\frac{\partial x}{\partial \rho} = \cos\phi$ and this $\frac{\partial x}{\partial \phi} = -\rho sin\phi$, and $\frac{\partial x}{\partial z} = 0$. Similarly, you can calculate $\frac{\partial y}{\partial \rho}$, which will be $\sin\phi$, $\frac{\partial y}{\partial \phi} = \rho\cos\phi$, and $\frac{\partial y}{\partial z} = 0$, okay and then you can compute because z equals z there is no transformation okay along the z axis there is okay it remains same. So, you can simply have these two terms 0 and $\frac{\partial z}{\partial z}$ equals 1.

So, with that, you can see this is the Jacobian matrix that can be used to transform the functions. and the variables from the cylindrical to the Cartesian coordinate system.

Next, we can look into converting spherical to Cartesian coordinates, okay? So, the Cartesian and spherical coordinates are basically related like this: x can be written as $r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$.

Spherical to Cartesian

- The Cartesian and spherical coordinates are related through:

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

- The Jacobian Matrix is then:

$$[\mathrm{A}] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \Leftrightarrow \begin{matrix} \frac{\partial x}{\partial r} = \sin\theta\cos\phi & \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \\ \frac{\partial y}{\partial r} = \sin\theta\sin\phi & \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi & \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \\ \frac{\partial z}{\partial r} = \cos\theta & \frac{\partial z}{\partial \theta} = -r\sin\theta & \frac{\partial z}{\partial \phi} = 0 \end{matrix}$$

$$[\mathrm{A}] = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

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Just for a quick recap, theta is nothing but the polar angle, and phi is basically your azimuthal angle on the xy plane, Right. So, once you know that, you can calculate the Jacobian matrix as A; this is a 3 by 3 matrix. So, you can calculate this as: $[\mathrm{A}] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}$, because your previous coordinate system is r, theta, phi, and the new coordinate system that is x prime, y prime, and z prime is basically your x, y, and z, Okay. So, now you can calculate all nine of these elements. So, I will just take one or two examples.

So, this is the first one: $x = r\sin\theta\cos\phi$; this is the first relation. So, if you want to take the partial derivative with respect to r, you simply have $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, okay. Similarly, if you take another one, say x and phi. So, this is the relationship. So, you take $\frac{\partial x}{\partial \phi}$, which will be $-r\sin\theta\sin\phi$, right.

Accordingly, you calculate all these 9 elements, and you get that this is the Jacobian matrix for this particular case.

So, this Jacobian matrix will help you transform, you know, the functions and the operations from the spherical to Cartesian coordinate system.

Cartesian to Cartesian

- Coordinates in Cartesian space can be transformed according to:

$$x' = x'(x, y, z) \qquad y' = y'(x, y, z) \qquad z' = z'(x, y, z)$$

- The Jacobian matrix is defined as:

$$[\mathrm{A}] = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix}$$

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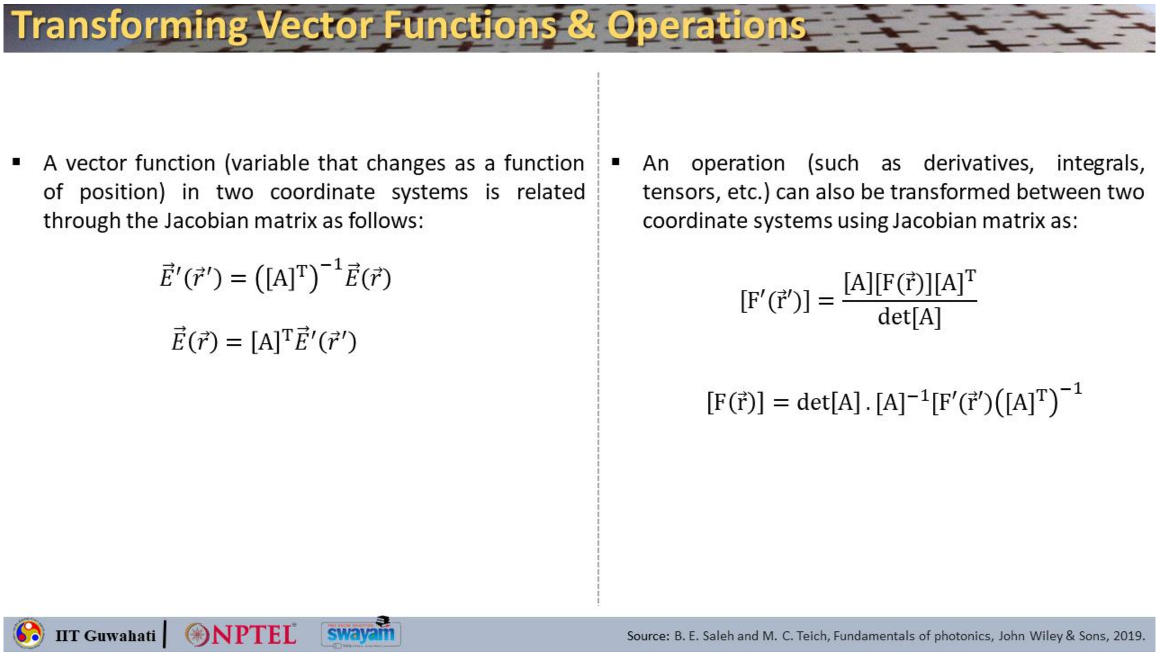
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

You can also, you know, transform between the two Cartesian coordinate systems. So, you can say that you know the new one, x prime, will be a function of x, y, and z, okay. So, this is how the Jacobian matrix will look. So, x prime, y prime, and z prime are basically the new axes, and x, y, z are the old axes.

So, this is how the transformation looks.



Now I have been repeating that the Jacobian matrix can help you transform vector functions and operations. So, let us see how it is done. So, you take a vector function that is basically a variable that changes as a function of position, okay, in two coordinate systems, that can be related to the Jacobian matrix through this kind of relation. So, $\vec{E}'(\vec{r}')$ will be $([A]^T)^{-1}\vec{E}(\vec{r})$, Okay. So, you can simply write E; this can also be any field, including electric fields for that matter, okay. You can write E' is nothing but $[A]^T\vec{E}'(\vec{r}')$, okay. So, that way any vector function can be, you know, transferred from one coordinate system to another. Another coordinate system if you know the Jacobian matrix A. Similarly, if you take any operation, such as a derivative or an integral, you know that it can also be transformed between the two coordinate systems using the Jacobian matrix. So, you can write you know $[F'(\vec{r}')]$ that is an operation that you are doing okay. So, $F'(\vec{r}')$ can be written as $\frac{[A][F(\vec{r})][A]^T}{\det[A]}$, which is the operation in the previous coordinate system; then you have A transpose divided by the determinant of A. So, that way you can write that you know this old one can be nothing but $\det[A] \,.\, [A]^{-1}[F'(\vec{r}')([A]^T)^{-1}$ that is the already transformed the function or the operation in the transform coordinate system into A transpose inverse. So, what I am showing here is that, you know, using the Jacobian matrix, you can transfer or convert the vector function. or any operation between the two coordinate systems, and that is possible because of the form invariance of Maxwell's equations.

Form Invariance of Maxwell's Equations

Maxwell's Equations – Form Invariant

- In any coordinate system, Maxwell's equation can be written as:

Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$	$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$	$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$
$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$	$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$	$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$

- We can transform Maxwell's equations to a different coordinate system, but still they have the same form:

$$\nabla' \times \vec{H}' = j\omega[\varepsilon']\vec{E}'$$

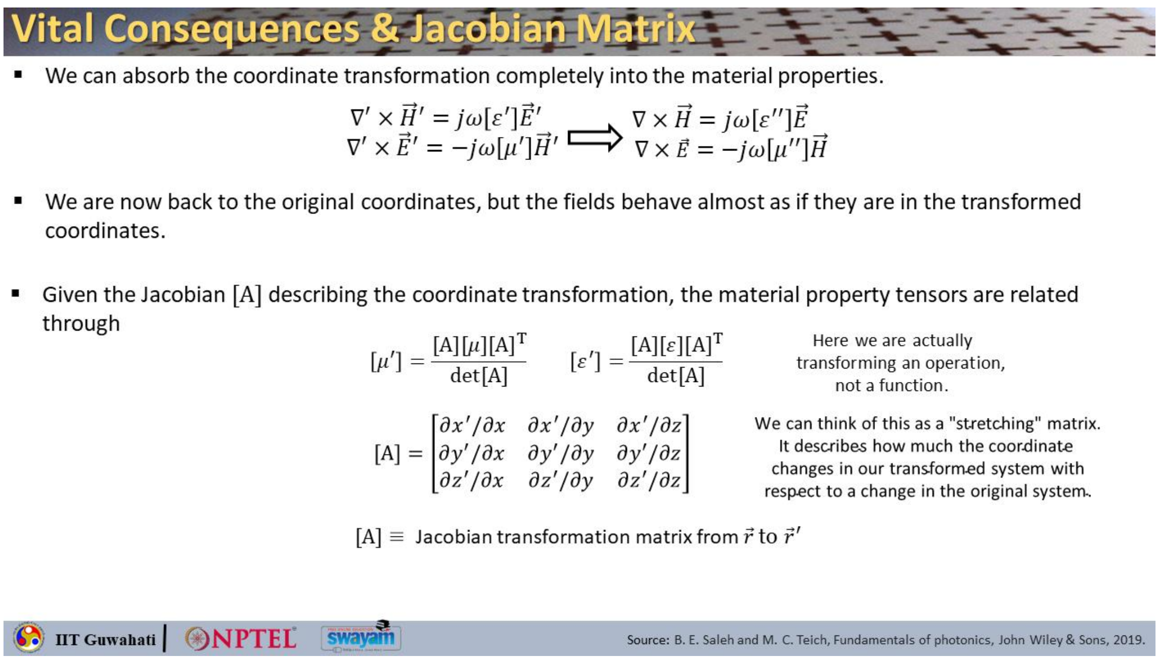
$$\nabla' \times \vec{E}' = -j\omega[\mu']\vec{H}'$$

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, if you write Maxwell's equations in any coordinate system, it basically looks like this. So, in the Cartesian coordinate system, you have seen that the $\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$ and $\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$, okay.

Now, if you change it to a cylindrical coordinate system, it looks exactly the same, okay? If you go to a spherical system, it looks exactly the same. So, we can transform

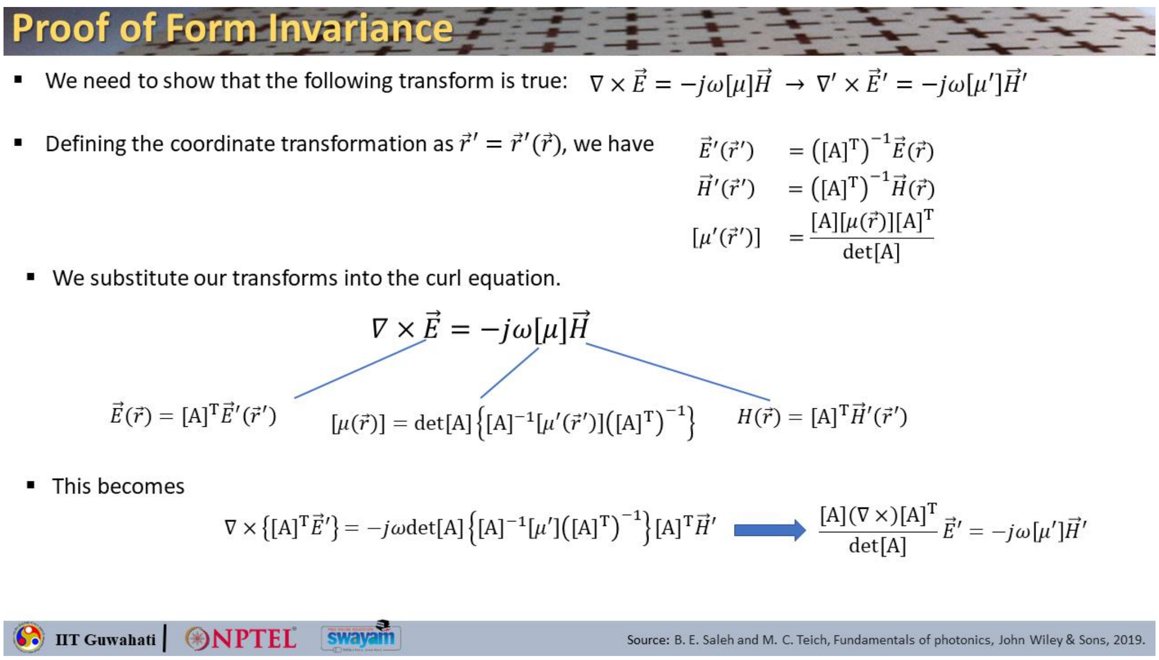
Maxwell's equation to a different coordinate system, but they will basically look exactly the same. So, you can write $\nabla' \times \vec{H}'$ that will be equal to $j\omega[\varepsilon']\vec{E}'$and $\nabla' \times \vec{E}' = -j\omega[\mu']\vec{H}'$. So, that is basically the curl of H and the curl of E in the new coordinate system, right? So, the vector fields are also transferred to the new coordinate system and the operation that You are taking the curl that is also transformed to the new coordinate system. Both actually follow the same form because you can write the curl of H in the new coordinate system as $j\omega[\varepsilon']\vec{E}'$. The curl of E in the new coordinate system can be written as $-j\omega[\mu']\vec{H}'$, Okay.



So, we can basically absorb the transformation of coordinate transformation completely into the material properties, right? So, that is the vital consequences of this Jacobian matrix and using this kind of coordinate transformation. So, you can write the same sort of Maxwell's equation in the new coordinate system, but what you can do. You can basically push all this, you know, coordinate transformation into the material property like this. and you go back to your old coordinate system. Now you can go back to the old coordinate system, but you know in this case the fields will basically behave.

As if they are in the transformed coordinate system because this and this are now equivalent. Okay. So, what are these new ones? So, I am basically talking about the new material properties. So, these are basically this, okay? So, I am just, you know, writing mu prime $\{[\mu']\}$, that is, the new material property tensor will look like $\frac{[\mathrm{A}][\mu][\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}$ and then you have epsilon prime $\{[\varepsilon']\}$, which will be equal to $\frac{[\mathrm{A}][\varepsilon][\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}$.

So, we are basically now transforming an operation here not a function right. What is it that you can always know from which coordinate system you are going to the new one? So, you know what your x prime, y prime, and z prime are; you know your old ones, x, y, and z, from that. You calculate the Jacobian transformation matrix, which basically tells you that you are moving from a vector space of, you know. Or you can say a dimension space from r to r prime, right?



Now let us prove this form of invariance because we just consider it to be correct.

Now let us see how it is done. So, if you take the $\nabla \times \vec{E}$ $i.e.$; curl of E, you can write $-j\omega[\mu]\vec{H}$, okay. Now, when you write this in the new coordinate system, you can simply write $\nabla' \times \vec{E}' = -j\omega[\mu']\vec{H}'$, okay. So, let us now define the coordinate transformation: r prime, which is nothing but a function of r, right? So, you can write $\vec{E}'(\vec{r}')$ as $([\mathrm{A}]^{\mathrm{T}})^{-1}\vec{E}(\vec{r})$; this we have already seen for the magnetic field. Also, you can write in the same way, and for you, know the material property $[\mu'(\vec{r}')]$. You can write it as $\frac{[\mathrm{A}][\mu(\vec{r})][\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}$, okay.

Now we substitute our transforms into this curl equation. So now instead of E okay you can write instead of E you can write now $[\mathrm{A}]^{\mathrm{T}}\vec{E}'(\vec{r}')$ okay. This mu can be written from here, which is nothing but, you know, the determinant of A, okay? And then you have $[\mathrm{A}]^{-1}[\mu'(\vec{r}')]([\mathrm{A}]^{\mathrm{T}})^{-1}$ and H, also you can write from here it is nothing but $[\mathrm{A}]^{\mathrm{T}}\vec{H}'(\vec{r}')$. So, this is basically now becoming as if you are doing, you know $\nabla\times\{[\mathrm{A}]^{\mathrm{T}}\vec{E}'\}$, and it is giving you $-j\omega\det[\mathrm{A}]\{[\mathrm{A}]^{-1}[\mu']([\mathrm{A}]^{\mathrm{T}})^{-1}\}[\mathrm{A}]^{\mathrm{T}}\vec{H}'$.

So, you can see that this is basically taking the form of it. So, you can divide it by the determinant of A, and then you can take this guy to the other side. So, you will see that you are basically getting this kind of form that you have, you know, $\frac{[\mathrm{A}](\nabla\times)[\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}\vec{E}'$, which is equivalent to $-j\omega[\mu']\vec{H}'$, right.

Proof of Form Invariance

- Recall the form of transforming an operation

$$[F'(\vec{r}')] = \frac{[\mathrm{A}][F(\vec{r})][\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}$$

- We see that the group of terms around the curl operation indicates this is just the transformed curl.

$$\underbrace{\frac{[\mathrm{A}](\nabla\times)[\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}}_{\nabla'\times}\vec{E}' = -j\omega[\mu']\vec{H}' \;\Rightarrow\; \nabla'\times\vec{E}' = -j\omega[\mu']\vec{H}'$$

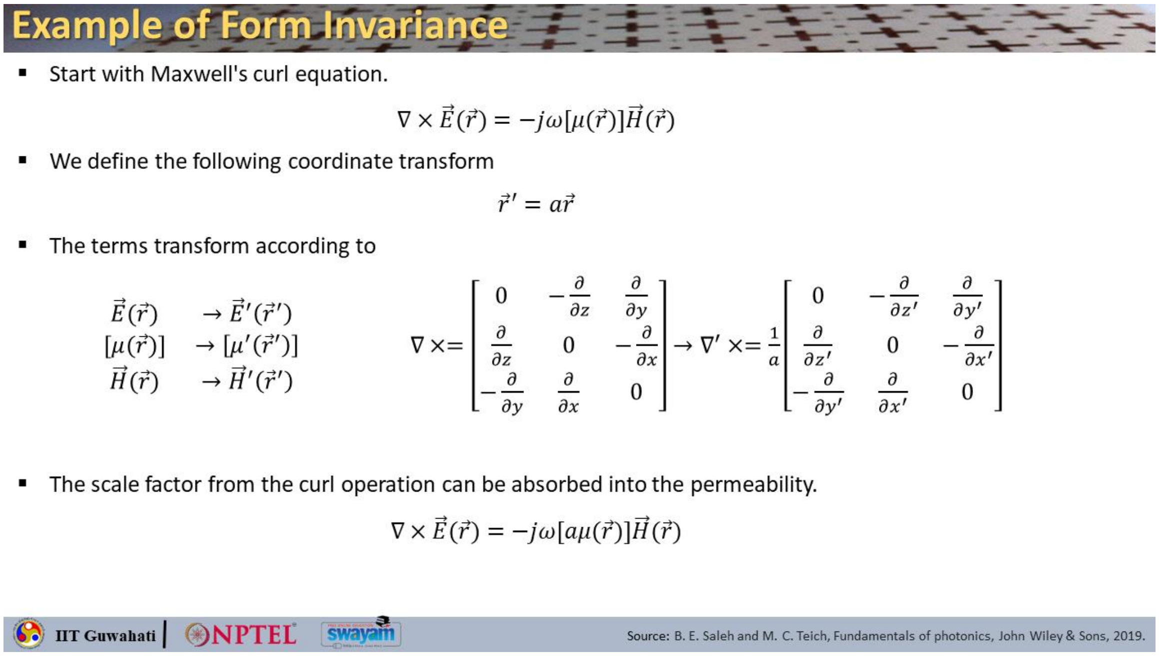
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, if you recall the form of transforming an operation, it looks like this: You have $[F'(\vec{r}')]$

will be equal to $\frac{[\mathrm{A}][F(\vec{r})][\mathrm{A}]^{\mathrm{T}}}{\det[\mathrm{A}]}$. So, here also you are getting the same thing. So, you are basically doing instead of this function, you are basically having the curl operation, right.

So, this is nothing but your curl prime, okay, from the previous one. This is your curl prime, right and then you have E prime, okay? And on the right side, you have

$-j\omega[\mu']\vec{H}'$. So, you can simply write this equation as you know curl prime or $\nabla' \times \vec{E}'$, that is nothing prime but the curl of E in the new coordinate system $-j\omega[\mu']\vec{H}'$, right.



So, if you have started with the Maxwell's curl equation, you would have written this that: $\nabla \times \vec{E}(\vec{r}) = -j\omega[\mu(\vec{r})]\vec{H}(\vec{r})$. Now, because you are using this transformation, you are moving to the r prime coordinate system, okay? And the relation between the r prime and r system is that the r prime is the scaled version of the previous system. So, r prime is nothing but a.r, and now you can actually do this shifting, so you can write $E\ (r)$ in terms of the new electric field.

mu can be the mu prime in the r prime coordinate system; H can be H prime. So, all these things can be done, and you also need to write what the curl operation is in the new coordinate system. So, the curl prime operation will have this kind of transformation. So, you can calculate that, and you will see you are getting a 1 by a factor coming out of this matrix. So, if you put it back into the equation that has all the transformed fields and operations.

You will see that this kind of scaling that you have done is okay. So, curl prime will come with a 1 by a, and that a you can actually push into the material parameter. And you can simply go back to your old coordinate system and write $\nabla \times \vec{E}(\vec{r})$, which is in the original coordinate system, will be equal to $-j\omega[a\mu(\vec{r})]\vec{H}(\vec{r})$, okay. that means whatever is the transformation you are doing that can be just pushed into the material property. So, the new magnetic permeability will have the scaling factor of okay, and you can go back to the old coordinate system.

So, that will basically behave as if the original system has got this kind of a transformation.

So, with that, we will conclude here. If you have any queries regarding this lecture, you can drop an email to this email address. Mention the course title and the lecture number in the subject line. Thank you.