Course Name: Introduction to Microwave and Optical Metamaterials
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Week-4
Lecture-20

Lec 20: Effective Medium Theory-II

Hello everyone, welcome to Lecture 20 of the online course on Introduction to Microwave and Optical Metamaterials. In this lecture, it will be a continuation of the Effective Medium Theory Here we will complete the derivations of the Maxwell Garnett theory and we will also discuss the Bruggeman effective medium theory and Nicolson Ross Weir method. So, here will be the lecture outline; as I mentioned, we will complete this derivation. We will look into the equations for Bruggeman effective medium theory. We will discuss the limitations of the Maxwell-Garnett theory. We will introduce another interesting methods method of Nicolson Ross Weir and we will see the application areas where you can use this method and what are the limitations associated.

So, to continue from the previous lecture, we have seen that if the material is excited by an external electric field within the quasi static approximation ok. You can say that such a field can be considered constant by the length scale of each sphere. That means that it is possible when the sphere is much, much subwavelength. That means the wavelength of the light is much larger than the dimensions of the sphere.

Lecture Outline

- Maxwell Garnett Effective-Medium Theory
- Bruggeman Effective-Medium Theory
- Limitations of Maxwell Garnett Theory
- Nicolson-Ross-Weir method
- Limitations of Nicolson -Ross-Weir method

So, you can consider it to be the field to be quasi-static. So, in that case, what will happen is that the average operator could integrate by a sufficiently large volume. In order to provide an accurate description of the average fields in the original medium. So, you can write the average displacement field to be equal to ε_0 ε MG and then the average electric field. So, what is this? You have already seen that this is the scenario where you can apply Maxwell Garnett approximation that you have some inclusions which are again sub wavelength and size.

Maxwell Garnett 'Effective-Medium' Theory

- If the material is excited by an external electric field, in the quasi-static approximation such a field can be considered to be constant on the length scale of each sphere.
- The average operator integrates over sufficiently large volumes in order to provide an accurate description of average fields in the original medium. Hence, one can write:

$$\langle \mathbf{D} \rangle = \varepsilon_0 \varepsilon_{\mathrm{MG}} \langle \mathbf{E} \rangle$$

$$\langle \mathbf{D} \rangle = \varepsilon_0 \varepsilon_{\mathrm{h}} \langle \mathbf{E} \rangle + \langle \mathbf{P} \rangle$$

where $\varepsilon_{\mathrm{MG}}$ is the effective (relative) Maxwell Garnett permittivity that models the original mixture, as shown in Figure.

Figure: Illustration of Maxwell Garnett homogenization.

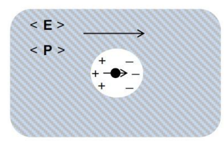
Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.

They are placed within this host medium, which has a permittivity of ε h. So, the permittivity of the inclusions are given as ε i, you can assume them to be sub-wavelength spheres and then finally, you can obtain a effective medium which has got a permittivity of ε MG. Now, this equation can also be written as $\varepsilon_0 \varepsilon_{\mathrm{h}} \langle \mathbf{E} \rangle + \langle \mathbf{P} \rangle$. So, you can actually separate out the polarization ok. So, you can write down the contribution of the electric field in the host medium + the polarization that is coming because of the inclusions right.

So, this is how you can use ε MG that will give you the effective relative permittivity that can model the actual mixture. Now, couple of important things here that the average dipole response which is this average P is nothing, but n * small p right. Here small p is typically not equal to ph that is the host dipole moment, it has to be now calculated in the presence of all other dipoles ok. That means the evaluation of this dipole moment can be classically performed by evaluating the local electric field Ei. That means this is basically the field that is felt by each individual dipole in the presence of all other dipoles.

Maxwell Garnett 'Effective-Medium' Theory

- The average dipole response is $\langle \mathbf{P} \rangle = N\mathbf{p}$, where the dipole moment $\mathbf{p} \neq \mathbf{p_h}$ (host) is now calculated in the presence of all the other dipoles.
- The evaluation of $\mathbf{p}$ is classically performed by evaluating the local electric field $\mathbf{E_L}$, which is the field locally "felt" by each dipole.
- This field is the average field $\langle \mathbf{E} \rangle$ augmented by a contribution due to the average polarization that surrounds each dipole, also known as the '**Lorentz field**'.
- To find the field $\mathbf{E_L}$ acting on a single dipole, a simple model of the mixture is adopted.
- A fictitious spherical boundary separates a macroscopic background with average polarization $\langle \mathbf{P} \rangle$ from a microscopic spherical cavity surrounding the dipole at the center of the sphere, as shown :

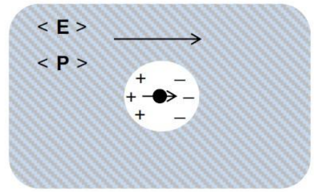


IIT Guwahati | NPTEL swayam Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.

Right. So, this field is basically the average field this average E augmented by a contribution due to the average polarization that basically surrounds each dipole and that is also known as Lorentz field. Now, to find this particular local electric field that is acting on a single dipole, a simple model of the mixture can be adopted. So, let us assume this particular scenario where a fictitious spherical boundary separates the macroscopic background with average polarization P from a microscopic spherical cavity here okay that is surrounding the dipole okay at the center of the sphere. So, this is the dipole, and this is the sphere that is within this particular microscopic cavity that lies within this macroscopic background. Now, what is the role of this one? So a microscopic spherical cavity here can separate the homogeneous background from a microscopic phase in which the dipole is basically included.

So because of this dipole that is a the electric field will be the charge separation will be like this and this is the field. Now the field at the dipole location which is at the centre of the sphere is basically the local electric field Ei. Now, how you can write this Ei? You can write this as the average field + the average polarization / 3 ε_0 ε h. And you can also write that the dipole moment small p will be equal to know that is proportional to the electric field with the polarizability α. So, you can write it as $\varepsilon_0 \varepsilon_h \alpha \langle \mathbf{E} \rangle + \alpha \frac{\langle \mathbf{P} \rangle}{3}$.

Maxwell Garnett 'Effective-Medium' Theory



- A fictitious spherical cavity separates a homogeneous background from a microscopic phase in which a dipole is included.
- The field at the dipole location, the center of the sphere, is the local field $\boldsymbol{E}_\mathbf{L}$.
- The local field can be straightforwardly written as:

$$\mathbf{E}_L = \langle \mathbf{E} \rangle + \frac{\langle \mathbf{P} \rangle}{3\varepsilon_0 \varepsilon_h}$$

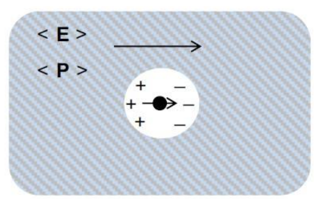
- And the dipole moment as: $\mathbf{p} = \varepsilon_0 \varepsilon_h \alpha \langle \mathbf{E} \rangle + \alpha \frac{\langle \mathbf{P} \rangle}{3}$
- We can now retrieve the Maxwell Garnett permittivity in terms of the polarizability α and the number density N:

$$\varepsilon_{MG} = \varepsilon_h \left(1 + \frac{N\alpha}{1 - \frac{N\alpha}{3}} \right)$$

Now, we can now retrieve this Maxwell Garnett permittivity which is our effective permittivity in terms of this polarizability α and then N which is basically the number density which tells you how many such dipoles are there in a particular volume. So, you can write $\varepsilon_{MG} = \varepsilon_h \left(1 + \frac{N\alpha}{1 - \frac{N\alpha}{3}} \right)$. Now, in the case of a very diluted medium, you can consider this term to be equal to 1. So, in that kind of approximation the effective permittivity ε MG will simply look like ε_h 1 + N α right. and the same expression can easily be obtained when you consider your Ei to be equal to the average electric field.

Maxwell Garnett 'Effective-Medium' Theory

$$\varepsilon_{MG} = \varepsilon_h \left(1 + \frac{N\alpha}{1 - \frac{N\alpha}{3}} \right)$$



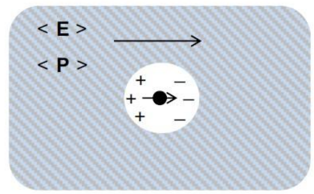
- For very diluted media: $1 - \frac{N\alpha}{3} \approx 1$
- The effective permittivity is simply $\varepsilon_{MG} \approx \varepsilon_h (1 + N\alpha)$.
- The same expression can easily be obtained when the local field is $\mathbf{E}_\mathbf{L} = \langle \mathbf{E} \rangle$.
- This approximation is fully justified in diluted mixtures where the interaction between dipoles is weak.
- The following form of the relation in equation: $\frac{N\alpha}{3} = \frac{\varepsilon_{MG} - \varepsilon_h}{\varepsilon_{MG} + 2\varepsilon_h}$

Now this particular approximation will happen in a very diluted medium means there are very less number of inclusions in the host medium okay and this is possible because the interaction between the dipoles are weak. So, your local electric field is nothing but the average electric field, right? So, with this you can also write that $\frac{N\alpha}{3} = \frac{\varepsilon_{MG} - \varepsilon_h}{\varepsilon_{MG} + 2\varepsilon_h}$. So, this particular relation is also called the Clausius-Mosotti formula or Maxwell's formula or the Lorentz-Lorentz formula okay. So, in this particular equation if you substitute the value of the polarizability α that is coming from the quasi-static theory. So, you can find out the polarizability as 3 V, V is basically the volume of the sphere ε i - ε h / ε i + 2 ε h.

Maxwell Garnett 'Effective-Medium' Theory

The Clausius-Mossotti formula, Maxwell's formula, or the Lorentz-Lorenz formula.

$$\frac{N\alpha}{3} = \frac{\varepsilon_{MG} - \varepsilon_h}{\varepsilon_{MG} + 2\varepsilon_h}$$



- Substitution of the expression of the polarizability α (from quasi-static theory):

$$\alpha = 3V\frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}$$

in the **Clausius-Mossotti relation** gives:

$$3V\frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}\frac{N}{3} = \frac{\varepsilon_{MG} - \varepsilon_h}{\varepsilon_{MG} + 2\varepsilon_h} \longrightarrow \frac{\varepsilon_{MG} - \varepsilon_h}{\varepsilon_{MG} + 2\varepsilon_h} = f\frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}$$

$\therefore f = NV$

volume fraction of the inclusions (in this case spheres) in the medium

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Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.

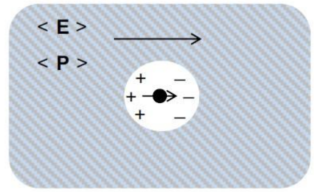
So, in that case the Clausius-Mossotti relation looks like this. So, you have $3V\frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}\frac{N}{3} = \frac{\varepsilon_{MG} - \varepsilon_h}{\varepsilon_{MG} + 2\varepsilon_h}$. So, from this you can actually see that there is N and V okay if you basically multiply this you are getting f which is basically the volume fraction okay of the inclusions. So here are these spheres in the host medium. So you can simplify this expression.

So NV can be written as okay or you can say so this side so NV will be written as f. Okay, that is the volume fraction. So, this particular equation is also known as the Rayleigh formula that relates the effective permittivity of this effective medium ε MG to the constituents permittivity which is basically ε i and ε h okay. For the parameter which is known as volume fraction f that is calculated as n v. So, v is the volume n is the number right.

Maxwell Garnett 'Effective-Medium' Theory

Rayleigh formula

$$\frac{\varepsilon_{\mathrm{MG}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{MG}} + 2\varepsilon_{\mathrm{h}}} = f\,\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}}}$$



- Relates the effective permittivity to the constituents' permittivities and to the parameter $f = NV$, which is the volume fraction of the inclusions (in this case spheres) in the medium.

- The **Maxwell Garnett** formula is derived from the above equation, and it is written as follows:

$$\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}\left[1 + 3f\,\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right]$$

Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.

So, this basically gives you the volume fraction of the inclusions in a particular medium. So, if you if you try to solve this and try to find out what will be the absolute value of ε MG. So, that you can obtain from this equation and you can write it as $\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}\left[1 + 3f\,\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right]$. So, this is how using the value of ε i ε h and f you can calculate what is the effective permittivity of the medium.

So, this is a very important formula. So, this is called the Maxwell Garnett formula and this simple formula represents the classical approach of homogenizing composite media and it is widely used in many applications. Now, it is interesting to notice that the only parameters for retrieving this Maxwell coordinate permittivity are nothing but the individual permittivity of the inclusions and the host and also you need to know the volume fraction that means in the host how much volume is actually occupied by the inclusions. Once you know that you can use this formula and find out what will be the effective permittivity of this composite medium. Now there are a couple of interesting observations that this formula does not require the spheres to be of the same size and they do not need to be located at a specific position or location.

So, you do not need to make a periodic array or something like that; there is no such requirement. The only requirement is that the spheres should be sub wavelength that means the wavelength in the medium should be much larger than the size of this inclusions. So, you can also see that Maxwell Garnett theory predicts that when the volume fraction goes to 0 the effective medium will be same as the host medium, when the volume fraction goes to 1 then the effective permittivity will be same as that of the inclusion permittivity ε i make sense right. So, till now we have seen that the Maxwell Garnett formula represents a valid homogenization model for mixtures with well-defined host medium and inclusions. But there are some cases in which things get mixed up.

Maxwell Garnett 'Effective-Medium' Theory

Maxwell Garnett formula

$$\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}\left[1 + 3f\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right]$$

- This simple formula represents the classical approach to homogenizing composite media, and it is widely used in many applications.
- It is interesting to notice that the only necessary parameters for retrieving the Maxwell Garnett permittivity are the permittivities of inclusions and host medium, along with the volume fraction of the inclusions.

Figure: Illustration of Maxwell Garnett homogenization.

We will see them. Now Maxwell Garnett's results are very accurate for relatively small values of the inclusion volume factor F. Now, for aggregate mixtures with random distributions of two or more constituents, You will see that the effective medium theory which are based on statistical formulation becomes more suitable something like this ok. So, here this is the case of an inhomogeneous mixture. So, any region in the mixture can be either this material or the other, okay. So, you can always model this as effective medium which is ε Br.

Maxwell Garnett 'Effective-Medium' Theory

Maxwell Garnett formula

$$\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}\left[1 + 3f\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right]$$

- The formula **does not require that the spheres are of the same size and located at specific positions** (e.g., periodic arrays).
- The only requirement is that the wavelength in the medium should be much larger than the size of the inclusions.
- The Maxwell Garnett theory predicts that $\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}$ for $f \rightarrow 0$, and $\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{i}}$ for $f \rightarrow 1$.

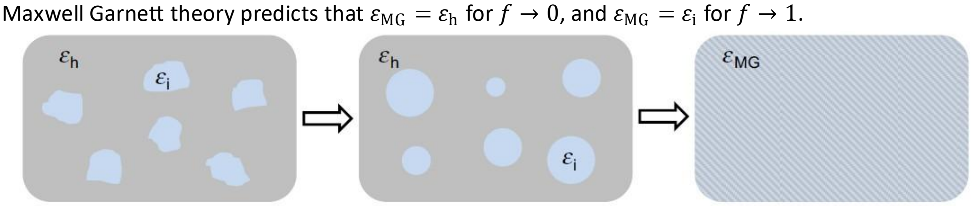


Figure: Illustration of Maxwell Garnett homogenization.

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So, what you can see that there are two phase inhomogeneous material and a space filling random mixture has happened between the two phases. So, it is clearly marked here that one phase will have dielectric constant of ε i and if you consider that to have a filling fraction of f. Then the other phase, which has a dielectric constant of ε h, will have a volume fraction of 1 - f, okay. So, on the right here you can see this is the Bruggeman theory model of this structure. So, it has got a background medium of effective permittivity ε Br that is hosting very small spheres okay whose dielectric constant is ε i with probability f or they can be ε h with probability 1 - f.

Bruggeman 'Effective-Medium' Theory

- The **Maxwell Garnett formulas** represent a valid homogenization model for mixtures with a well-defined host medium and inclusions
- These results are more accurate for relatively small values of the inclusion volume factor f.
- For aggregate mixtures with random distributions of two or more constituents, effective medium theories based on a statistical formulation are more suitable.
- The classic theory for this class of inhomogeneous mixtures is the **Bruggeman Theory**.

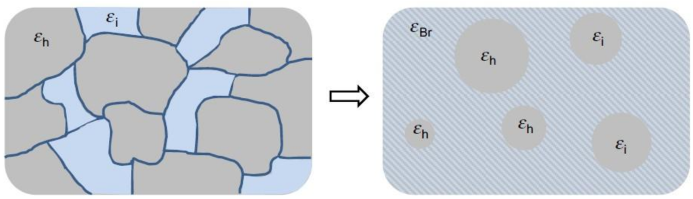


Figure: Illustration of Bruggeman Theory Model.

IIT Guwahati | NPTEL swayam Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.

So, you can consider a two phase microstructure of the type that is shown here where the constituent with permittivity ε i has a volume fraction of f and the constituents with permittivity ε h will have a volume fraction of 1 - f. So, you can now model this mixture as a continuous medium hosting a distribution of small spherical inclusions of the two different type of dielectric permittivities ε i and ε ok and the probability of finding spheres with permittivity ε i will be f. So, the probability of finding spheres with permittivity ε h will be 1 - f okay. So, this is how where things vary from you know the previous case to this

Bruggeman 'Effective-Medium' Theory

- Consider a two-phase microstructure of the type illustrated in the figure, where the constituent with permittivity ε_i has volume fill factor f, and the constituent with permittivity ε_h has volume fill factor $1 - f$.
- This mixture is now modeled as a continuous medium hosting a distribution of small spherical inclusions of two different dielectric permittivities.
- **The probabilities of finding spheres with permittivity ε_i and ε_h are f and $1 - f$**, respectively, which correspond to the volume fill factors of the two phases in the original mixture.

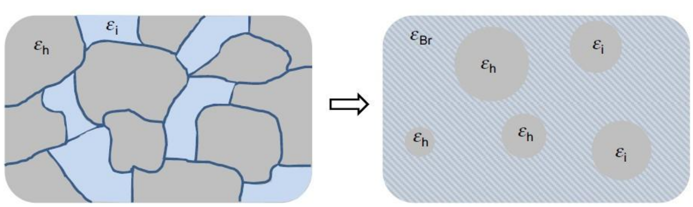
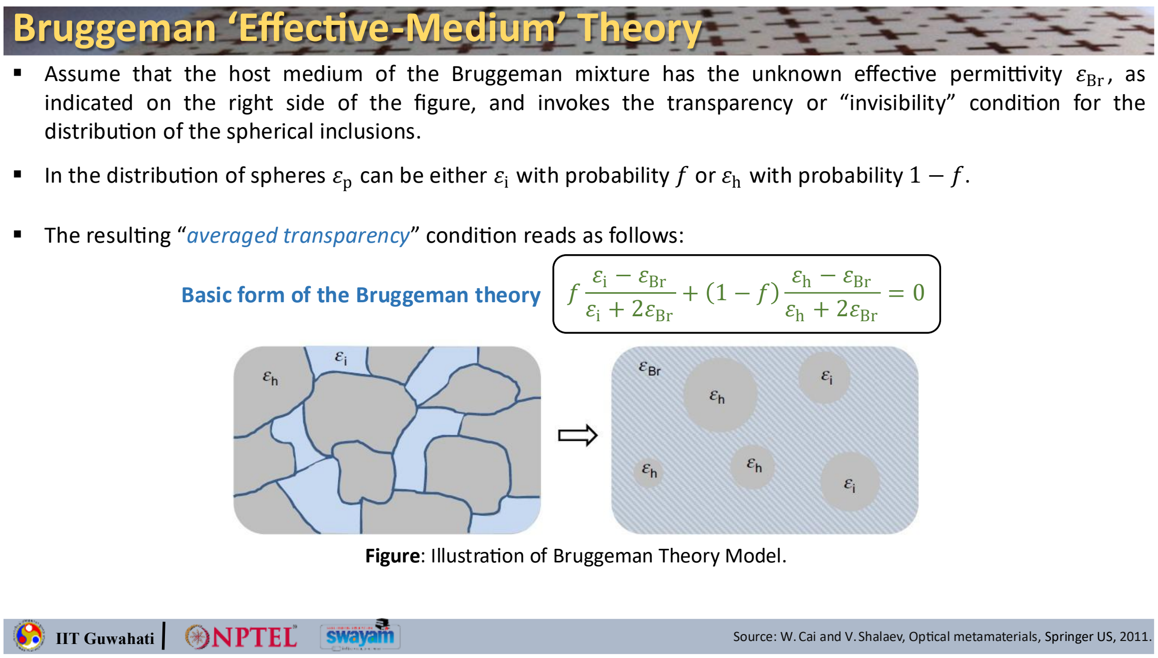


Figure: Illustration of Bruggeman Theory Model.

IIT Guwahati | NPTEL swayam Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.

Bruggeman case. So, let us assume that the host medium of the Bruggeman mixture has unknown effective permittivity which is ε Br.

Okay that we are supposed to find out and that will give you the transparency or invisibility condition for the distribution of the spherical inclusions right. So, if you consider the distribution of spheres, it can be okay. It can be that ε i occurs with probability f or ε h occurs with probability 1 - f. So, you can find an average transparency condition that will satisfy this equation, okay? So, it says $f\frac{\varepsilon_i-\varepsilon_{Br}}{\varepsilon_i+2\varepsilon_{Br}}+(1-f)\frac{\varepsilon_h-\varepsilon_{Br}}{\varepsilon_h+2\varepsilon_{Br}}=0$. So, this one is known as the Bruggeman theory okay.



So, you can also do the maths and find out what will be ε Br from this particular equation. Now if the formula for spherical inclusions now in case you have multiphase aggregates not only two phase. So, you have a greater number of elements mixing together. So, you can also extend this formula to this kind of multiphase aggregate and there you will write that summation small m = 1 to capital M f m ε h - ε Br ok this will be ε m basically ok. So, that will tell you the fill factor, okay.

So, I think this is the correct equation. So, you will see that ε summation by small m starting from 1 to capital M. So, there are total m number of phases materials present okay. So, the probability of finding each phase will be fm the permittivity is $\frac{\varepsilon_m-\varepsilon_{Br}}{\varepsilon_h+2\varepsilon_{Br}}=0$. So, as you can as I have already told that fm is basically the fill factor of the mth constituent of the mixture and capital M is the total number of phases right.

Bruggeman 'Effective-Medium' Theory

Bruggeman theory $$f\frac{\varepsilon_{\mathrm{i}}-\varepsilon_{\mathrm{Br}}}{\varepsilon_{\mathrm{i}}+2\varepsilon_{\mathrm{Br}}}+(1-f)\frac{\varepsilon_{\mathrm{h}}-\varepsilon_{\mathrm{Br}}}{\varepsilon_{\mathrm{h}}+2\varepsilon_{\mathrm{Br}}}=0$$

- The formula for spherical inclusions can be easily extended to multiphase aggregates by adding more terms in the above relation yielding:

$$\sum_{m=1}^{M} f_m \frac{\varepsilon_{\mathrm{m}}-\varepsilon_{\mathrm{Br}}}{\varepsilon_{\mathrm{h}}+2\varepsilon_{\mathrm{Br}}}=0$$

where f_m is the fill factor of the m^{th} constituent of the mixture, and M is the number of phases.

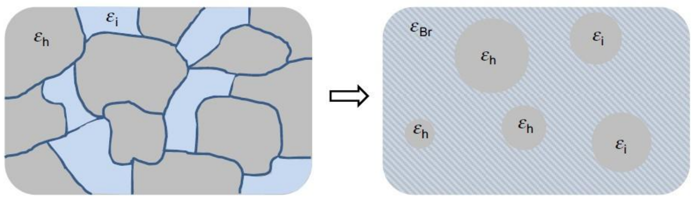


Figure: Illustration of Bruggeman Theory Model.

So, this tells you the effective permittivity that you can get from the Bruggeman theory. Now, let us look into some important limitations of the Maxwell Garnett theory and the other one. So, the first thing that is to be mentioned here is that the particles have a very small depolarization factor. So, when you have a low depolarization factor that basically implies a elongated shape of the particle and that will result in a stronger particle-particle interaction ok. Now, this is what you do not want you actually do not want your ε l sorry El the local electric field to be very different from the average electric field right.

Limitations of Maxwell Garnett Theory

- Limitation of the Maxwell Garnett theory — <u>particles with very small depolarization factors</u>.
  - A low depolarization factor implies an elongated shape of the particle with the result of stronger particle-particle interactions.
  - In this situation, which is similar to the scenario of a mixture with large inclusions' fill factor, the Bruggeman prediction should be adopted.
- Another scenario in which the Bruggeman theory provides a more realistic electromagnetic description is:
  - For mixtures with large differences in the permittivities of the constituents *e.g.*, metal-dielectric mixtures, where a percolation phenomenon above a threshold of the metallic phase occurs.
  - This threshold is the critical metal filling factor above which there is formation of long-range connectivity between metal grains, and the optical response of the mixture changes abruptly.

So, in such a scenario, it is similar to the case of a mixture of large inclusions field factor. you can go for the Bruggeman prediction right. So, whenever this Maxwell theory will not be sufficient you can go for the Bruggeman method. Another scenario in which the Bruggeman theory provides a more realistic electromagnetic description is when there are mixtures with large differences in the permittivity of the constituents. Something like metal dielectric mixture where you will see that a percolation phenomena above a threshold of the metallic phase takes place or happens.

Effective Medium Theory: Nicolson-Ross-Weir method

- This is a homogenization method based on the inversion of **Fresnel formulas** relative to the transmission and reflection coefficients through slabs of homogeneous media.
- The technique was conceived to estimate the complex permittivity and permeability of an unknown material from the measured transmission and reflection spectra of a finite thickness sample.
- It was originally proposed in the time domain for pulsed measurement systems and then adapted to higher resolution, frequency domain systems.
- The transmission and reflection spectra may be retrieved with experiments or with numerical simulations.

Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, this is basically percolation is nothing but moving movement and filtering of fluids through porous medium. So, these are the two cases where you will see that Bruggeman theory produces a much more closer response as compared to Maxwell Garnett theory. Now, this threshold in the critical metal filling factor above which the formation of long range connectivity between the metallic grains or the optical response of the mixture can change abruptly. So, there is a limit within which this kind of theory will be valid. So, now let us look into another method called Nicolson-Ross-Weir method.

So, this is a homogenization method based on the inversion of the Fresnel formula. Relative to the transmission and reflection coefficients through slabs of homogeneous media. So, this techniques was conceived to estimate the average or you can say the complex permittivity and permeability of an unknown material from the measured transmittance and reflectance / a spectrum ok for a finite thickness sample. So, I will show you with some examples So, it was originally proposed in the time domain for pulsed measurement systems and then later it was adapted to higher resolution and then to frequency domain systems. The transmission and reflection spectra may be retrieved from experiments or numerical simulations.

Effective Medium Theory: Nicolson-Ross-Weir method

- A slab of thickness d of the unknown natural or artificial mixture is modeled as a slab of a homogeneous medium with effective (relative) permittivity ε_{eff} and permeability μ_{eff}.
- It is supposed that the thickness of the homogeneous slab is equal to d.
- The (complex) reflection and transmission coefficients R and T under normal incidence plane wave excitation are somehow known, via an experiment, a theoretical prediction, or a numerical simulation.

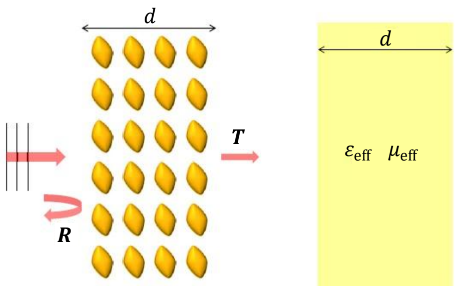


Figure: On the left is a slab of a composite medium with thickness d illuminated at normal incidence.

R and T are the complex reflection and transmission coefficients.

On the right is the homogenized slab with effective (relative) permittivity ε_{eff} and permeability μ_{eff}.

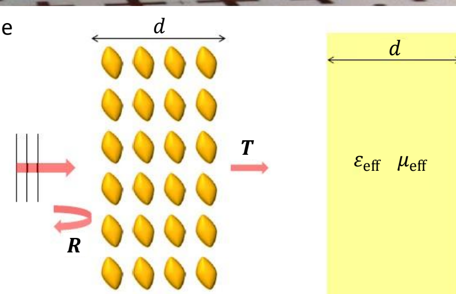
So, here is how it looks. So, let us assume a slab of thickness d which is made of some unknown natural or artificial mixtures. This can be modeled as a slab of homogeneous medium which has got effective permittivity and permeability ε effective and μ effective ok. So, here you can see that this is the actual one; the actual medium is basically a composite medium, okay. And then there is a wave that is falling on this medium, and you have got the reflectance and the transmittance, right? Now, you are basically mapping these properties of transmittance and reflectance and trying to guess what would be the effective permittivity and permeability which could have given you that kind of reflectance and transmittance. So, the first thing we assume is that the thickness of this slab will also be D.

Effective Medium Theory: Nicolson-Ross-Weir method

- The Fresnel reflection (R) and transmission (T) coefficients may be written in the following form:

$$R = \frac{\Gamma(1 - e^{-2ikn_{\text{eff}}d})}{1 - \Gamma^2 e^{-2ikn_{\text{eff}}d}}$$

$$T = \frac{(1 - \Gamma^2)e^{-ikn_{\text{eff}}d})}{1 - \Gamma^2 e^{-2ikn_{\text{eff}}d}}$$



which are dependent on the:

- effective refractive index of the slab $n_{\text{eff}} = \sqrt{\mu_{\text{eff}}\varepsilon_{\text{eff}}}$
- free-space wavenumber $k = \omega/c$
- reflection coefficient Γ across the first interface between the input medium and the semi-infinite homogeneous slab with relative parameters $(\varepsilon_{\text{eff}}, \mu_{\text{eff}})$.

Okay and the complex reflection and transmission coefficients that you have got capital R and capital T under normal incidence plane wave ok. If you know this you can either you can know this by experiment or by numerical simulation. And once you know, you can actually correlate them to the Fresnel reflection and the transmission coefficient with this. So, here you can see this equation has two new terms; one is n effective, and then you have γ, right? So, what are these two things n effective is basically coming from the effective refractive index of the slab which you can calculate a square root of μ effective and ε effective. Then you have the free space wave number that is appearing here k that is ω/c and then you have γ which is basically the reflection coefficient across the first interface between the input medium and if you assume that this particular slab is a semi infinite one which has got permittivity ε effective and μ effective right.

Effective Medium Theories: Nicolson-Ross-Weir method

- At normal incidence: $\Gamma = \dfrac{\eta_{\text{eff}} - \eta_0}{\eta_{\text{eff}} + \eta_0}$

  $\eta_{\text{eff}} = \sqrt{\mu_{\text{eff}}\varepsilon_{\text{eff}}}$ is the effective intrinsic impedance of the homogeneous slab

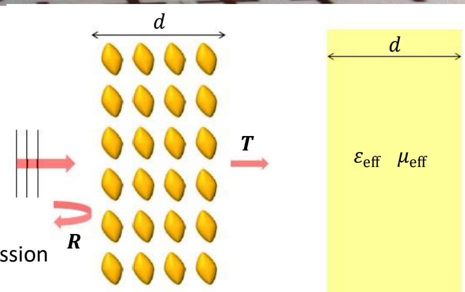
  η_0 is the intrinsic impedance of the input/output medium.

- The inversion of equations of R & T leads to the following expression of the effective impedance:

$$\eta_{\text{eff}} = \pm\eta_0\sqrt{\frac{(1+R)^2 - T^2}{(1-R)^2 - T^2}}$$

and the following expression for the quantity:

$$Q = e^{-ikn_{\text{eff}}d} \rightarrow Q = \frac{T}{1 - R\dfrac{\eta_{\text{eff}} - \eta_0}{\eta_{\text{eff}} + \eta_0}}$$



$$R = \frac{\Gamma(1 - e^{-2ikn_{\text{eff}}d})}{1 - \Gamma^2 e^{-2ikn_{\text{eff}}d}}$$

$$T = \frac{(1 - \Gamma^2)e^{-ikn_{\text{eff}}d}}{1 - \Gamma^2 e^{-2ikn_{\text{eff}}d}}$$

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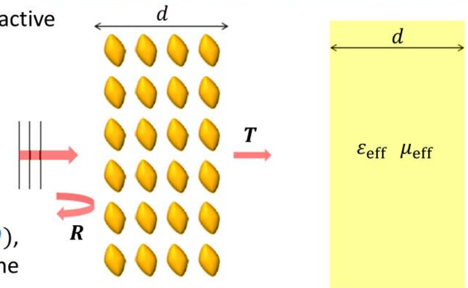
Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, what will the reflection coefficient be from this interface? So, that actually gives you this. Now if you look at only normal incidence you can write $\Gamma = \frac{\eta_{\text{eff}} - \eta_0}{\eta_{\text{eff}} + \eta_0}$. Now, what is η effective that is coming from square root of μ effective ε effective and this is nothing but the effective intrinsic impedance of the homogeneous slab. H naught obviously it is the intrinsic impedance of the input or the output medium you can consider this as vacuum impedance as well right. Now the inversion of the equations of R and T will lead to this that you can now calculate η effective from η_0 and when you have the values of R and T given you can find out what is your effective intrinsic impedance of this homogeneous slab. And once you know this you can calculate the expression for this quantity Q that will be given as $e^{-ikn_{\text{eff}}d}$. So Q can be written in terms of T, R and n effective in this particular form. So, $\frac{T}{1 - R\frac{\eta_{\text{eff}} - \eta_0}{\eta_{\text{eff}} + \eta_0}}$. So, it is not n this is η effective this is the intrinsic impedance right. So, this is the effective impedance of the slab, and this is the intrinsic impedance of the input of the output medium.

Effective Medium Theories: Nicolson-Ross-Weir method

- From equations of η_{eff} and Q, one can write the effective refractive index as follows:

$$n_{\text{eff}} = \frac{i}{kd}\log(Q) = \frac{1}{kd}\{i\text{Log}|Q| - [\text{Arg}(Q) + 2m\pi]\}$$



$$\eta_{\text{eff}} = \pm\eta_0\sqrt{\frac{(1+R)^2 - T^2}{(1-R)^2 - T^2}}$$

$$Q = \frac{T}{1 - R\frac{\eta_{\text{eff}} - \eta_0}{\eta_{\text{eff}} + \eta_0}}$$

- Here $\log(Q)$ is the complex, multiple-valued logarithm of (Q), $\text{Log}|Q|$ is the ordinary real logarithm of $|Q|$, $\text{Arg}(Q)$ is the argument in the principal branch ($m = 0$), and $m = \pm 1, \pm 2 \ldots\ldots$ indicates the branch of $\log(Q)$.

- The effective parameters can finally be written as follows:

$$\varepsilon_{\text{eff}} = \frac{kn_{\text{eff}}}{\omega\eta_{\text{eff}}} \qquad \mu_{\text{eff}} = \frac{kn_{\text{eff}}\eta_{\text{eff}}}{\omega}$$

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Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, once you know the value of η effective and q, you can write down you know $n_{\text{eff}} = \frac{i}{kd}\log(Q) = \frac{1}{kd}\{i\text{Log}|Q| - [\text{Arg}(Q) + 2m\pi]\}$. Now here you will see that the log Q is the complex multivalued logarithm of this quantity Q. This is basically the ordinary real logarithm of the modulus of Q. Argument is basically giving you the principal branch that is when m = 0 and then m can give you the other integer branches as well. So, you can also have + - 1 + - 2 which are basically indicating the branch of log Q.

So, once you know this n effective you can ah effective parameters can be written as ε effective will be $\varepsilon_{\text{eff}} = \frac{kn_{\text{eff}}}{\omega\eta_{\text{eff}}}$ and you will can write effective permeability, $\mu_{\text{eff}} = \frac{kn_{\text{eff}}\eta_{\text{eff}}}{\omega}$. So, that way from reflection and transmission coefficient you can find out what is the effective permittivity of this material. So, here we will be able to find out what the effective permittivity and effective permeability of the material are. Now, there are some limitations to this factor as well. So, you will see that you know while the choice of sign for the effective impedance and the reflective index does not alter the value of the effective parameters which are extracted by these equations ok.

Limitations of Nicolson-Ross-Weir method

- This method has several limitations:
  - While the choice of sign for the effective impedance and refractive index does not alter the value of the effective parameters extracted via equations:

$$\varepsilon_{\text{eff}} = \frac{kn_{\text{eff}}}{\omega\eta_{\text{eff}}} \qquad \mu_{\text{eff}} = \frac{kn_{\text{eff}}\eta_{\text{eff}}}{\omega}$$

  - There is an intrinsic ambiguity in the definition of $Re(n_{\text{eff}})$ in equation:

$$\eta_{\text{eff}} = \frac{i}{kd}\log(Q) = \frac{1}{kd}\{i\text{Log}|Q| - [\text{Arg}(Q) + 2m\pi]\}$$

owing to the multiple-valued complex logarithm $\log(Q)$ and the choice of the branch order m.

  - This problem may be solved in very thin slabs in which the effective wavelength is larger than $2d$.

- The Nicolson-Ross-Weir method has been extended to the characterization of mixtures in the case of oblique plane wave incidence for the study of spatial dispersion effects in metamaterials.

Source: W. Cai and V. Shalaev, Optical metamaterials, Springer US, 2011.
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, there is an intrinsic ambiguity in the definition of this equation which is nothing but real of n effective. So, it is coming from this equation and that is mainly coming from the fact that the multivalued complex algorithm log Q and it also depends on the choice of the branch order M. And this problem may be solved in very thin slabs in which the effective wavelength is much larger than 2d ok. So, this particular method Nicholson-Ross-Weir method can also be extended to the characterization of mixtures in the case of oblique plane wave incidence here we have just shown it for the normal incidence. So, that can be used for the study of spatial dispersion effects in metamaterials.

So, the application of homogenization principle considered for bulk natural and artificial mixtures may be misleading for the homogenization of very thin films. For structured films or artificial surfaces, usually more sophisticated techniques have been developed. Based on the interaction of non-local effective surface susceptibility and all of those. So, we will not cover and go into those details. We will basically look into the normally or regularly used effective medium theory approaches which are going to be very useful for defining the properties when you will be going further and designing the metamaterials.

So, with that we will conclude this and if you have got any query regarding this lecture drop an email to this email address mentioning the lecture number and the course title on the subject line. Thank you.

Thank You