Course Name- Nanophotonics, Plasmonics and Metamaterials

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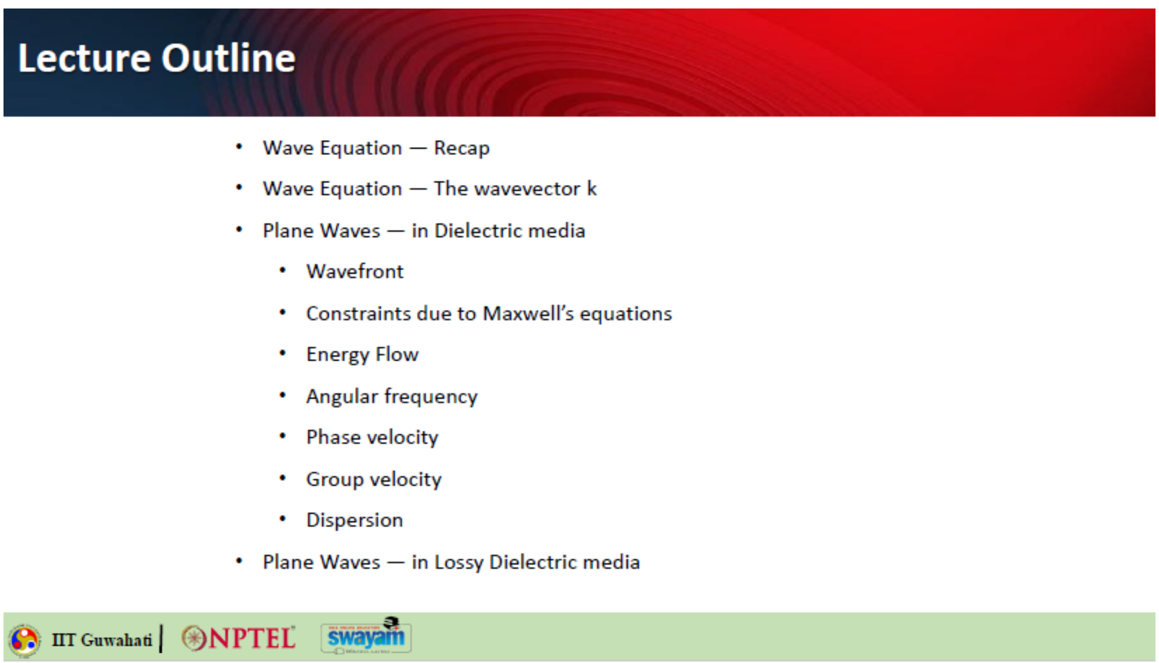
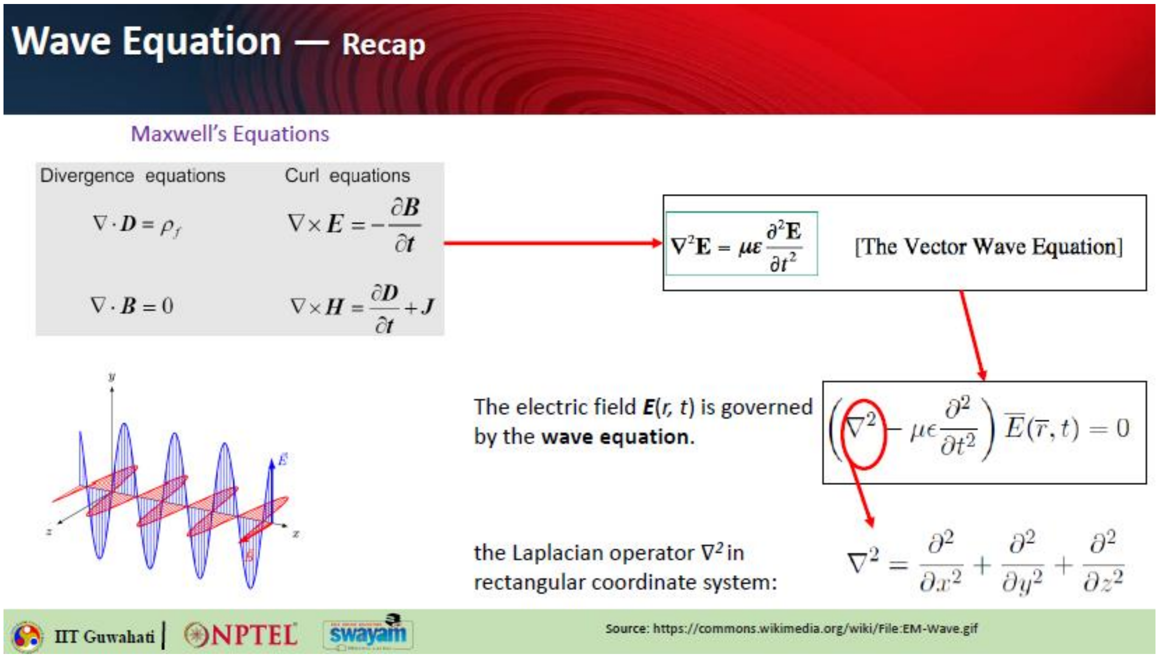
Week-02

Lecture -06

Hello everyone, welcome to lecture 6 of the online course on Nanophotonics, Plasmonicsand Metamaterials. So today's lecture will be on electromagnetic waves in dielectric media. So here is the outline of the lecture. We'll have a quick recap of the wave equation. Then we'll introduce the concept of wave vector k. We'll look into plane waves in dielectric medium, discuss about the concept of wavefront, then constraints due to Maxwell's equations, energy flow, angular frequency, phase velocity, group velocity, dispersion, and finally we'll look into plane waves in plane waves in lossy dielectric media.



So from the last lecture, you must remember that we have discussed how to derive wave equation from Maxwell's equation. So for your quick reference, we have again reproduced the four Maxwell's equations here, which can be tabulated as differential equations and curl equations. Now from the curl equation, we have seen that if we take the curl of curl of this vector, we can introduce curl of B. B you can write as μH.



So curl of H can be replaced by this term and finally we come up to this vector wave equation. That is $\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$. So it's a second order partial derivative and this is the vector wave equation. Now this can also be written in this form, like instead of writing E vector using bold phase, one can actually write it as this kind of notation. That is also fine.

And you can explicitly write that this vector E is a function of position r, that's a vector r, and time t. And when you take both the terms on left hand side, you will get this minus. So this is basically the governing equation of the electric field and this equation is nothing but your wave equation. So you can actually have a look at the electromagnetic wave here, which is propagating along x direction. And you see that the B field, that is the magnetic field is oscillating along z and the electric field is oscillating along y direction.

And this particular parameter, that is nabla square or del square parameter, is called Laplacian parameter. So, if you consider Cartesian or rectangular coordinate system, you can write $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. So these are all known factors. I'm just quickly giving you an overview. Now let us see how to write wave equation in frequency domain.

Wave Equation — frequency-domain

- If we assume a time-harmonic wave propagating along x-direction. Then, to obtain the frequency-domain wave equations, we use the **Fourier transform** with an $e^{i\omega}$ time dependence.

$$\boldsymbol{E}(x,t) = \boldsymbol{E}\,(x)e^{i\omega t} \longrightarrow \text{Phasor form (where, } \omega \text{ is the angular frequency)}$$

- The derivative of $e^{i\omega}$ with respect to time is $i\omega e^{i\omega}$. Thus, we can easily convert the time-domain wave equations to the frequency-domain by replacing $\delta/\delta t$ with $i\omega$ and $\delta^2/\delta t^2$ with $-\omega^2$. The wave equations are therefore given by:

$$\frac{\delta^2 E(x)}{\delta z^2} - \mu\varepsilon\frac{\delta^2 E(x)}{\delta t^2} = 0 \quad \Rightarrow \quad \left[\frac{d^2E(x)}{dz^2} - \mu\varepsilon\,(i\omega)^2 E(x)\right]e^{i\omega t} = 0$$

$$\text{or} \quad \frac{d^2E(x)}{dz^2} + \left(\frac{\omega}{v}\right)^2 E(x) = 0, \quad where\ v = velocity\ of\ wave = 1/\sqrt{\mu\varepsilon}$$

- Now, we will introduce a new quantity called "wave number" $k = \omega/v = 2\pi f/v$

- Thus, the Helmholtz wave equation can be written as: $\frac{d^2E(x)}{dz^2} + k^2E(x) = 0$

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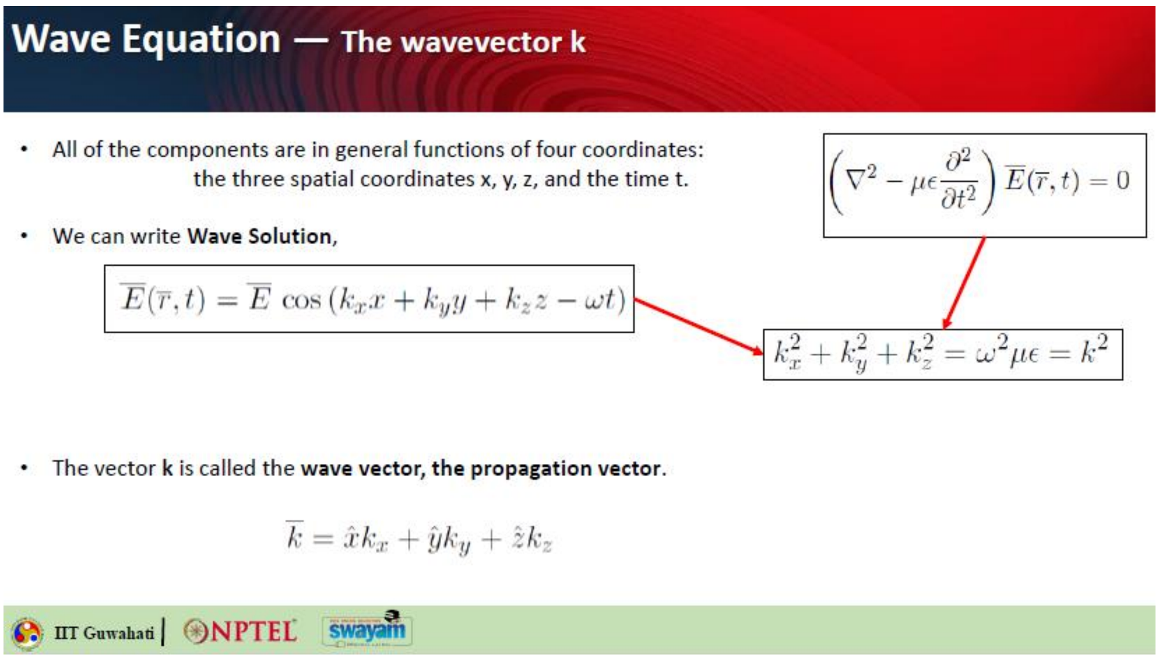
Now if you assume a time harmonic wave propagating along x direction, as was shown here, then to obtain the frequency domain wave equation, you have to do Fourier transform. So we can use Fourier transform with e to the power i omega time dependence (t). So there should be t here. So you can write E(x) of t, because here we assume that the propagation is along x direction. So, it is $\boldsymbol{E}(x)e^{i\omega t}$.

And this form is also known as phasor form. So, we have introduced one new term here, which is omega, that is nothing but the angular frequency. Now if we take derivative of $e^{i\omega t}$ with respect to time, so you can assume t here, you will get i omega $e^{i\omega t}$. That means using this you can easily convert the time domain wave equation into frequency domain wave equation by replacing $\delta/\delta t$ with $i\omega$ and you can replace $\delta^2/\delta t^2$ with $-\omega^2$.

So, you can start with this one, this is your wave equation. And you can finally write that instead of $\delta^2/\delta t^2$, you can write $(i\omega)^2$. And that will actually give you, i square will give you minus 1, that minus minus gives you plus. So what do you have? You have actually mu epsilon omega square. So, if you take, velocity of wave v as $1/\sqrt{\mu\varepsilon}$, you can write this term as $\left(\frac{\omega}{v}\right)^2$.

And these terms remain as it is. So you have actually everything in frequency domain now, so you are able to get this kind of equation. So you see that there is a dependence of omega over v. And this particular term, we can introduce a new term here that is called wave number or k. So, k can be written as omega over v.

Or it can be, if you write omega as 2 pi f, where f is the linear frequency, you can write $k = \omega / v = 2\pi f / v$, v is the velocity of the wave. So this equation in the form of wave vector or wave number, this is wave number, so you can also call this equation as Helmholtz equation or Helmholtz wave equation. So this is for a particular case where the wave is propagating along x direction. So, you can also consider a generic form of this equation, and you can write $\frac{d^2 E(x)}{dz^2} + k^2 E(x) = 0$.

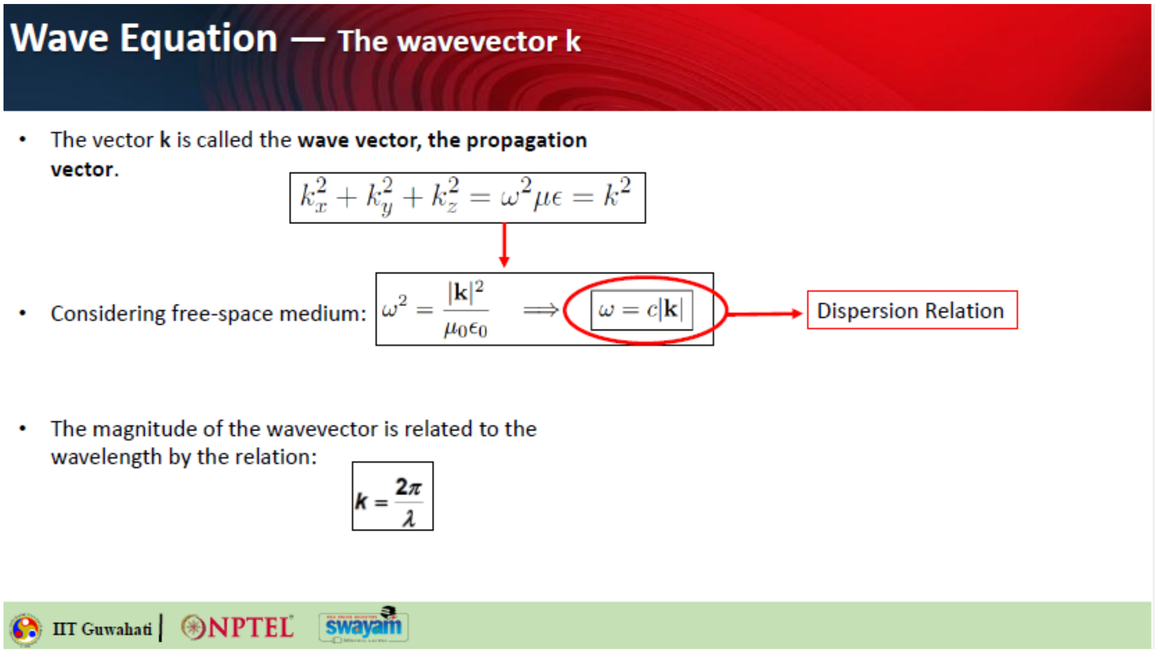


So that will be Helmholtz equation in generic form. So we understood wave equation, and then from there we know how to go into the frequency domain and we can obtain the Helmholtz equation. Now let us look into this wave vector concept in more details. So all of the components in general are basically functions of four coordinates. So there are basically three spatial components x, y, and z.

These are the spatial coordinates and there is also time. So if you actually consider electric field, it is a function of position as I told you before and time. So, you can write this $\overline{E}(\overline{r},t) = \overline{E}\cos\left(k_x x + k_y y + k_z z - \omega t\right)$. This could be a solution to the wave equation. Now here k_x , k_y, and k_z are nothing but the x, y, and z components of the wave vector k.

Omega is the angular frequency. So, when you put this form into this particular equation, you will see that you turn up getting $k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu\epsilon$. So omega square mu epsilon, this part you can write again as k square. So what do you get here? You actually get to see the same equality that we have seen before. So, you can write $\omega\sqrt{\mu\epsilon} = k$.

So $1/\sqrt{\mu\epsilon} = v$. So that way you can find omega by v is basically k. It is the same thing. So this wave vector k is also known as propagation vector because it tells you the direction of wave propagation. So, you can actually write this vector k as I mentioned k_x along x cap, k_y along y cap, and k_z along z cap.



So these are the different components along the three-unit vectors. Now so we have seen the derivation here that from wave vector or the propagation vector, how we can find out the relationship between omega and k. So, if you consider free space, then these values μ and ε, they will be μ_0 and ε_0. That is the permeability and permittivity of the free space. So you can simply write that omega square is basically k square or if you think about the amplitude only, so you can take modulus of k whole square.

That will be divided by these two terms. So these two comes in the denominator. So, for free space they are μ_0 and ε_0. And 1 over square root of mu naught epsilon naught is nothing but c. So, when you take square root on both sides, you get $\omega = c|\mathbf{k}|$.

So this relationship is also called the dispersion relation. So omega k relationship is known as dispersion relation. So this tells you the relationship between the wave vector or modulus of wave vector is nothing but the wave number and the angular frequency. So here the value is also shown, the magnitude of the wave vector that is when you take the modulus of this k vector, you can write it as simple $k = \frac{2\pi}{\lambda}$. What is λ? λ is the wavelength of light.

Here in this case we are considering everything in vacuum. So here it will be wavelength of light in vacuum. If you consider any media, it will be the wavelength of light in that particular media. Now let us see how plane waves propagate in dielectric media. So, the direction of propagation is actually given by the scalar product of the wave vector k and the position vector r.

Plane Waves — in Dielectric media

- The scalar product of the wave vector **k** and the position vector **r** gives

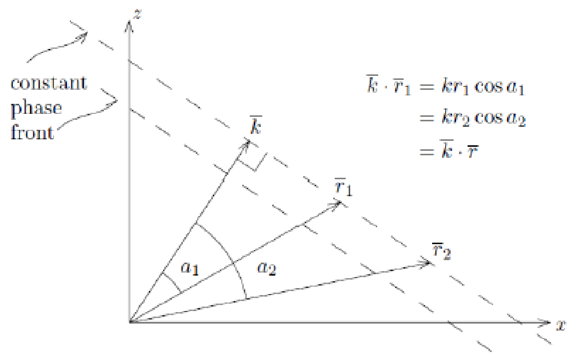
$$\bar{k} \cdot \bar{r} = k_x x + k_y y + k_z z$$

- A constant phase front is determined by
  k· r = constant,
  which indicates that the front is a plane perpendicular to the **k** *vector.*

- The phase front is a plane and the amplitude of the electric field on the plane is a constant.

- Thus,

$$\bar{E}(\bar{r}, t) = \bar{E}\cos(k_x x + k_y y + k_z z - \omega t)$$ → Uniform Plane Wave.



Source: J. A. Kong, Electromagnetic Wave Theory, 2008, EMW Publishing.

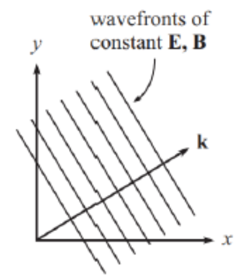
So this one basically tells you the direction of propagation. So, when you compute $\bar{k} \cdot \bar{r}$, r is the position vector that can be written as $k_x x + k_y y + k_z z$. So that actually gives you a phase front because k dot r is a constant.

So here also you can see k dot r. In both cases you will see that k dot r actually coming along this direction or you can say these are the constant phase fronts. So phase front is basically nothing but a plane and the amplitude of the electric field on this plane is a constant. So we can assume the electric field E(r,t) to have this particular form in this case. So that was the solution of our wave equation also. So, E(r,t) can be written as $\bar{E}\cos\left(k_x x + k_y y + k_z z - \omega t\right)$.

So when all the amplitude are equal, we can consider it as a uniform plane wave. So first of all when the phase front is a plane, that is a plane wave, and when the uniform amplitude is there along all the points on this phase front, we call it as a uniform plane wave. So here is the definition. So uniform plane waves are nothing but waves with constant phase fronts. So, the phase fronts, they are basically planar and the amplitude E_0 is also uniform across this plane.

Plane Waves — wavefront

- **Uniform Plane Waves:** Waves whose constant phase fronts are planar (plane waves) and whose amplitude (E_0) is uniform.
- A surface over which the phase of a wave is constant at a given instant is referred to as a **wavefront**. A **wavefront** of a plane wave is obviously an infinite plane perpendicular to the direction of propagation.
- A plane wave is non-uniform if its phase front is a plane but the amplitudes of the field are not constant.
- Since the constant phase front must be perpendicular to **k** at all times, we conclude that this phase front propagates in the direction of **k**.
- Every point on a given plane is equivalent, as far as **E** and **B** are concerned.
- How do these wavefronts move as time goes by?
  The must always be perpendicular to **k**, so all they can do is move in the direction of **k**.



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Source: https://scholar.harvard.edu/files/david-morin/files/waves_electromagnetic.pdf

And what is this wavefront? That is nothing but a surface over which the phase of the wave is constant at a given instant, and that is referred to as wavefront. So a wavefront of a plane wave is obviously a infinite plane that is perpendicular to the direction of propagation. And that is also the other case possible that when the plane wave is non-uniform, it means its phase front is still a plane because it is a plane wave, but the amplitude of the field are not constant. Now since the constant phase front must be perpendicular to the wave vector k all the times, so we can conclude that the phase front also propagates in the same direction as your wave. So phase front has to travel in the same direction of your k.

And at every point on a given plane is equivalent. So you can also say like as far as E and B are concerned, all these points are actually same. So the question comes, how do these wave fronts move as time goes on? So the answer is these wave fronts must be always perpendicular to the wave vector k. So, they will actually have to move in the same direction of k.

Plane Waves — Constraints due to Maxwell's equations

- So, let's now see how Maxwell's equations further constrain the form of the waves.
- We'll look at Maxwell's equations in order and see what each of them implies. Using the Maxwell's first equation for a source-free region,

$$\nabla \cdot \mathbf{E} = 0 \implies \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$
$$\implies ik_x E_x + ik_y E_y + ik_z E_z = 0$$
$$\implies \boxed{\mathbf{k} \cdot \mathbf{E} = 0}$$

E is **perpendicular** to the direction of wave propagation

- Similarly, the second of Maxwell's equations gives

$$\boxed{\mathbf{k} \cdot \mathbf{B} = 0}$$

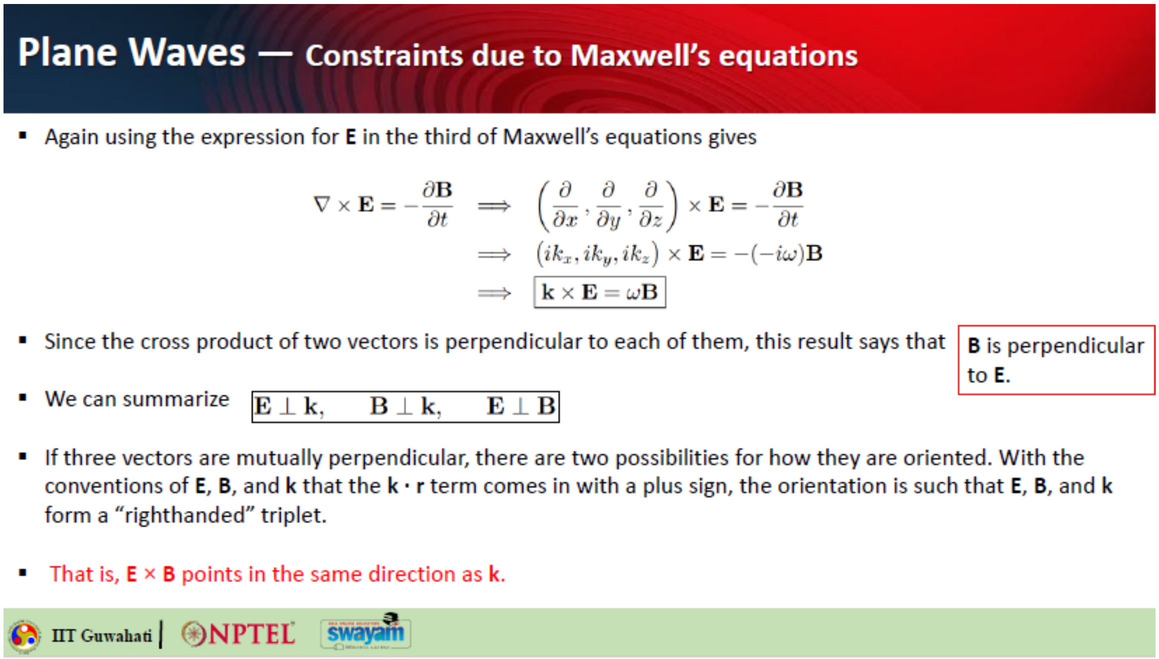
B is **perpendicular** to the direction of wave propagation

I hope that is clear. So now let us see that what are the constraints to this plane wave being put by the Maxwell's equation because Maxwell's equation tells you about the coupled electric and magnetic fields. So depending on the relationship, there may be some constraints So now let us see how Maxwell's equation can further constrain the form of the waves. So let us look into the Maxwell's first equation for a source free region. So here is the Maxwell's first equation when there is no source that is rho is 0. Charge density, surface charge density is 0.

So you can simply write $\nabla \cdot \mathbf{E} = 0$. Now $\nabla \cdot \mathbf{E}$ can be written as $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$. And this partial derivative with respect to space can be converted into ik_x. That is how you take their Fouriers. And you can write them in terms of the wave vectors. So, this one you can write ik_x , this one you can write ik_y , this one you can write ik_z.

So from this equation, you can also write that this is, you take i common and because it is 0 on the other side, so it simply becomes $k_x E_x + k_y E_y + k_z E_z$. So this is nothing but the dot product of k vector and E vector. So k and E, the dot product is 0. So when the two vectors have dot product 0, it means they are perpendicular to each other. It means the electric field is basically perpendicular to the direction of wave propagation.

And the same thing can also be shown for the second Maxwell's equation that k dot B is becoming 0 because you will start with $\nabla \cdot \mathrm{B} = 0$ and you can do the same exercise and you will write the same conclusion that B magnetic field is also perpendicular to the wave propagation. We are saying B also is magnetic field because B= μH. So they can be actually, B is particularly it is basically magnetic flux density, but this also tells you the direction of the magnetic field. Now here, let us see the expression of E in the third Maxwell's equation.



So, this is the third Maxwell's equation where you have $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

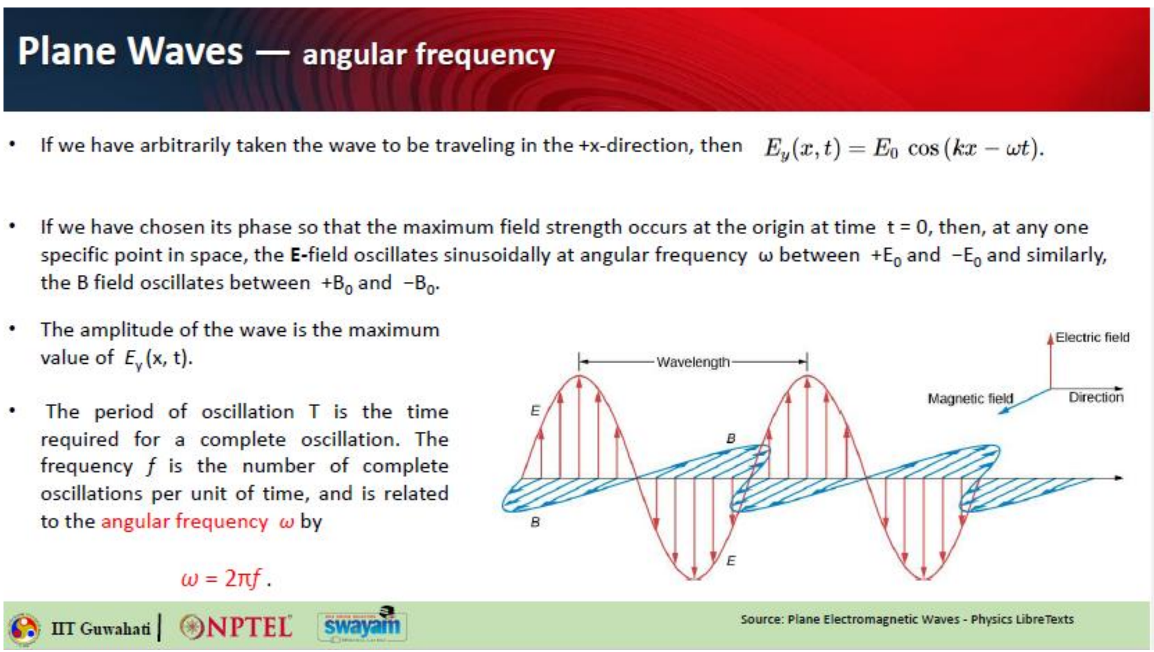
Curl can be written as this vector, del vector can be written as $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. You are doing a cross product with E. So that actually can be converted into this domain, okay. So, this is nothing but ik_x. So $\frac{\partial}{\partial x}$ is nothing but ik_x in the Fourier space, okay, and you obtain this one.

Time derivatives, $\frac{\partial}{\partial t}$ can be written as $-i\omega$, okay. So, once you see this, you can write this is nothing but $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$. It means the cross product to two vectors is perpendicular to each of them and that is happen only when B is perpendicular to E. So that means these are the conditions that a plane wave have to follow because of the Maxwell's condition. That electric field will be perpendicular to the propagation direction, magnetic field will be perpendicular to the propagation direction, and electric field and magnetic field will also be orthogonal or perpendicular to each other.

So this kind of orientation basically form a right-handed triplet, okay. So you can take one, this finger along E, this is B, so this will be showing your k direction, wave propagation direction. So it means E cross B, if you do this particular cross product, you will get the same direction as k. It means the wave is finally propagating along the k direction. Now with that we have to understand when a wave is propagating, there should be some energy associated with it and which direction the energy is flowing.

So when electromagnetic wave travels in space, it carries energy, and the energy density is always associated with the electric field and magnetic field. And there is something called Poynting vector which is named after John Henry Poynting, and this particular vector is used in order to demonstrate the energy flux density of an electromagnetic field. So Poynting vector is nothing but it is a result of the vector product of the field's electric and magnetic component. So mathematically you can write it as P equals E cross H. Now H can be written as B over mu, so this is how it looks like,

$$\frac{\vec{E}\times\vec{B}}{\mu}.$$



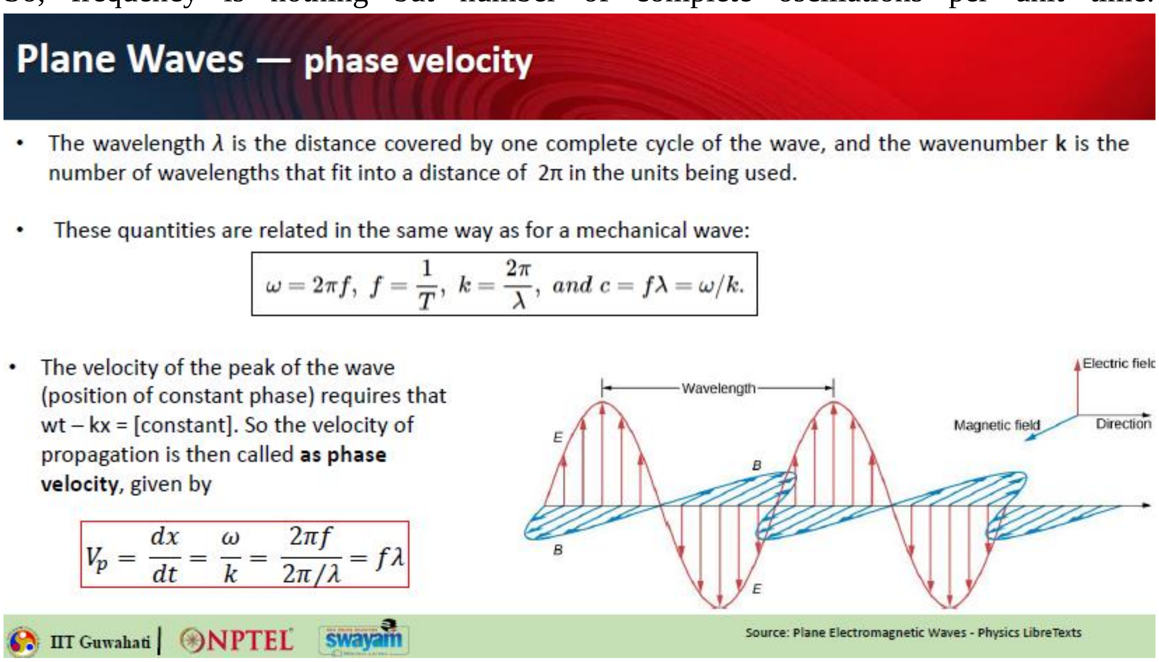
So Poynting vector again is a vector which is basically cross product of two vectors, so it will be in the direction perpendicular to the plane that is containing E and H. And Poynting vector is also called instantaneous energy flux density since it represents the rate of energy transfer per unit area, so it has got a unit of watt per meter square. So the next attribute to the plane wave is the angular frequency. Now if we have arbitrarily taken a wave that is travelling along say positive x direction, then we can write this. So E_y, so you are talking about the electric field component along y direction, x is the propagation direction, and t is the time dependence, so it can be written as $E_0 \cos(kx-\omega t)$

.

So E naught is the amplitude and that is varying as a cosine function. So, if we have chosen its phase so that the maximum field strength occurs at the origin at time t equals 0, so that is why cos function has been taken, that we assume that the maximum field strength is happening when t= 0. Then at any point of space you can say that the electric field oscillates at an angular frequency of omega between plus E_0 and $-E_0$. Let us assume that this maximum amplitude is E_0 and on the negative side it is $-E_0$, so you can say it is oscillating between plus E_0 to $-E_0$.

And what is the frequency? Frequency is omega. Similarly, for magnetic field also, B field also you can say that it is basically oscillating between plus B_0 and $-B_0$. So this is how it has been defined. So, the amplitude of the wave is maximum value of $E_y(x,t)$. So, we are writing this electric field as E_y because this is the direction, y direction of the electric field vector (xt) because they are propagating along x and t is the time dependence. So, the amplitude of the wave E_0 is basically the maximum amplitude of this particular field.

So we also can define the period of oscillation that is defined as capital T. So that is nothing but the time taken for a complete oscillation from here to here. And once you know capital T, that is the period of oscillation, you can also find out the frequency f. So, frequency is nothing but number of complete oscillations per unit time.

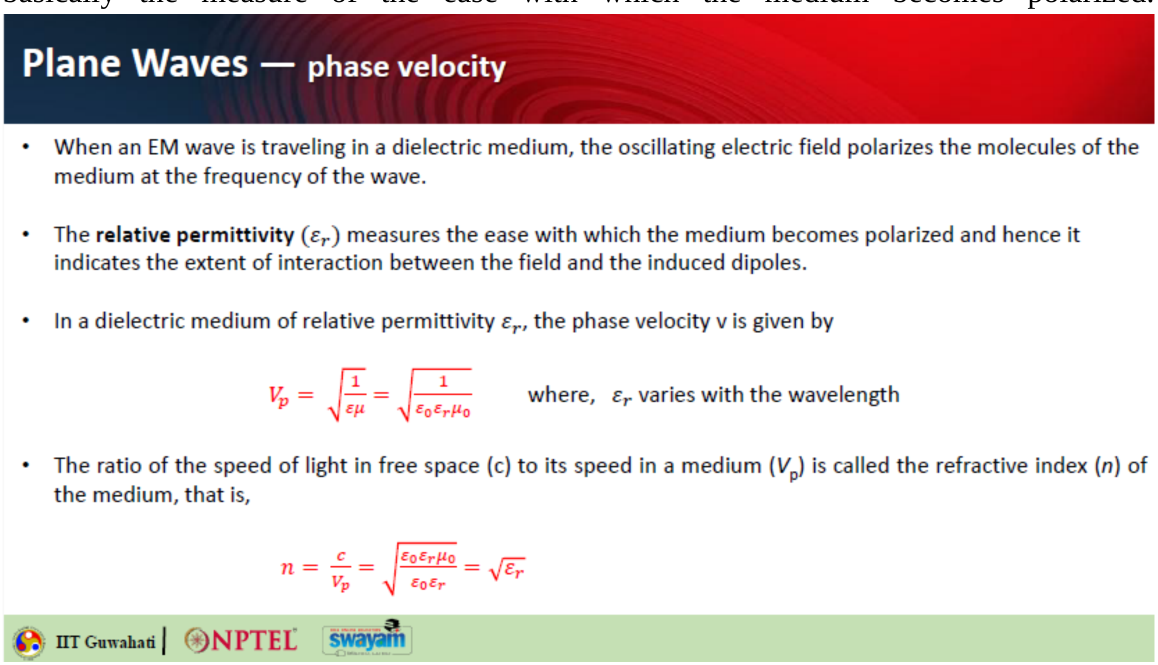


So it is basically inverse of this time period T. So from that you can also find out what is the angular frequency omega. So $\omega = 2\pi f$ and f can be written as 1/T. So, omega can be written as 2 π/T. Clear? So these are the couple of important concepts.

So we have seen wave vector, we have seen the angular frequency. Now let us look into another important attribute of plane wave which is phase velocity. So the wavelength lambda is a distance covered by one complete cycle of the wave. And the wave number k is basically the number of wavelengths that can fit into a distance of 2π. So that is how we have defined this wavelength and wave number. So, relationship between k and lambda we have seen, $k = 2\pi / \lambda$.

So the quantities of electromagnetic wave is basically similar to that of a mechanical wave. So, we know $\omega = 2\pi f$, f is nothing but 1/T, $k = 2\pi / \lambda$ and c, speed is nothing but $f\lambda$ which is also written as $\frac{\omega}{k}$. Now if you see this particular relationship, you can also understand that the velocity of the peak of the wave, that is the position of the constant phase. So, the velocity requires that omega t minus k_x will be constant here, because the slope is 0. So, you can write the velocity of the propagation is then defined as phase velocity.

So you can write $V_p = \frac{dx}{dt} = \frac{\omega}{k}$. Now omega you can write as 2πf and k you can write as $2\pi / \lambda$, so you get $V_p = f\lambda$. So that is the phase velocity. Now when an electromagnetic wave is travelling in a dielectric medium, the oscillating electric field basically polarizes the molecule of that medium at a frequency of the wave. And relative permittivity is basically the measure of the ease with which the medium becomes polarized.

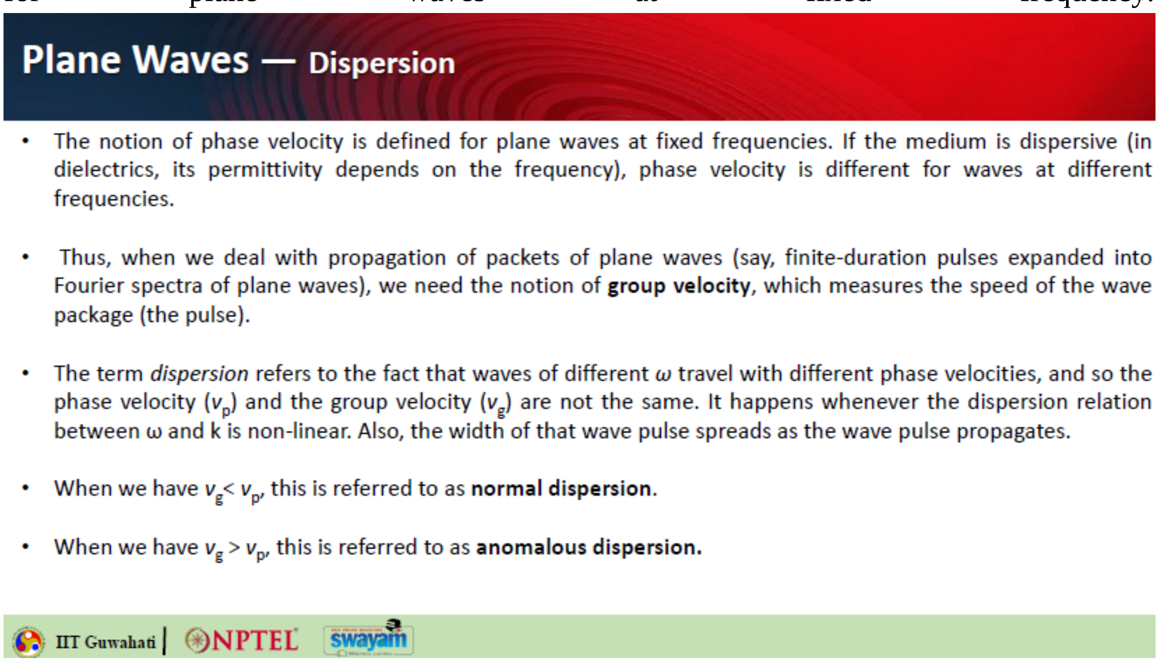


And hence it also indicates the extent of interaction between that medium and the electromagnetic field. Now if you consider a dielectric medium to have a permittivity of

ε_r and a phase velocity, then you can write the phase velocity V_p will be $\sqrt{\frac{1}{\varepsilon\mu}}$. Now ε is nothing but $\varepsilon_0\varepsilon_r$ where ε_r is the relative permittivity. And this is how you can write.

So in this case ε_r also varies with wavelength. So this will be the phase velocity. So phase velocity is also giving you one important thing, that is the ratio of the speed. So once you know this particular phase velocity, you can also find out the ratio of the speed of light in free space that is given by c and the speed in a medium that is phase velocity. And this ratio is nothing but the refractive index.

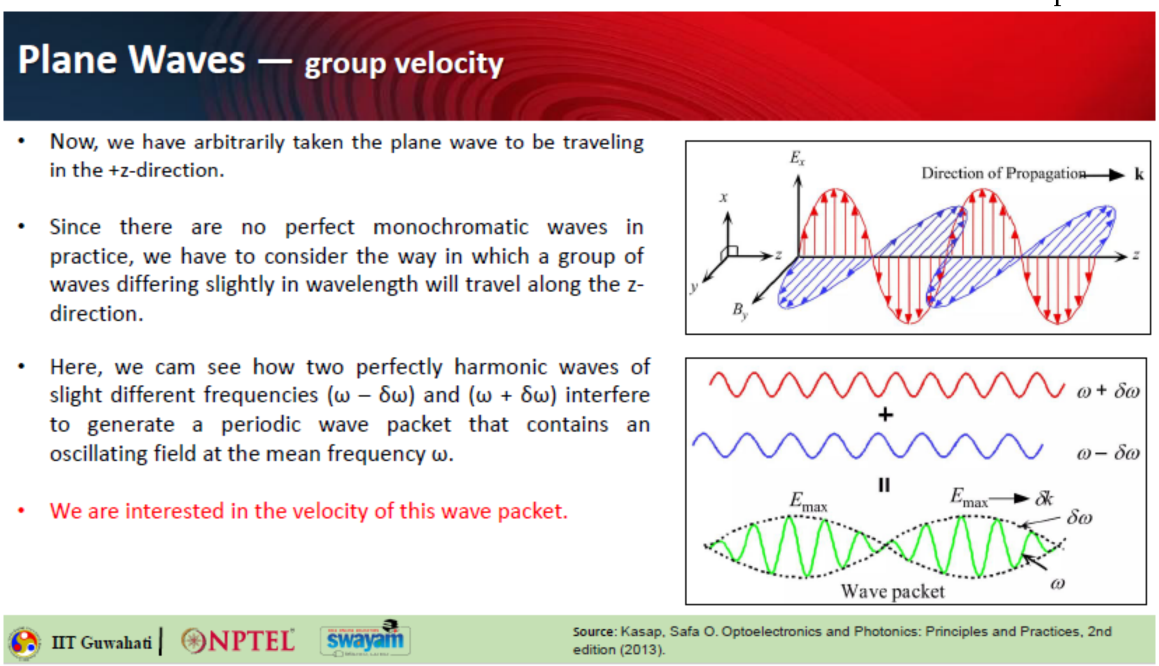
So that is the definition of refractive index. So $n = \frac{c}{V_p}$, this is the phase velocity. So, you can write $\sqrt{\frac{\varepsilon_0\varepsilon_r\mu_0}{\varepsilon_0\varepsilon_r}}$. So, you can cancel out the terms and you can write it as $\sqrt{\varepsilon_r}$. So that is how in a lossless medium, refractive index is same as $\sqrt{\varepsilon_r}$. You can also think about dispersion in the sense that the concept or the notion of phase velocity is defined for plane waves at fixed frequency.



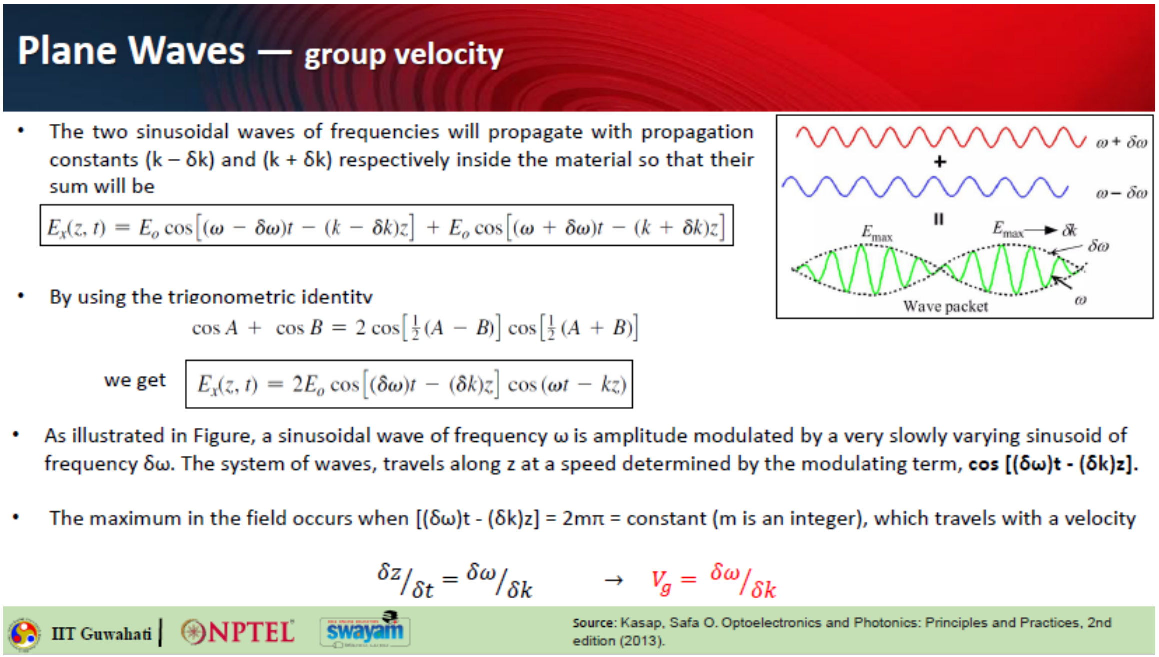
Now if the medium is dispersive, that means in dielectrics, its permittivity can be dependent on the frequency. In that case, the phase velocity is basically different for waves with different frequency. So if that is the case, how do we deal with it? Because different frequency will have a different phase velocity. So when we deal with propagation of packets of plane wave, that means we are discussing about finite duration pulses. And when you convert it into Fourier spectra of plane waves, you will see that there are different frequency components present.

So we need the concept of group velocity to measure what will be the average speed you can say of that particular package or the pulse. So, the term dispersion refers to the fact that the waves of different frequency travel at different phase velocities and so the phase velocity V_p and the group velocity v_g, they will not be identical. And it happens whenever the dispersion relationship between omega and k becomes non-linear. So if the relationship is linear, then phase velocity will be same as group velocity.

Something like if you take free space, the dispersion relation is omega equals c k. So, if you take V_p that is ω/k or you take group velocity that is $\frac{\partial\omega}{\partial k}$, they will give you the same value c. So in that case, they are same. But then if the medium has got a nonlinear dispersion relationship, you will get the two velocities to be different. So if the group velocity is less than phase velocity, that is basically a normal dispersion. And if it is other way, like group velocity is more than the phase velocity, you can call it as anomalous dispersion.



So let us look into the concept of group velocity in little bit more details. So let us assume in this case that there is a plane wave traveling in plus z direction. And since there is no perfect monochromatic wave in reality, we have to consider that a group of waves differing slightly in wavelength is actually traveling. So it is not exactly one particular wavelength, there will be little bit spread in the wavelength and they are all traveling in the same direction. So you can actually think of in terms of frequency and say that two perfectly harmonic waves with slightly different frequency, one to have $\omega-\delta\omega$ and another can have frequency $\omega+\delta\omega$, they actually propagate together and they interfere and generate a periodic wave packet which contains an oscillating field at the mean frequency mu, something like this. So, they actually create a wave packet where the field inside is basically oscillating at a frequency of omega.

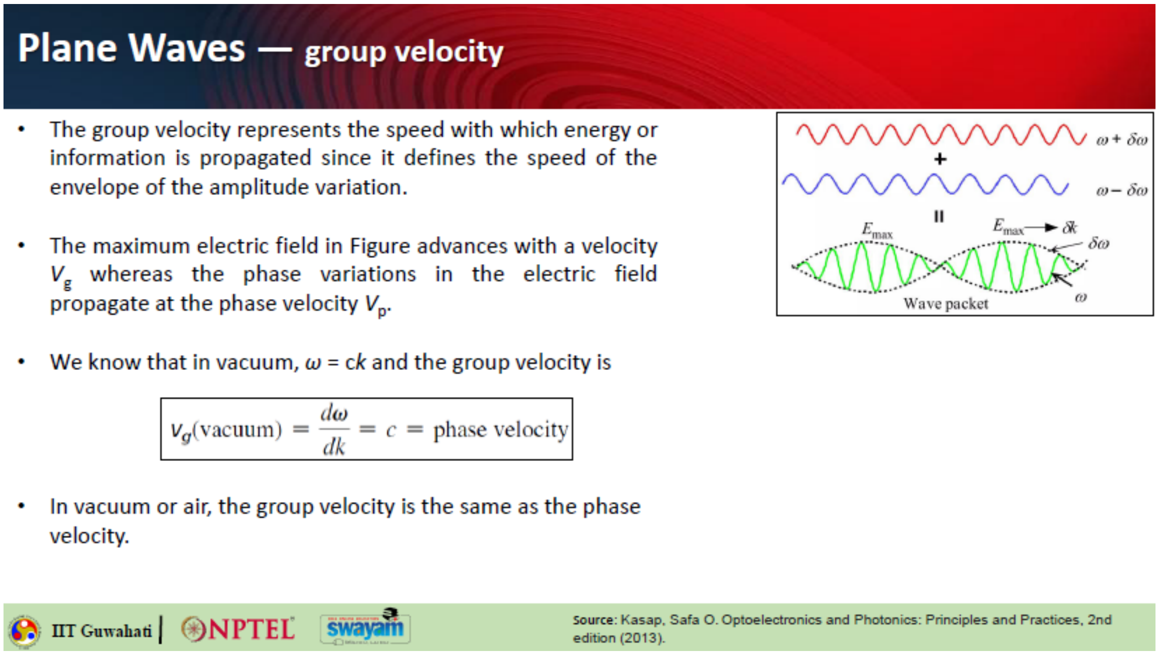


And what we are interested because it is a group or a packet, we are interested at what speed this packet itself is traveling. And for this packet to determine this velocity, you must focus on the maximum electric field here. So let us look into this again. So, if you consider two sinusoidal waves of frequencies will propagate with propagation constant $k-\delta k$ and $k+\delta k$ inside a material so that the sum will add up like this. So, you will have Ex(z,t) can be written as $E_o \cos[(\omega-\delta\omega)t-(k-\delta k)z]$.

This is one wave. The other wave is $E_o \cos[(\omega+\delta\omega)t-(k+\delta k)z]$. So, these are the two waves that we have seen here and when you do the interference or you add them up together, you can use the trigonometric identity $\cos A + \cos B$ and you can write it in terms of this. So, you can finally see that you get $2E_o \cos[(\delta\omega)t-(\delta k)z]$. So that will be one oscillation and the other one is $\cos(\omega t - kz)$. So, you will see that the sinusoidal wave of frequency omega is amplitude modulated here by a slowly varying sinusoid which has got a frequency of $\delta\omega$.

And the system of waves, this one, they will travel along the z direction which is determined by the modulating term. So here the modulating term is nothing but $\cos[(\delta\omega)t-(\delta k)z]$. So in this case, the maximum in the field will occur. So, this is the maximum.

So maximum in the field will occur when $(\delta\omega)t-(\delta k)z$ will be constant. That is basically $2m\pi$. What is m? m is integer. In that case, you can find out what is the velocity. So, if you do $\delta z / \delta t$, you will come up with $\delta\omega / \delta k$.



So this is the definition of the group velocity. So, group velocity V_g should be written as $\delta\omega / \delta k$. So this is kind of a derivation of how do we come to this particular relationship of the group velocity. Now the group velocity represents the speed with which the energy of the information is propagating since it defines the speed of the envelope of the amplitude variation. The maximum electric field in the figure advances that is E_{max} that advances with a velocity of V_g and whereas the phase variation in the electric field, so these are the phase variation in the electric field, they actually propagate with a phase velocity of V_p. So once again the amplitude, the maximum amplitude that is the electric field amplitude that propagates with V_g through this envelope and the phase variations, they do this first oscillation and they have this phase velocity of V_p.

So in vacuum we know the dispersion relation is $\omega = ck$. So, in vacuum the group velocity is also $\frac{d\omega}{dk}$ if you do you will get c that is same as phase velocity. So as I mentioned in vacuum or air group velocity is same as phase velocity. So, this we have already seen that the maximum electric field in that figure advances with the group velocity of V_g whereas the phase variations in the electric field that propagated with phase velocity of V_p. Now for an electromagnetic wave in a medium k is the propagation constant inside the medium.

Plane Waves — group velocity

- The maximum electric field in Figure advances with a velocity V_g whereas the phase variations in the electric field propagate at the phase velocity V_p.
- For an EM wave in a medium, *k* is the propagation constant inside the medium, which can be written

  $k = 2\pi n/\lambda_0$ where λ_0 is the free pace wavelength.

- The group velocity then is not necessarily the same as the phase velocity V_p, which depends on ω/k and is given by c/n.
- The group velocity $V_g = \delta\omega/\delta k$ depends on how the propagation changes in the medium, δk, with the change in frequency δω, and δω/δk, is not necessarily the same as ω/k when the refractive index has a wavelength dependence.

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So it can be written as $k = 2\pi n/\lambda_0$. What is n? n is the refractive index of that medium and lambda naught is the free space wavelength. There is a typo here, it's space. So, the group velocity then is not necessarily the same as the phase velocity V_p ω/k and is given by c/n. So, the group velocity definition $V_g = \delta\omega/\delta k$ depends on how the propagation changes in the medium δk. That is how the propagation constant changes in the medium with the change in the frequency that is doh omega.

Plane Waves — group index

- Suppose that the refractive index of the dielectric medium $n = n(\lambda_0)$ is a function of (free space) wavelength λ_0.

  Consider $\omega = 2\pi c/\lambda_o$ and $k = 2\pi n/\lambda_o$.

- We can easily find the **group velocity** by first finding $d\omega$ and dk,

Differentiate $\omega = 2\pi c/\lambda_o$ to get $d\omega = -(2\pi c/\lambda_o^2)d\lambda_o$, and then differentiate $k = 2\pi n/\lambda_o$ to find

$$dk = 2\pi n(-1/\lambda_o^2)d\lambda_o + (2\pi/\lambda_o)\left(\frac{dn}{d\lambda_o}\right)d\lambda_o = -(2\pi/\lambda_o^2)\left(n - \lambda_o\frac{dn}{d\lambda_o}\right)d\lambda_o$$

$$v_g = \frac{d\omega}{dk} = \frac{-(2\pi c/\lambda_o^2)d\lambda_o}{-(2\pi/\lambda_o^2)\left(n - \lambda_o\frac{dn}{d\lambda_o}\right)d\lambda_o} = \frac{c}{n - \lambda_o\frac{dn}{d\lambda_o}}$$

$$v_g(\text{medium}) = \frac{c}{N_g}$$

in which the **group index of the medium** N_g can be given as:

$$N_g = n - \lambda_o\frac{dn}{d\lambda_o}$$

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That is why this ratio is important for finding out the phase velocity and it is not necessarily same as ω/k when the refractive index has a wavelength dependence. So, if you get n is different for different different wavelength so there are material which are dispersive in nature. That means the refractive index is not same throughout and different wavelengths they have a slightly different refractive index. For those kind of material also group velocity and phase velocity will not remain same.

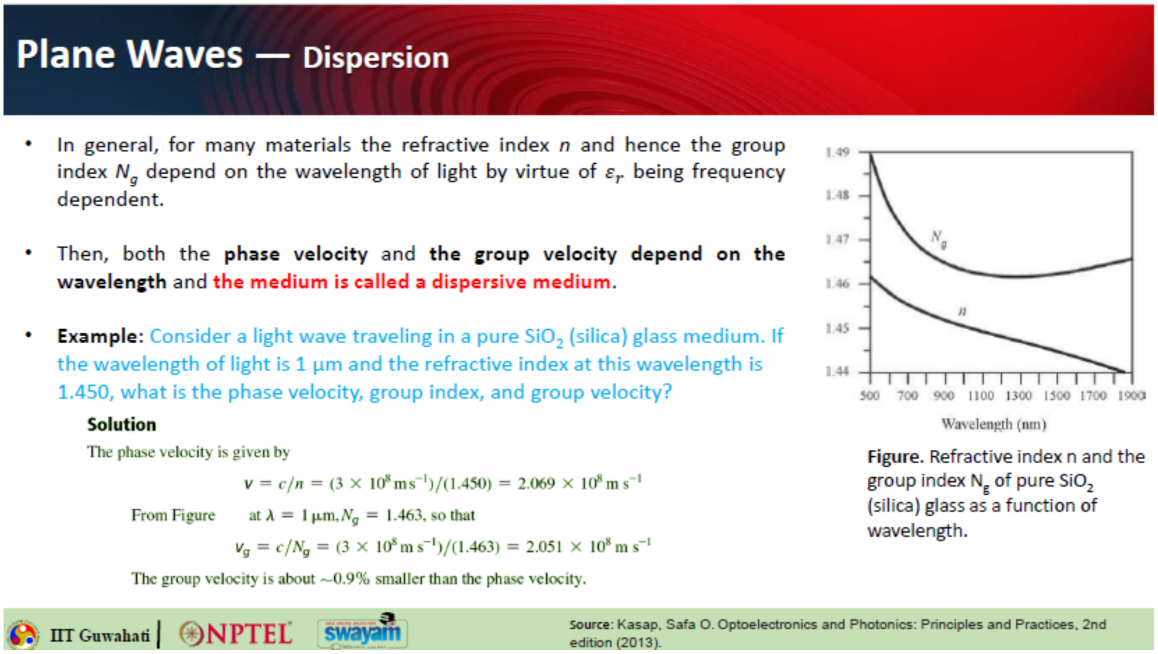
So that brings a new concept here that is called group index. So if you consider the refractive index of the dielectric medium n is basically a function of the wavelength. So, n is basically $n(\lambda_0)$. That is the refractive index is a function of free space wavelength (λ_0). So, in that case you can consider $\omega = 2\pi c / \lambda_o$ and k is nothing but $2\pi n / \lambda_o$.

So k is basically the wave vector inside that particular dielectric medium of refractive index n. Now we can find out the group velocity but by first finding you need doh omega and doh k. So, you take this equation $\omega = 2\pi c / \lambda_o$ and to get $d\omega$ you differentiate it. So that gives you $-\left(2\pi c / \lambda_o^2\right) d\lambda_o$. And then you take $k = 2\pi n / \lambda_o$ and there also you can do the differentiation you can write doh k equals.

So this is how you differentiate it. So here this is also a function of (λ_0). So that is why you have to do it in two parts and finally you can add up together. I am just not reading out these terms. You can simply see this is a very simple calculation. And finally, you can write $v_g = \dfrac{d\omega}{dk}$. So doh omega you can bring this term and doh k you can bring this term and then when you make the common terms cancel out each other you will be left out $\dfrac{c}{n - \lambda_o \dfrac{dn}{d\lambda_o}}$.

That is (λ_0) that is the free space wavelength times the derivative the way the refractive index is changing with (λ_0). So that term comes into picture. So that is how you can write vg is nothing but $\dfrac{c}{N_g}$ that is the group index. So group index belongs to a plane wave of certain frequencies traveling inside a particular dispersive medium.

So you can write also what is Ng that is the group index of this medium. So, this particular medium n has got a dependence of lambda for that you can find out what is the group index of that medium and it can be given as $n - \lambda_o \dfrac{dn}{d\lambda_o}$. So now let us look into the dispersion. So, in general for many materials the refractive index n and hence the group index N_g that depend on the wavelength of light by virtue of ε_r being frequency dependent. So in those cases the phase velocity and the group velocity both depend on wavelength and these are the kind of medium which is called dispersive medium where both phase velocity and group velocity is dependent on wavelength. Now if you consider a light traveling in a pure silica glass medium and if the wavelength of light is 1 micron and the refractive index at that wavelength is 1.



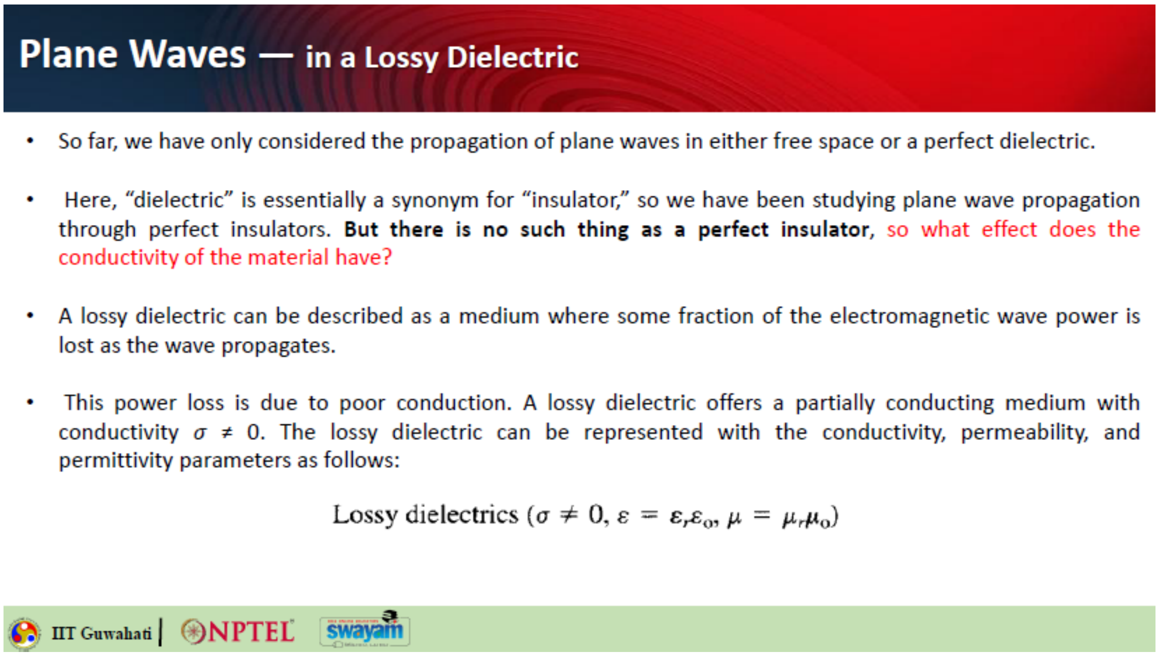
450 micron then what will be the phase velocity, group index and group velocity. So you can actually look into the dispersion curve of silica given here. So here is the refractive index how it is varying with wavelength and also how the group index is varying with wavelength. So, you can find phase velocity using this formula $v=c/n$ that is very simple.

So $v=c/n$, c you already know n at 1 micron is already given 1.450 you put it you get this value. So that is the speed, phase velocity. Now what is group index? At $\lambda=1$ this one so here you can see your N_g is 1.463. So, once you know N_g that is your group index you can find out group velocity V_g that is c/N_g .

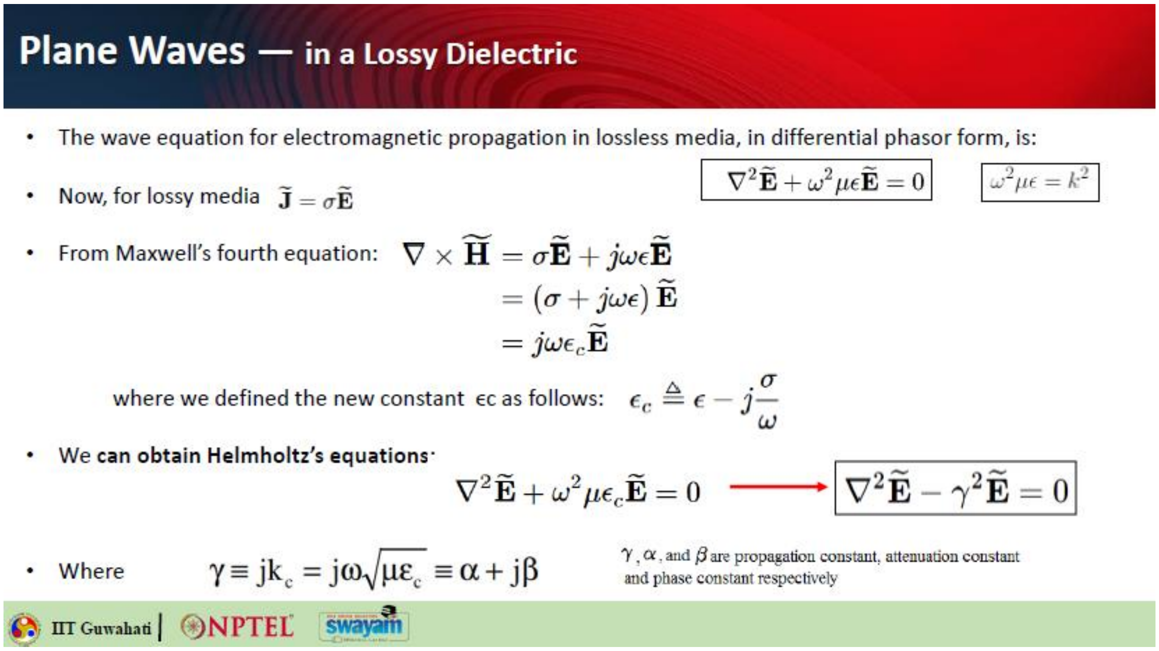
c you already know you put the value of N_g you can find out what is the group velocity. So here if you compare these two you will see that the group velocity is roughly 0.9 times smaller than the phase velocity and that is usually the case. So this is the case of normal dispersion. Now till now we have considered like the plane wave propagation in either free space or perfect dielectric that is lossless dielectric.

But you know in this case dielectric we have essentially used it as a synonym for insulator. So, we have been studying plane wave propagation through perfect insulators.

But in reality there is no such thing called perfect insulator. So what effect will the conductivity of the material bring into the picture? That means if the dielectric is basically lossy. Now a lossy dielectric can be described as a medium where some fraction of the electromagnetic field power is basically lost when the wave is propagating.



So that is how it is lossy. Now this power loss is basically due to poor conduction. A lossy dielectric offers a partially conducting medium with a non-zero conductivity. So sigma becomes non-zero. The lossy dielectric can be represented by its conductivity, permeability and permittivity using this kind of values. So, you can denote a lossy dielectric as $\sigma \neq 0$ and then $\varepsilon = \varepsilon_r \varepsilon_o, \mu = \mu_r \mu_0$.



So once you do that you will have some changes in the wave equation. So the wave equation for electromagnetic propagation in lossless media we have seen. This was the value $\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\epsilon\tilde{\mathbf{E}} = 0$. You could have written this part as k^2 and we would have called this as Helmholtz equation.

So that is all fine. Now if you consider that for lossy medium J is nothing but σE. So, if you go back to Maxwell's fourth equation and you write $\nabla \times \tilde{\mathbf{H}} = J + \frac{\partial D}{\partial t}$. So, D can be written as εE and $\frac{\partial}{\partial t}$ can be written as $j\omega$. So that way you can write you take E common you can write $\sigma + j\omega\epsilon$. So in the other cases we have seen sigma was zero in the lossless case.

But here you are actually having a finite value of this is a non-zero. So let us denote this term as a complex permittivity. So, you are writing as ε_c. So, it's nothing but $j\frac{\sigma}{\omega}$. So, you can actually define this new constant ε_c can be written as $\epsilon - j\frac{\sigma}{\omega}$. So, once you do that you can put it back in the form of Helmholtz equation and this time the Helmholtz equation changes.

So only change will be $\nabla^2\tilde{\mathbf{E}} + \omega^2\mu$. Instead of this epsilon you will have this ε_c. And this term can also be written as $-\gamma^2\tilde{\mathbf{E}}$. So let us see what is this gamma. So, this gamma is nothing but the propagation constant and it is defined as jk_c .

So k_c can be written as $\omega\sqrt{\mu\varepsilon_c}$. And gamma can also be written in two parts like $\alpha + j\beta$ where alpha is basically the attenuation constant and beta is basically the phase constant or the propagation constant. So you can equate the real part to the real part and imaginary part to the imaginary part and find out what are these. So, you can briefly say gamma is nothing but $\alpha + j\beta$. So, alpha is nothing but the attenuation constant which is a measure of the spatial rate of decay of the electromagnetic wave in the medium.

That means the rate at which it is getting decayed. So, it can be written in terms of N_p/m or you can also write in dB/m. Phase constant is basically a measure of the phase shift per length or you can also call it as phase constant or wave number. So that we have already seen. So, once you understand these two properties in terms of mu, epsilon and sigma you can represent alpha and beta in this particular form.

Plane Waves — in a Lossy Dielectric

$$\gamma = \alpha + j\beta$$

- **Attenuation constant (α)** is the measure of the spatial rate of decay of the electromagnetic wave in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m).
- **Phase constant (β)** is the measure of the phase shift per length and is called the phase constant or wave number.
- In terms of dielectric material property, α and β can be given as:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2}-1\right]}$$

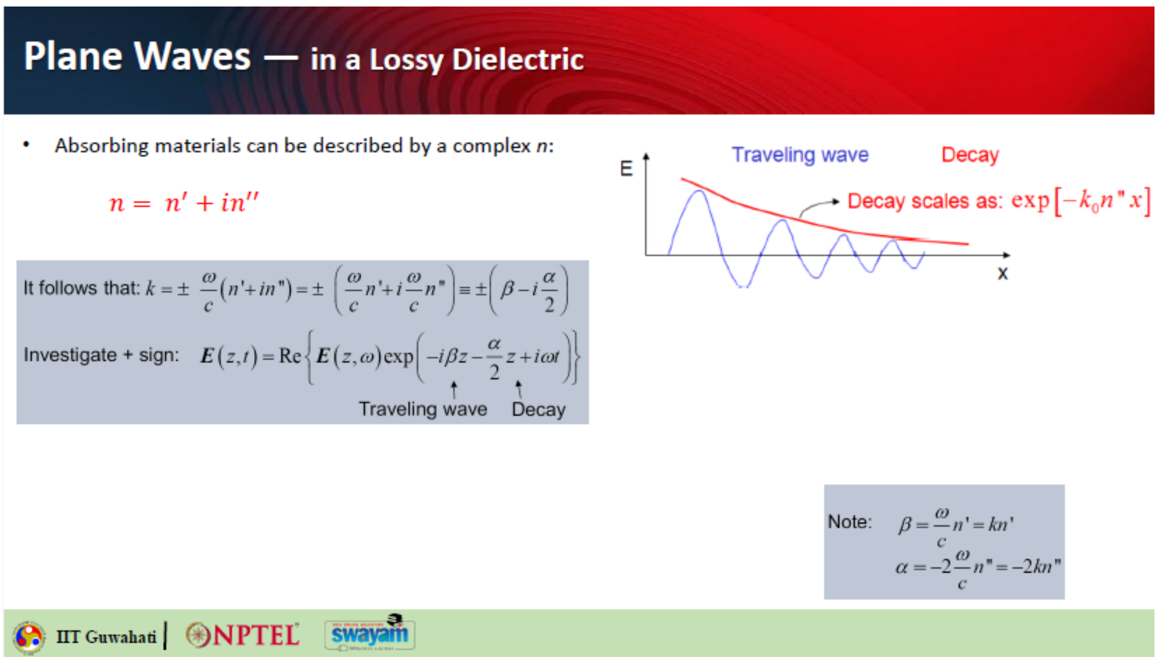
$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2}+1\right]}$$

- The intrinsic impedance of the medium, given by: $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

Note: the intrinsic impedance of free space is $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377$ [Ohms]

So they are very identical equation only this is minus and this is plus. And what you have seen that from this you can also obtain what is the intrinsic impedance of the medium that is eta that is given by $\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$. Now in the case of free space where conductivity is zero, this turns out to be this term goes out $j\omega$ $j\omega$ goes out you simply have omega this is $\frac{u_0}{\varepsilon_0}$ and η_0 that is the free space intrinsic impedance will be nothing but $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ that is a standard value everybody knows it's 120π or 377Ω . So, if you consider the material to have a permittivity so if the material is absorbing we can say the material also has a complex permittivity. Similarly, in terms of refractive index you can say that the material any absorbing material has got a complex refractive index *n*.

So *n* now will have two parts that is real part and imaginary part so n' and n''. And this is how they can be correlated. So, you can write what is *k*, *k* is $\pm\frac{\omega}{c}(n'+in'')$ and then you can split them out in real and imaginary part and then equate this with $\beta - i\frac{\alpha}{2}$. So that way you can also find out so if you only investigate the wave vector or wave number with positive sign in that case you can write down the wave equation or the electric field expression like this. So, E(z,t) will have real Ez omega exponential $-i\beta z$ so that tells you about the traveling wave $-\frac{\alpha}{2}z$.



So this terms tells you the decay constant with z. So I think this is z. So anyways this is correct because if the propagation direction is x the decay term here is also correct. It is nothing but exponential. You can see this term beta is nothing but kn' .

So decay term is basically this one. It is beta and alpha so beta is the travelling wave term. So, beta is $\frac{\omega}{c}n'$. Alpha is $-2\frac{\omega}{c}n''$. So that is nothing but $-2kn''$.

So from here you can also see this is $\alpha/2$ so 2 cancels out. It is only $-k_0 n'' x$. *x* is nothing but the propagation direction. Here we have considered *x* propagation direction.

Though here we have taken the equation along *z*. If it is along *x* you make it to *x* this will be *x* and this will also be *x*. So that is fine. So, what I mean to say that here the wave will be propagating but then along with the propagation the amplitude will not remain same. It will keep on decaying and this will be the rate of decay. So, with that we will stop here and in the next lecture we will discuss about polarization optics and in case you have any query on this particular topic and any questions you can drop an email to this email address. Thank you.