Applied Linear Algebra
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Week 01
Linear Combinations and Span

Hello everyone, we're going to continue our study of linear algebra. In the first lecture we saw a very general introduction to what this abstract notion of a vector is and a definition for an abstract notion of a vector space and all that. Now we're going to build on that and keep going further. And in this particular lecture, we'll study some very crucial fundamental ideas in all vector spaces. These play a very important role in vector spaces in general, so these are the basics. So you should know them really really well, okay?

So we'll be studying specifically linear combinations, which in many ways people would say is the essential part of linear algebra. We'll look at things called span, subspaces which are very important and also the very important notion of linear dependence and independence, okay? Once again I am going to appeal to what you know about vectors a little bit from before, okay? And this will be a pretty quick introduction to some of the core ideas. I will emphasize the important things, the ideas by themselves at some level are simple. So they're not very complicated. You don't have to be very worried about them, okay? So let's get started.

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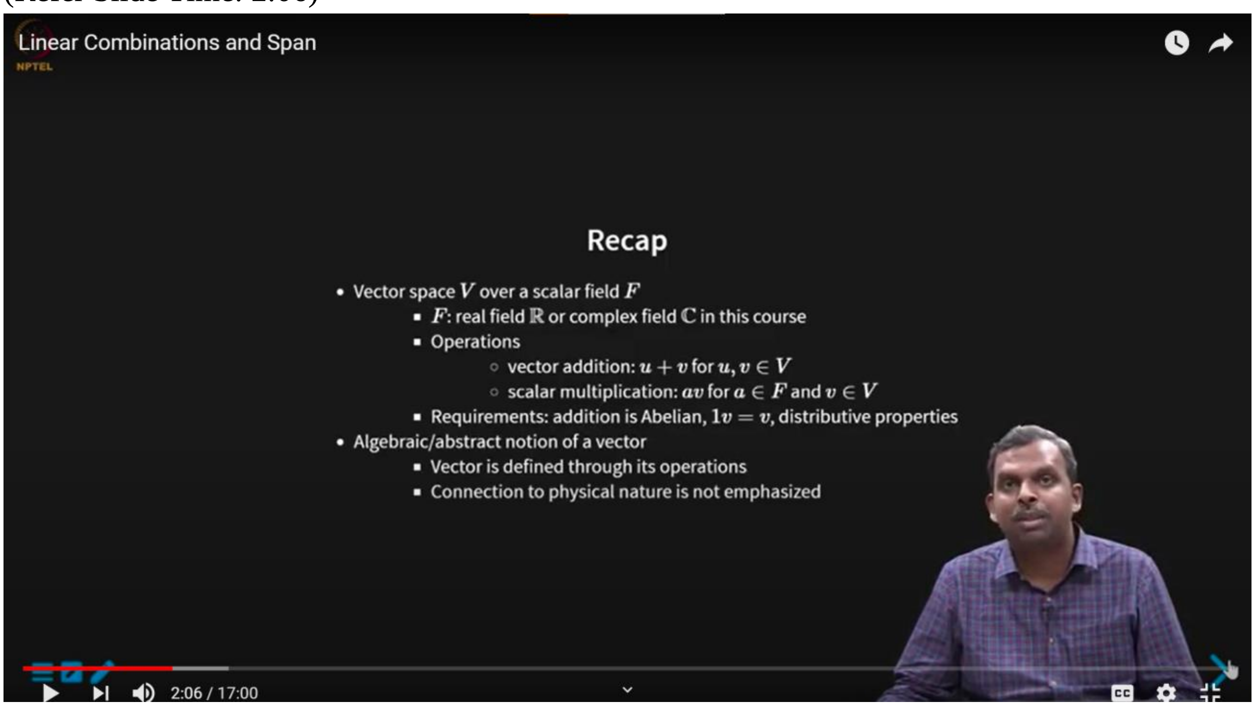

Okay, a quick recap. Like I said we are going to think of a vector space V over a scalar field F. So that will be a constant thing that I carry throughout this course, the scalar field F is going to be either the real numbers or the complex numbers for us in this course. But quite a few results that we do will hold for any scalar field F by the way. But in this course we'll primarily think about real numbers and complex numbers.

Okay, the two operations we spoke about - the vector addition, the scalar multiplication, both of them are very important. And then these various requirements we had for a particular set of vectors to be a valid vector space, okay, what are those requirements? The addition should be Abelian, there should be these distributive properties etc. etc. okay? So that is a quick recap about the previous lectures. And then let us move on to studying more things today.

Okay, so like I said the first and most important thing that we'll study in this course is this notion of linear combinations, okay? So we have already seen that there are two important operations in a vector space - the addition and the scalar multiplication. In linear combination, we do this very simple thing of combining the two, okay? So we do scaling and addition, okay? And the definition is given up on top here. If you have a set of vectors v_1, v_2, v_3 etc. You number them like that and then you have a set of scalars a_1, a_2, a_3, like that. Then the linear combination of these vectors with those scalars is $a_1 v_1 + a_2 v_2 + \cdots$ okay?

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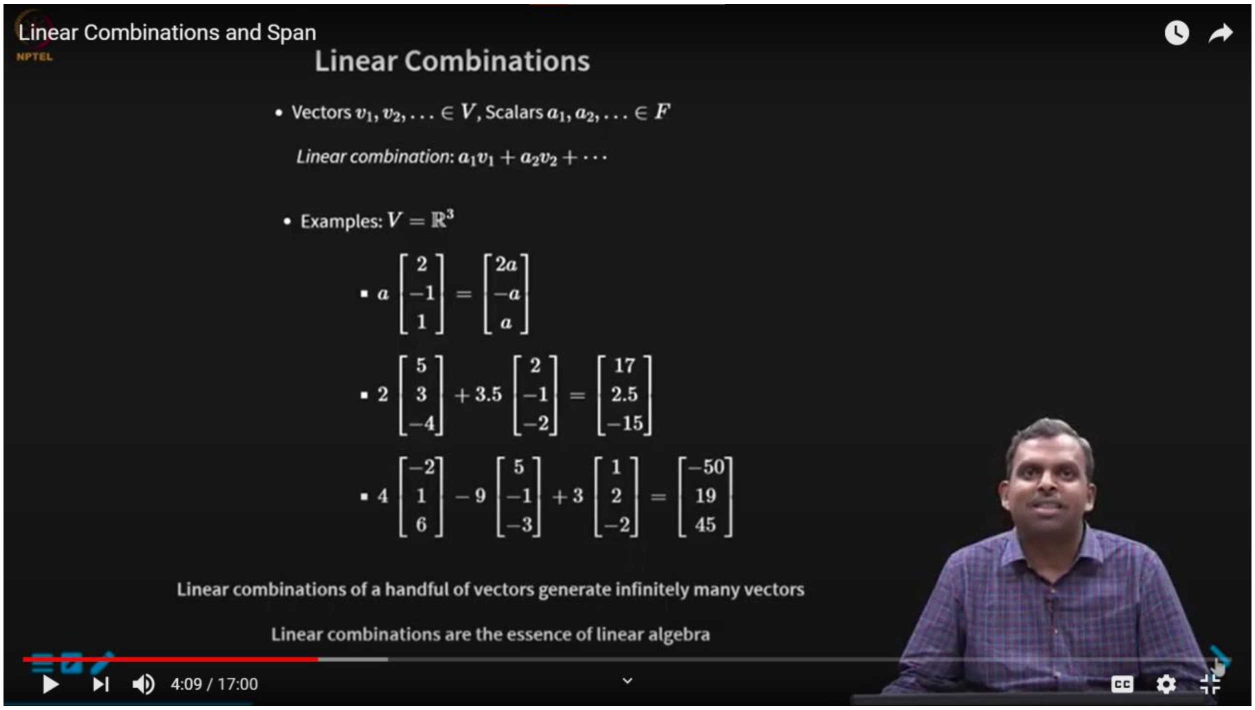

And to illustrate it for you I am showing you some very simple examples. This is just a basic operation. So if you take the $\mathbb{R}_3$ vector space for instance, you can do linear combinations with just one vector. That would not be too interesting, it just does scaling, okay? And you could also do linear combinations with more than one vector, which is scaling and addition. Three vectors, four vectors, like that any number of vectors you can do linear combinations, okay?

So you notice here what is interesting about linear combinations is - supposing you have two vectors in $\mathbb{R}_3$ right? And you start looking at all possible linear combinations of them. By only changing the scalars a_1 and a_2, okay, you get a whole bunch of vectors right? So a whole lot of vectors you will get just by that simple operation. So these linear combinations give you this power by varying scalars to get a large number of vectors from a small set of vectors, okay? So that is one way one reason why linear combinations are very important and powerful etc.

But it's not wrong to say that linear combinations are the essence of linear algebra. So that's the essential linear property. So anytime somebody says something is linear, operation is linear, I'm looking at linear operations - invariably they mean linear combinations, they're looking at scaling and adding, okay? So that is the notion of linearity, okay? So that is the definition. I am sure you must have seen some form of this definition before but let’s formally define it in this fashion okay?

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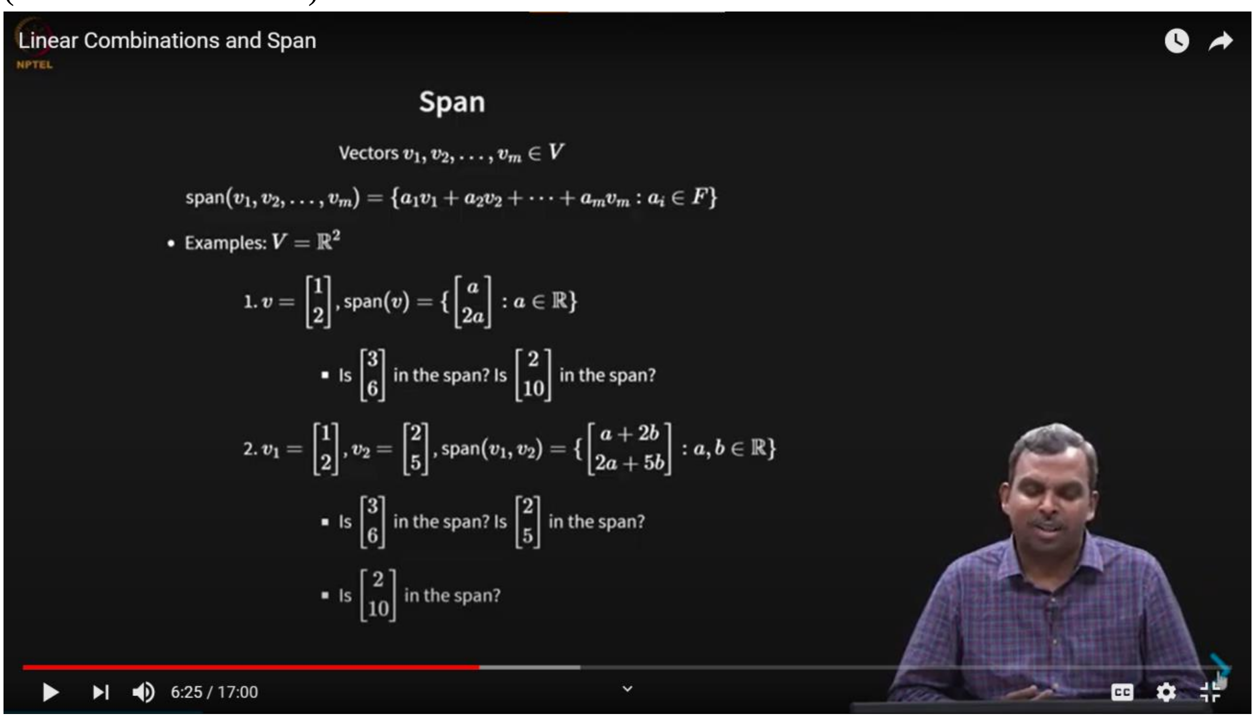

So the next comes the idea of span, okay? So this is sort of related to what I alluded to in the previous slide. As in, you start with a small number of vectors and you can do these linear combinations to generate a whole bunch of other vectors, okay? So you collect all those vectors together and put them together in a set and that becomes the span of the vectors you started with, okay? So supposing for instance you start with v_1, v_2 till v_m from your vector space V. The span of those vectors, right? Span means the extent or reach, right? Something like that. So the span of those vectors is all possible linear combinations that you can do with them, okay? And the scalars can be anything. You can pick any scalar, you take all possible linear combinations you will get the notion of the span, right? So that's what's most important.

And I've shown here a few examples okay? Before I go through the examples one by one, it's interesting at least for small examples like $\mathbb{R}_2$ and $\mathbb{R}_3$ to think of span and vectors on the... and draw pictures of them, right? So think of them on the graph or the 2D plane or the 3D space right? Once it crosses 3 it's difficult to visualize, but at least for $\mathbb{R}_2$ and $\mathbb{R}_3$we can visualize okay? And the visualization is very good sometimes, it gives you a lot of intuition about what would happen. But it's very limited. Beyond a certain point it doesn't really help you. But you can take some ideas from here and move on ahead.

So let me begin by illustrating some of these notions of span using pictures, okay? So these pictures for $\mathbb{R}_2$, the picture is very easy to draw. $\mathbb{R}_3$ is a little bit more complicated but let me, at least for $\mathbb{R}_2$, show you some pictures. So for instance here I'm looking at span of (1, 2) and the span of (1, 2) along with (2, 5) right? So let me show you how this would look on a graph. So usually when people think of $\mathbb{R}_2$, okay, so they think of the two dimensional plane. So you can think of maybe the x axis here, and the y axis here. So let me just draw it along those lines as much as possible. I will draw it along those lines okay? So hopefully you can see this okay? So this is the x-axis, this is the y-axis. So if you want me to... so maybe I should just do this part alone a little bit better, okay? Along the line, there you go, okay? So and then you have these coordinates, I mean this would be 1, this would be 2, this would be 3, so on, right? And then this would be 1, here this would be 2 here, so on, okay? So now if I want to plot the point (1, 2) I am going to get the point here, right? So this is the point (1, 2) ok? So that is the vector.

So usually when you might, you know, for instance, be used to drawing an arrow like this, right? And then thinking of that as the vector, okay? So in this class we will not do this, okay? So I will not draw these arrows. I will simply write the vector next to the x, okay? So this is my vector okay? And not anything else, okay? So keep that in mind, we will think of the (1, 2) as a vector like this, ok? So now I want to think of the span of (1, 2) right? So I have... That is the question that was asked in the slide. I want to think of the span of (1, 2) okay? What is the span of this vector? The answer that was given there was that it's going to be $(a, 2a)$ for all a in the real line right? So you change the value of a, you just keep doing $(a, 2a)$ right? So that is what it is and you know from your practice with coordinate geometry etc. that these points $(a, 2a)$ they
actually make a line, right? They go on like this, like that, and then on this side also they would go on like this. Okay maybe I didn't draw it very cleanly but you can see this is the span right? So the span is the straight line ok? So that is the line passing through (1, 2) and the origin. So this is the

origin, right? Okay, so the line connecting O and (1, 2) and if you extend it in both directions that becomes your span.

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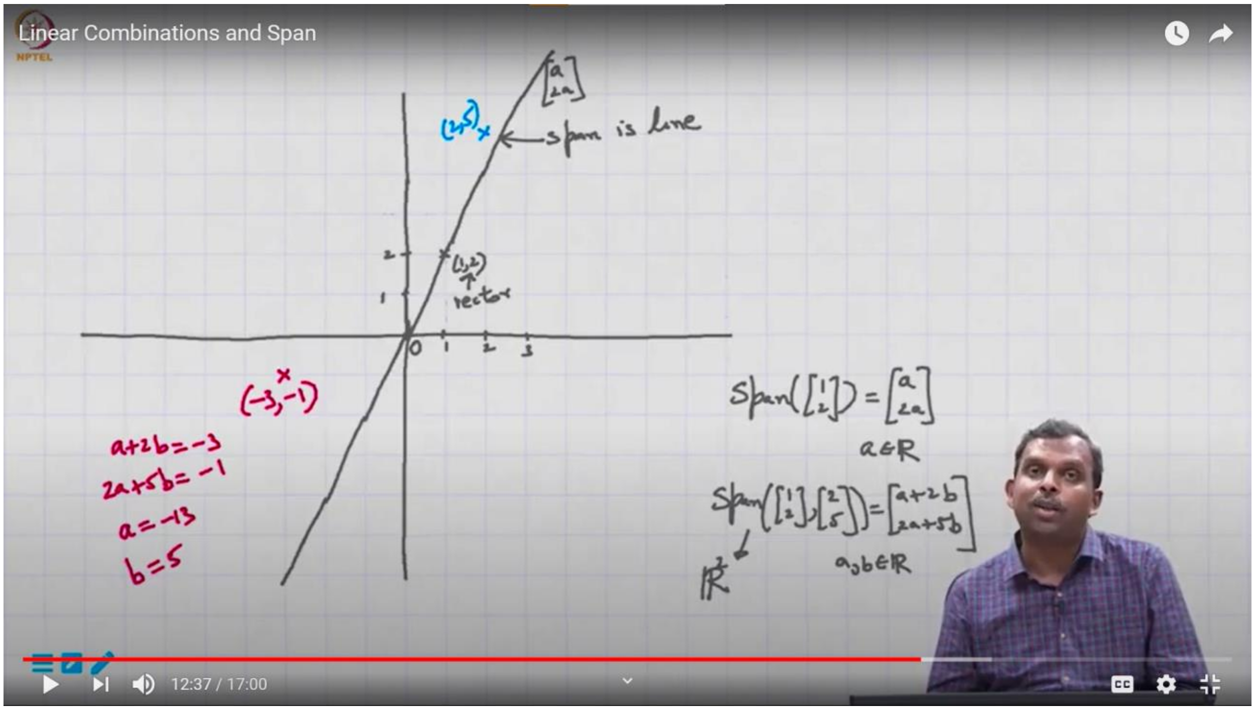

All right so that's a nice picture to imagine. So span of one vector in $\mathbb{R}_2$ is always going to be a line through the origin and passing through that vector okay? So the next question asked you for a span of (1, 2) and (2, 5). So (2, 5) is a point here. So maybe I will draw it in another colour. This is the point (2, 5) okay? Now what will happen if you do span of these two things, okay? Span of (1, 2) and (2, 5) - this is going to be $(a + 2b, 2a + 5b)$, right? And then a and b are just real numbers, you can take all possible real numbers and then combine them you would get this. Seems simple enough, but pictorially what is going to happen here? You take the (1, 2) point and the (2, 5) point and you're allowed to scale them and add them. If you keep doing that what's going to happen? It may not be very obvious to you right now, but it turns out any point can be reached by this method, okay? So we'll see some more interesting ideas of why that is true later on, but if you think about it for a little while maybe it will be clear to you, that I can get any point in $\mathbb{R}_2$with just (1, 2) and (2, 5), as the span of (1, 2) and (2, 5) okay? So this set is actually the entire plane $\mathbb{R}_2$ okay?

So even though these two things seem to be pretty close to each other... So for instance if you want to illustrate something, so let us say maybe you know.... Maybe if you want to say - can I get the point... What point is this? (-3, -1) okay? So you can take this as an exercise, you can try to find a, b such that $a + 2b = -3$ and $2a + 5b = -1$ okay? So this kind of thing you might have solved quite often. You know how to solve this system of linear equations - you can find a equals, right... So you can, if you want to find a and b, let me think about this for a little while…

You can multiply the first equation by 2 and then subtract these two to other. So you get b is 5. And once b is 5 you will get a to be -13, okay? So you see, if you take the linear combination -13(1, 2) + 5(2, 5) you will end up getting (-3, -1), okay? So this way you can solve for the whole thing and sort of convince yourself that whatever point you want in $\mathbb{R}_2$ you can get as the span of (1, 2) and (2, 5). Okay so this is a nice example which tells you how, you know, span is very interesting. Even though you just have two vectors in $\mathbb{R}_2$ you can get the whole plane from them by just doing span, okay? So span is a powerful operation, okay?

So now, so there's lots of interesting questions like this you can ask. So maybe based on what I explained with the picture, you can answer the questions that are here. You know how to do this. So if you look at (1, 2), (2, 5) the span is the entire plane, so (3, 6) of course will be in the span. (2, 5) is in the span. It is easy to find a, b. And (2, 10) is also in the span you can find that, okay? On the other hand if you just take the vector (1, 2) okay? In the first case the span is only the line, okay? The line passing through (1, 2), so a point like (2, 10) will not be in the span. So that sort of gives you an illustration.

So now let us go to a slightly more complicated example. We'll go to $\mathbb{R}_3$ for instance, right? So notice what's going on here in $\mathbb{R}_3$. I have given you two vectors (1, 2, 3) and (2, 3, 4) and then I'm asking, okay... So span itself you can find quite easily. You multiply the first vector by a, second vector by b, all possible combinations you will get. You will get a nice, you know, algebraic description for the span, okay? And then if somebody comes and asks you - is (2, 8, 10) in the span, okay? How do you answer this question? You can set up some equations and see if something is possible or not, right? So you can try to do that. So some equation solving is involved and it's not very clear, you know? It's not immediate, it's not obvious, right?

On the other hand, look at the second example here. My first vector is (1, 0, 0), second vector is (0, 1, 0) and the third vector is (0, 0, 1), okay? Now if I want to look at span of $\{v_1, v_2, v_3\}$ right? Any question is easy to answer right? If I give you any (x, y, z) and ask is (x, y, z) in the span of (1, 0, 0), (0, 1, 0), (0, 0, 1) you can immediately say the answer is yes, right? Because I can scale v_1 by x, v_2 by y and v_3 by z and I will always get it. Okay, so that is a nice thing to remember here. So this is always equal to $xv_1 + yv_2 + zv_3$ right? Isn't it? So that's a good thing to know.

So what do I mean by that? So the point I am trying to make here is - depending on what the vectors are, the problem of finding out whether or not a particular vector is in the span may be either very easy or a little bit more difficult, okay? So in the first case (1, 2, 3), (2, 3, 4), (2, 8, 10) is in the span or not - you have to do some thinking, some calculations to answer. Maybe yes, maybe no, we don't know, right? On the other hand, if the vectors were as easy as (1, 0, 0), (0, 1, 0), (0, 0, 1) then it's very easy to answer whether it's in the span or not. So a lot of zeroes placed in a critical way help you a lot in answering these questions, okay? So this is something that you can remember. Later on we will come back and see how to make use of this.

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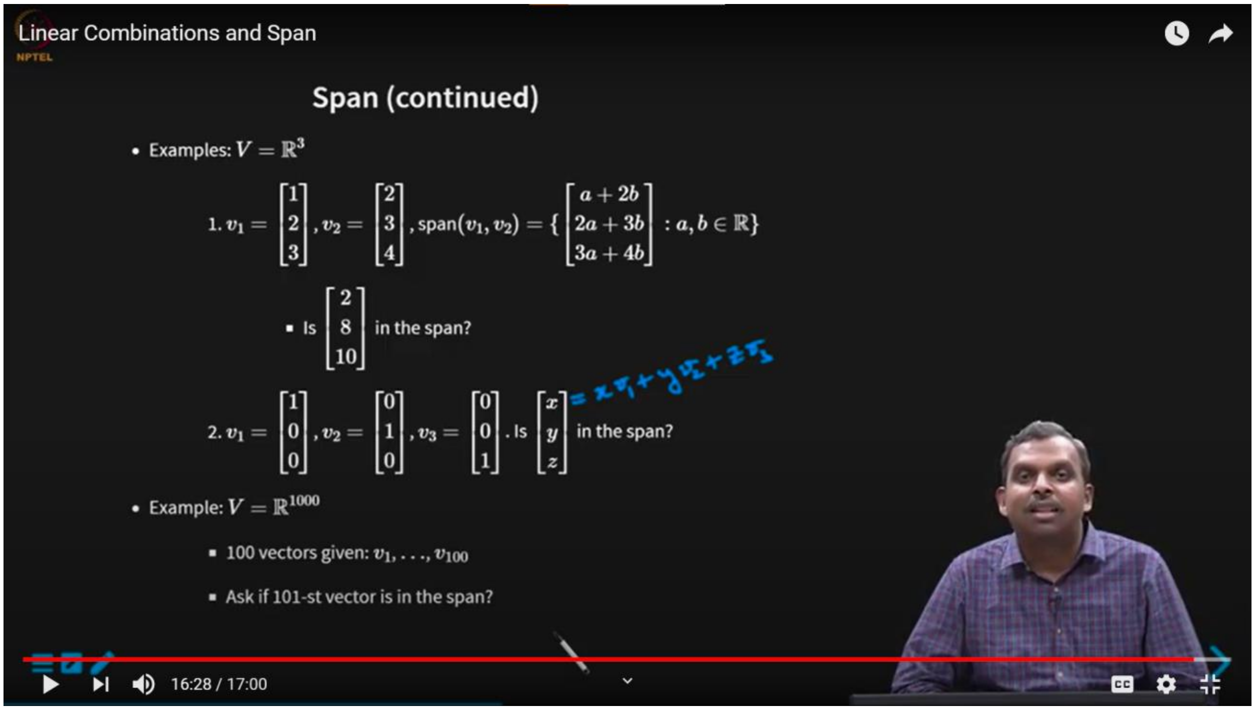

So the next example I have given here is really very intriguing. And today people talk about data science, big data, lot of data is available. So usually people deal with vectors that are really really big today. I mean minimum thousand entries. Thousand is nothing, okay? People say thousand is a medium size too. It's not really that large. Only when you go like ten thousand, million, people start getting really worried these days. Anyway let's even take thousand sized vector for this problem, and suppose I give you hundred vectors, okay, in, of length thousand right v_1, v_2, to v_{100}. And then somebody comes up with the 101st vector and asks - is this in the span of the first hundred vectors, okay? So any of these methods are not really going to help you much, right? Except if the vectors have a very special form right? If the 100 vectors have a very special form, then maybe you can answer. Otherwise how do you answer? I mean it's not very clear. You set up some big equations and hope to solve them okay? So all these things are very important questions, interesting questions and we will solve them in this course as we go along okay?