# Applied Linear Algebra

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## Week 05
## Polynomials and Roots

Hello and welcome to week 5 of the lectures of Applied Linear Algebra. This is the start of a new phase of this course. In the last four weeks we saw the basics and the foundations and linear operators, definitions of various spaces and what to do. How to do elementary row operations, column operations, how to do change of basis to get different types of matrices for linear maps and all that. Hopefully all that is reasonably clear, right? Now you're able to picture the linear map, you're able to understand it. Now, from now on we're going to start trying to understand operators. Operators are, you know, linear maps that act from one vector space to the same vector space, right? So $T: V \rightarrow V$. So those kinds of operators we will see. And I also pointed out that, you know, the nice thing about operators is: you can take powers of operators, you know? It's meaningful to do that and that gives you a very rich theory for understanding operators, classifying them and all that, okay? And naturally when you take powers, one of the tools that ends up showing up is this polynomials and roots of polynomials and let us sort of refresh ourselves with the basic ideas of polynomials. I think most of you would have read about polynomials, you have some familiarity with their form and structure and all that. So this lecture will be reasonably quick. I will point out the most important things about polynomials with real coefficients, complex coefficients, that's important to us for linear algebra, okay? So let's get started.

So as usual let's begin with a quick recap. We've been talking about vector spaces, matrices, linear maps. In particular one of the nice applications we've already seen is solving linear equations and we saw how one can use the notions of elementary row operations, column operations, your understanding of row space, column space, null space and left null space to simplify a lot of those things. Think about those things properly. So by now you should be comfortable in finding, given a matrix, finding the, you know, basis for the row space, column space, null space, left null space, all of those things should be easy for you now. We also saw determinants. We saw how to compute determinants. How determinants, you know, represent… Determinants are very interesting functions from square matrices to real numbers, to the scalar field, and they have very nice properties. And finally we saw change of basis for linear map. If you remember, we were at a point where we said we want to look at operators, right? Operators in particular. In a linear map when you went from $V$ to $W$, we saw that, you know, maybe we can do things quite easily there. But when it goes from $V$ to $V$, then it looked like we needed more work, we wanted to keep the input and output basis the same and still try and find a, sort of a simple matrix description for the linear operator. We wanted a lot of zeros in the matrix and it wasn't clear how to go about doing that. And the next few lectures will show you how to do that. It's really a very nice idea and that's what we're going to explore from this lecture on. And one of the tools we need for that is a good understanding of polynomials with real and complex complex coefficients and their roots and all that, okay? So that's what we are seeing in this lecture.

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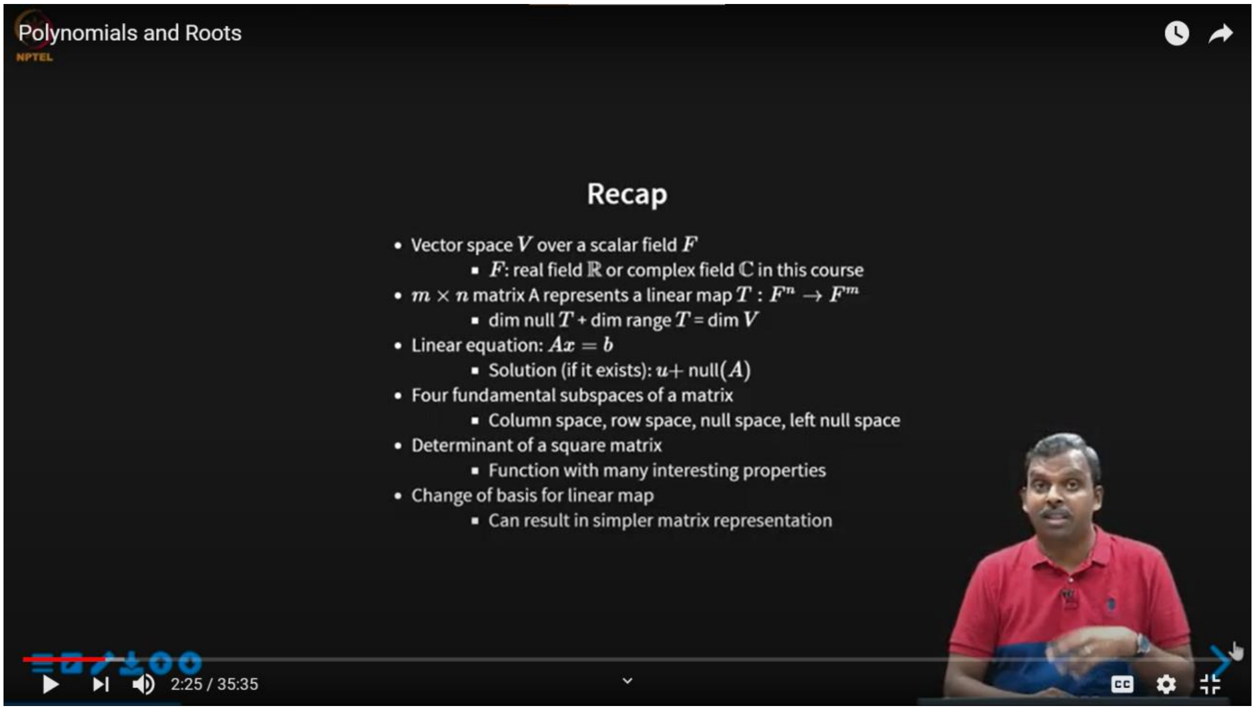


Okay. So let us begin with quite a few examples. And examples particularly you know, very classic examples in polynomials, things you might know from before. So the first polynomial we'll see are very... The simplest polynomial is simply a constant, right? So constant is a polynomial, it's supposed to be a degree zero polynomial. If $a$ is not zero, okay? Degree is zero if $a$ is not zero. If $a$ is zero, you have a zero polynomial, okay? So I'm thinking of a constant $a$ where $a$ comes from the field. Field could be real or complex, that's what we're thinking of, okay? If $a$ is non-zero then it's supposed to be degree zero. If $a$ is equal to zero, you have what's called a zero polynomial. Usually for the zero polynomial, we don't associate a degree. And just for completeness you can think of degree as minus infinity if you want, okay? Or minus one or something, okay? So just keep it as some degree like that.

You can see in terms of roots, okay? What are roots of a polynomial? The values of $x$ for which the polynomial evaluates to zero. That is the root. Now if $a$ is 0, then everything is a root, right? For the polynomial. On the other hand, if there is no root, I mean if $a$ is a constant, then there is no root, okay? There is no value for $x$ for which a will suddenly become 0, right? So that's the world of the constant polynomials. And it's very easy, okay? So constant polynomials are nothing but scalars. Now the next interesting case is this linear polynomial or, you know, polynomials of degree one. They are going to be of this form $ax + b$, and $a$ and $b$ are from the field. And $a$ is not zero, right? And you can quickly see from your knowledge of these linear functions and polynomials that they have just one root and that is $-b/a$, okay? $ax + b = 0$, you solve for $x$, you get $-b/a$, right? So that's a linear polynomial. I don't want to say too much more about it. You know how to sketch it etc. You must have studied this in quite a bit of detail. But they will show up in some very interesting ways. So first of all, linear polynomials are very nice. They have just one root and you can picture them very clearly. And it turns out you they are, I mean, they are very nice and they are useful and you can think of linear polynomials as being used as factors for larger polynomials as well, okay?

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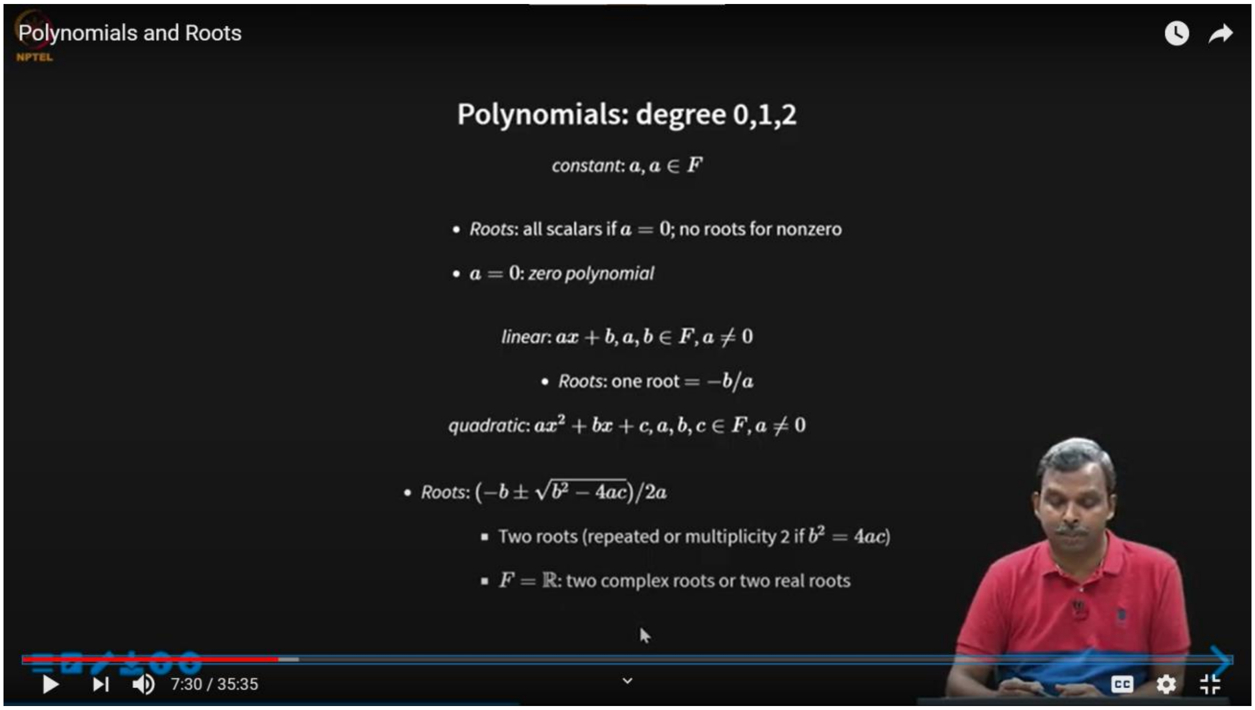


So the next interesting case is quadratic and this also you must have studied from school and all that. The general form is $ax^2 + bx + c$ and you want a not to be equal to 0. And then here the roots, there is an explicit radical form answer for the roots which is a very famous formula. All of you should know this. And if we look at the number of roots, you may get, you will get two roots, but they could be equal, right? You may have the same root occurring. So the quadratic might be of the form, you know, $(ax + b)^2$, right? Or, you know, $(x - a)^2$ or something like that. So in that case, you have the same root appearing twice in this formula, okay? So that's that kind of a thing is called multiplicity, okay? So we will use this word multiplicity a lot. If the same root occurs twice in a polynomial, we will call it a root of multiplicity 2. You can also have higher multiplicities, but for now multiplicity 2 is possible, okay? So you could have, for the quadratic itself you could have two distinct roots. Basically you could have two different roots, each of multiplicity one, that's possible. Or the two roots could be the same and you could have multiplicity 2, all right?

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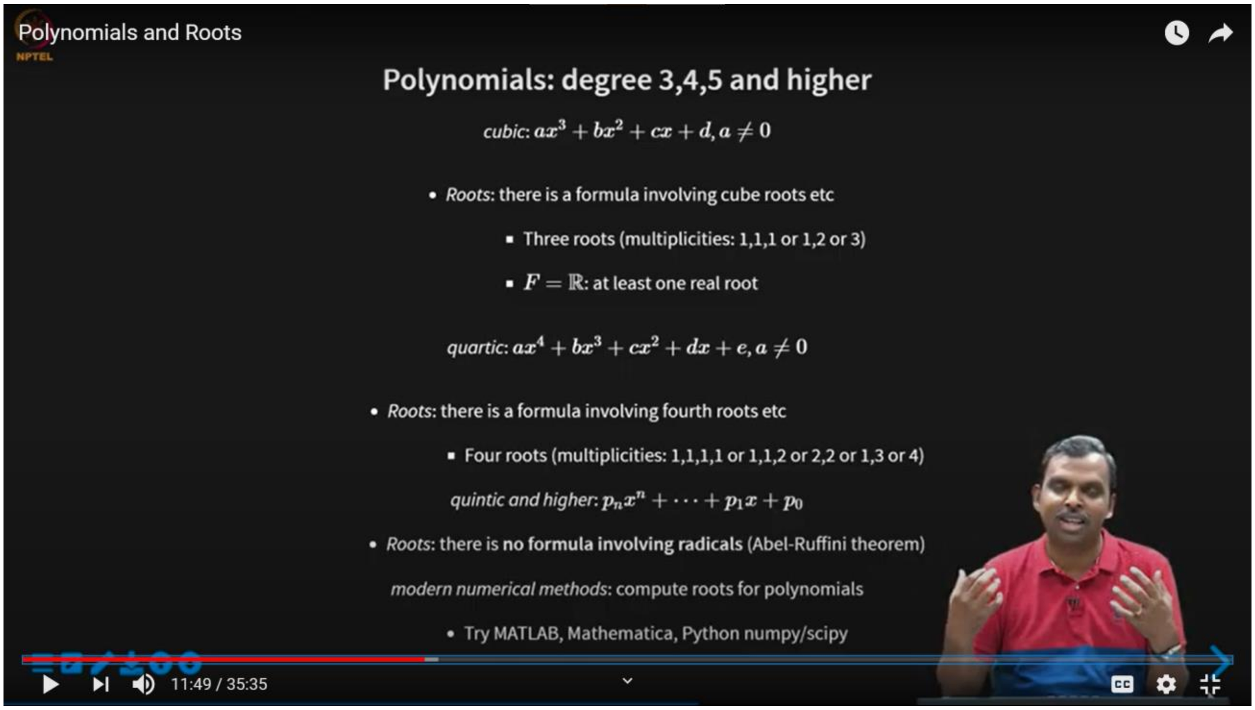


So now this is an interesting case. So if $\mathbb{F}$ is complex, that's okay, you're going to get two complex roots. No problem. Now what if $\mathbb{F}$ is real, okay? What if you have real coefficients? Will you always get real roots? Not necessarily, right? You may get complex roots also, okay? You may get two complex roots or you may get two real roots, okay? So this can also happen. So this is something interesting. And notice that even if your coefficients are real, you need to go to the complex field to get roots sometimes, right? So there are very easy examples here, okay? So this is all about degree 0, 1 and 2 polynomials, zero polynomial etcetera, all this is very classic. You can go back to very ancient works, both Indian, Egyptian and otherwise where people have studied these things. And polynomials continue to be studied through the modern age. Also particularly during the Italian Renaissance there used to be a lot of competitions where people had to solve, you know, polynomial roots and you got money for that, became famous, etc. So people looked at higher degrees, okay? The next natural higher degree to look at is cubic polynomials and that's the structure there. And you can see it's a natural extension. And it turns out there is a formula involving cube roots and other radicals for roots of the cubic equation, okay? So I am not going to go into that formula itself exactly. There is a process by which you can actually find all the three roots, you can give an explicit formula. And what is more interesting to us is... There will be three roots, right? For complex coefficients, you have three complex roots in general. And the multiplicities could be 1, 1, 1. So when I say 1, 1, 1, what do I mean? There are three distinct roots.

The three roots are different. That's possible. Or you could have a multiplicity of 1, 2. What does that mean? One, there are only two distinct roots, one of them appears twice, okay? And then... Or you could have 3 multiplicity roots. As in, one root appearing three times, okay? So in that case, the polynomial will be of the form some $(x - a)^3$ or something like that, right? So the same root appears multiple times. And then you have multiplicities of different types. So these are possible, okay?

So now here is a special case when the $\mathbb{F}$ is $\mathbb{R}$, okay? So when your coefficients are real, it turns out there should be at least one real root in this case, okay? So think about why that should be true. It turns out there should be at least one real root. But there could be, you know, one real root and two complex roots, that's possible. Or you could have three real roots. It's not possible not to have that one real root. If you have three different complex roots, it doesn't work, okay? So $\mathbb{F} = \mathbb{R}$ is an interesting special case here. But important thing is, even for cubic polynomials, there is an explicit formula involving cube root. So one can find the answer, okay? Same thing sort of carries over to quartic polynomials. If you have degree four polynomials, it turns out there is a formula involving fourth roots, maybe square roots etc. etc. Some radicals and all are involved, you can find an explicit formula for quartic polynomials also, okay? In this case, multiplicities are going to have many more possibilities, right? So many partitions of four are possible, you know? 1, 1, 1, 1, you could have four distinct roots. 1, 1, 2 or 2, 2, you could have two distinct roots each appearing twice. It's possible, all these cases are possible. One can go into all these things closely if you want. But quartic polynomials look like that, okay?

Now quintic and higher, it turns out you get into this very interesting territory, okay? So algebraically, classically, these were very, very interesting. Quintic and higher it turns out you can prove, there is this famous Abel-Ruffini theorem which proves that there cannot be, in general, there cannot be a universal formula involving radicals for all roots of quintic and higher polynomials, okay? So think about what that means, okay? So it's a very, sort of a surprising result to even try and imagine how one could prove it. Just because you have degree five or higher, in general all degree five polynomials, if you want one formula for it, it won't work. There is no such formula, okay? And that can be proven, that fact can be proven. It was done classically, it's been of great interest. I think those days people had, needed a formula to find roots, they didn't have powerful numerical methods and all that. So having these kinds of closed form expressions was very important in those days. The computer was not around, right? So today with computer, in the modern day computer world where you can really do a lot of number crunching, these kinds of classical results are not so interesting I would say, from, at least from an applied, you know, engineering point of view. Because, the reason is, you pick up any modern computing tool today, you can do the Python, Numpy, Scipy, or more expensive tools like Mathematica, Matlab, whatever you have access to. You input a polynomial, ask for the roots, it will give you the roots like that, okay? You could have even very higher degree polynomials etc., there are very powerful numerical methods today which can compute roots for you. So I will ask you to experiment with that. Pick your own numerical tool. You know there's lots of free tools out there. Google Colab for instance. We have a G-Suite account, so you can go and access Colab. It naturally connects to all the python toolboxes and libraries. Pick up Numpy, Scipy, and put in a polynomial, read the documents on how to do that and then try and compute the roots. You'll see the roots will be shown to you and you can see that, you know, it's quite quick and computing roots for polynomials, even if it's degree 100 or something, today people probably don't, are not really that afraid of that. But anyway these classical results are very interesting, particularly the Abel-Ruffini theorem is a very interesting area of study in mathematics. And hope you are interested. If you are interested you can read up more about it. I was referring to Arton's book on Algebra. That's a very good book at the correct level. So there there is a very nice exposition of this Abel-Ruffini theorem. How is it that this works, in case you are interested, you can read that. But I would also urge you to make sure you have a tool in your hand, right? Where you can key in a polynomial and get the roots, okay? So somewhere you should have that. It could be your calculator, it could be a computer, it could be something on the cloud, but you should have an answer for finding roots of a polynomial. It's very important. I mentioned the Italian Renaissance, there used to be these competitions where, you know, you give a polynomial to a guy and the person gives the roots, you get paid a lot of money, you know? I mean there were awards of that kind. So this day and age, nobody's going to give you an award, but still, you know, you should have that ready-made in your hand. Somebody gives you a polynomial, you should be able to find roots, right? So that should be possible, okay? So this is the general picture.

But what do we need in this class? So you remember in this class, we've been also doing proofs and formal development of the ideas in as rigorous a way as possible, mathematical way as possible. So what is it in theory that we need to know about polynomials and roots and factors and all these things, okay? So let us look at that in the next few slides, okay? So what is the theoretical picture of a polynomial? So that's the picture, right? Polynomial in one variable with coefficients from a field, that is what we are looking at. We will call it $p(x)$. $p_0 + p_1x + p_2x^2 + ... + p_nx^n$ and each $p_i$ will be in $\mathbb{F}$, okay? Every polynomial has a lot of things going on around it. There are the coefficients $p_0, p_1, p_2$ etc. Each thing is called a term. $p_0$ is the constant term. $p_1x$ is the degree 1 term, $p_2x^2$ is the degree 2 term etc. The polynomial itself has a degree that is the largest power of $x$ that shows up non-zero in the polynomial, right? So largest $n$ such that $p_n \neq 0$. Largest $i$ such that, okay, $p_i \neq 0$, okay?

Now degree of the zero polynomial is minus infinity, that is some convention that we use. It is not a big deal, if you don't remember that also, it's okay. There's also this leading term for a polynomial, okay? So the highest degree term along with its coefficient, it's called the leading term of the polynomial, okay? And $n$ of course is the degree of $x$, degree of $p(x)$, okay? So these are standard terms and we will use them a little bit as we go along, okay? Addition and multiplication of polynomials are very well known, I am not going to go into great detail here. In particular, I want to point out this little result which says if you multiply $a(x)$ and $b(x)$, the degree can only go up, right? You cannot go down as long as $b(x)$ is non-zero. Of course if $b(x)$ is zero, then the degree doesn't exist, no? So that's another problem. But generally the degree is going to go up. It's a very simple result of multiplication, okay? So this is addition, multiplication.

Of great importance to us though is the division algorithm, okay? So many of you might have forgotten how to divide two polynomials. Maybe you still remember, hopefully you remember. But I've put down here an algorithm which sort of precisely puts out how to do polynomial division, okay? So let me also write here. I will write an example of how the division happens. So let's say you're given a polynomial $p(x)$ which is, let's just take this example $2x^5 + 3x^4 + 2x^3 + x^2 + 8$, okay? So let's say this is my polynomial $p(x)$. I want to divide by this other polynomial, let's say, we'll call it $x^2 + 3x + 5$, okay? So this is basically, this algorithm here, division algorithm is an exact codification of your familiar polynomial division with remainder that you might be aware of, okay? So notice here $p(x)$ is this big polynomial inside. $q(x)$ is the quotient. Right now it's 0, nothing is there up there. $a(x)$ is what I am going to divide by, okay? $x^2 + 3x + 5$.

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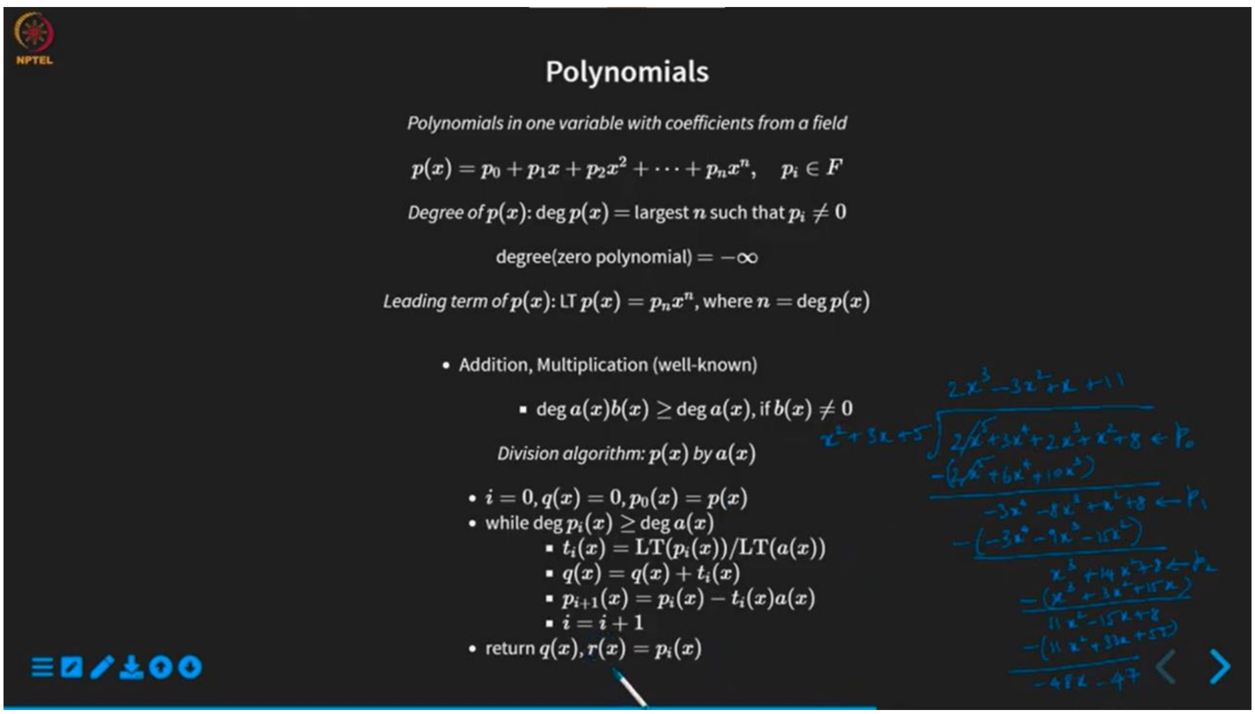


First thing you check is: you make $p_0$, I am calling it $p_0$ as $p(x)$ which is just basically the first one, that's $p_0$ for me, okay? So that's $p_0$, okay? And then I check if the degree of this guy is greater than the degree, greater than or equal to degree of $a$, okay? So 5 is greater than 2. So then what do I do? I find one term which is the leading term divided by this, okay? So that is $2x^3$, I put that, I add it to $q(x)$, the quotient that leading term, the ratio of the leading terms gets added to the quotient and then I have to subtract from $p_i$, the leading term, the term that I put, just added times

$a(x)$, right? So that is what I would do. This $-2x^5 - 6x^4 - 10x^3$, isn't it? So then you subtract, you see that the leading term will cancel, okay? So the way we have chosen this $t_i(x)$ is always to cancel the leading term. So the degree of $p_{i+1}$ will always be strictly lesser than the degree of $p_i$, right? So that's a thing that we'll maintain. So you will have $-3x^4 - 8x^3 + x^2 + 8$, okay? So this will be $p_1$, isn't it? Okay. And then you do this again, you check, you keep repeating it. Increment i and repeat. So you see that the leading term here is again higher, so you get $-3x^2$, okay? And then you multiply $-3x^4 - 9x^3 - 15x^2$, okay? And then you subtract these two guys, okay? Sorry, so you should be careful here... So let me put this carefully. This should be minus of this, okay? So just be careful about that. So this you get 0, you get $x^3$, you get plus $14x^2 + 8$, okay? This becomes your $p_2$. You carry on like this. So let me just do a couple of more steps, hopefully I have enough space for that. $+x$, you have $x^3 + 3x^2 + 15x$ and then when you subtract these two, you get $11x^2$. Let me write it a little bit to this side... $-15x + 8$, and then you get a $+11$ minus, again minus here, sorry, $-(11x\text{^}2 + 33x + 55)$. So hopefully I'm not making too many mistakes here. So if you add these two, subtract these two you're going to get $-48x$ and then $-47$ I think so, okay. There you go, okay? So that is the end of your division and at the end of the division you will, you will be left with the $p$, and then the degree would have gone down, right? So degree of $p$, I don't know $p_3$, $p_4(x)$ will become less than $a(x)$ and at that point you can stop, okay? And what you return is the quotient that you had on top and the remainder, okay?

What do we know about the about the remainder? The remainder is going to have a degree strictly less than the degree of $a(x)$, okay? So this is a standard polynomial division and polynomial division is an extremely important algorithm that we will use in theory quite a bit, okay? Simple algorithm, you can see how it works, okay? So I am assuming all of you would have some familiarity with that. Once you finished your division, you will get a remainder and you will get a quotient and you will get $a(x)$. So this $p(x)$ will be $a(x)$ times quotient plus the remainder, okay? You can check that that would be true, okay? And more important to us the degree of $r(x)$ is strictly less than degree of $a(x)$, okay? That is when you stop, all right? So this is the familiar division algorithm, I am just writing it down, I am codifying it, okay?

So division algorithm is important to us. In particular, you will see this is how division algorithm is described in theory, okay? Given two polynomials $p(x)$ and $a(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $p(x)$ becomes $q(x)a(x) + r(x)$, right? And with $r(x)$, the remainder could either be zero, or if it's non zero let's say, its degree should be less than the degree of $a(x)$, okay? So of course if you use the convention the degree is minus infinity, you can simply say degree $r(x)$ less than degree $a(x)$, but it's sometimes good to, you know, explicitly call out some things, if $r(x)$ is 0 in this case. It's an important case, so it's good to pick it up, okay? So to prove it, I think the proof here, the existence of $q(x)$ and $r(x)$ is easy, you can just use division algorithm. For the uniqueness maybe you have to work a little bit more, you can do that, it's not too bad. I'm skipping the proof of this because, I mean it just plainly uses the division algorithm, right? So you can go back, write down the division algorithm and you can see that it will work out.

Uniqueness maybe it's a little bit not clear to you but it's unique, there's no problem here. Particularly if it's… It's easy to see that it's unique, okay? So not let me not worry too much about it. One way of proving uniqueness is to assume there are two different ones and then show that they are the same etc. You can do such things, okay? So that is good, that is division algorithm, okay? So this is an important construct for us. $p(x)$ being, there existing always $q(x)$ and $r(x)$ such that $p(x)$ is like this, okay? There are also these other cases. What if degree of $a(x)$ is already greater than degree of $p(x)$? You can simply set the quotient to zero and the remainder to be equal to $p(x)$ itself, right? So it's easy to deal with those cases also. The corner cases are easy to deal with. So this is a result. And this is the theorem which is usually associated with the division algorithm for polynomials, okay?

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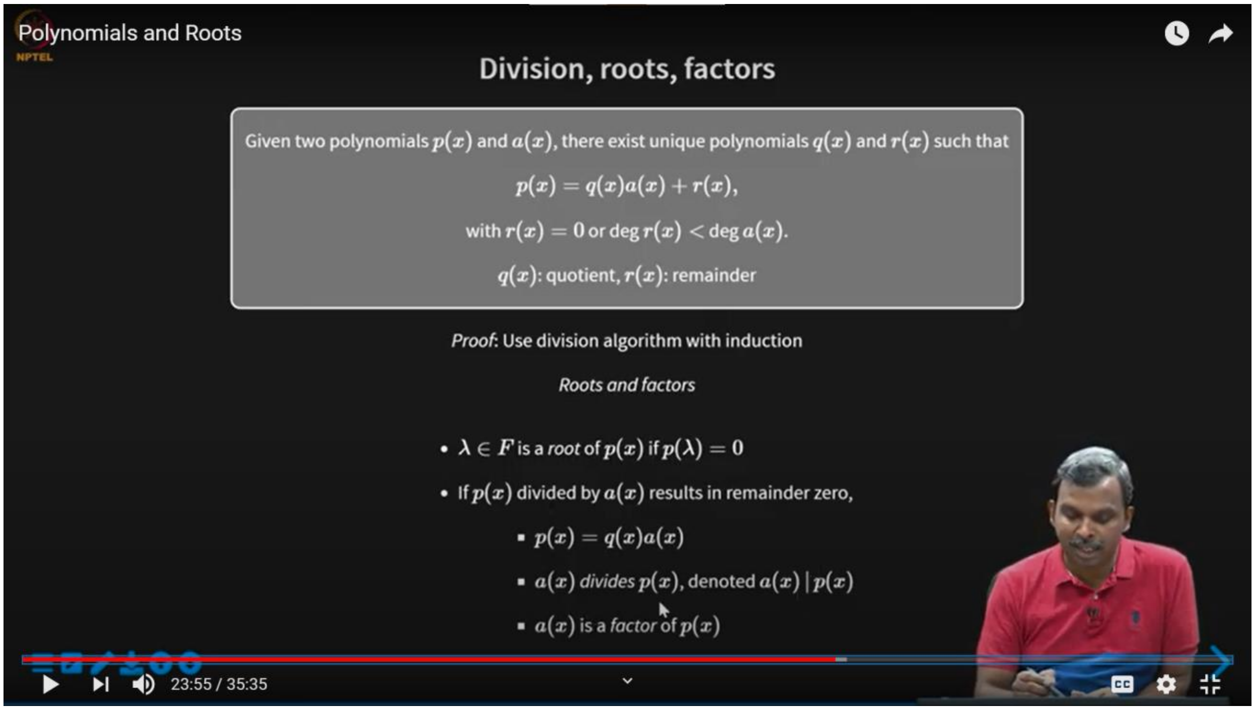


Now from here one can formally sort of think of roots and factors. What is the root of a polynomial? What is a root? Element from the scalar field such that $p(\lambda)$ evaluates to zero, okay? So $\lambda$ from the field is called a root of $p(x)$, $p(\lambda) = 0$, the value $\lambda$ when you plug it into $x$ you should get 0, okay? So that's root. So now notice what happens when you divide $p(x)$ by $a(x)$ and the remainder ends up being 0, okay? So that's a special sort of case and in that case you are able to factor $p(x)$, right? $p(x)$ becomes $q(x)a(x)$, okay? So that is a special case, okay? Whenever that happens, whenever remainder is zero we say that $a(x)$ divides $p(x)$, okay? So it is a factor of $p(x)$, it divides $p(x)$ and there is this notation that we will use. We'll simply say $a(x)|p(x)$. Meaning that $a(x)$ divides $p(x)$. What does this mean? If you do the division algorithm, take $p(x)$, divide by $a(x)$, finally you will get a reminder 0, okay? So it will perfectly divide, okay? So this being a factor is very important and there is a connection between factors and roots which is also very important for us, okay? And that is what we will see next, okay?

Okay, so before we see that connection, we'll see that soon enough, there is something called a Fundamental Theorem of Algebra, okay? So you might wonder, given a polynomial, given an arbitrary polynomial, should it have a root, right? Should it always have a root? Is it a case, is the case that every polynomial has a root? It turns out there is this fantastic theorem called the Fundamental Theorem of Algebra. Notice this course you've already seen two fundamental theorems, right? You have seen a Fundamental Theorem of Linear Maps which we continue to use all the time. Then here is a fundamental theorem which is a Fundamental Theorem of Algebra. It is much more famous than the Fundamental Theorem of Linear Maps. Let me point out that to you. This is used across the board in so many areas. In fact so many non-trivial results always finally boil down, particularly in applications, they always boil down to the fact, to this Fundamental Theorem of Algebra, okay? So you always have a root and you only have so many roots etc. These are powerful results and these are very, very important to know, okay? So what is this Fundamental Theorem of Algebra I was telling you? Every non-constant polynomial, okay? So if, of course if you have a constant polynomial which is non-zero, there are no roots, right? So that's just to rule it out. Every non-constant polynomial with complex coefficients has a complex root, okay? So that's the Fundamental Theorem of Algebra. So it is a solid theorem about existence, okay? It says that every non-constant polynomial has a complex root, okay? You cannot go without a complex root, okay?

So the proof of this is way beyond the scope of this course. It uses Complex Analysis. If you take a slightly advanced course in Complex Analysis, you will see a proof for this Fundamental Theorem of Algebra. It's very nice. So, but we are not going to see a proof of this in this class. We will accept this, okay? So we will say every non-constant polynomial will have a complex root. And what do I do to find that root? You go to your favorite numerical tool, it will give you the root. It gives you all the roots. So you can pick up the roots. So now this result is important to us, okay? So if $\lambda$ is a root of $p(x)$, then it turns out $(x-\lambda)$ is a factor of $p(x)$ and it also goes the other way around, okay? So these kind of degree one factors, $(x-\lambda)$ are called linear factors, okay? And linear factors, like I mentioned, linear polynomials are very nice and you want to understand polynomials in terms of linear factors. That is very good. So, and you notice that roots of a polynomial are strongly connected to linear factors of polynomials, okay? And the proof is actually very easy, you can divide $p(x)$ by $(x-\lambda)$, okay? So supposing you divide $p(x)$ by $(x-\lambda)$. What will you get? You will get a quotient, right? You will get a quotient and you will get a reminder. But what will be the remainder? I am not writing $r(x)$, I am simply writing an $r$ because remainder will be a constant polynomial. Why will it be a constant? Because $a$ has degree one, right? When you divide by $(x-\lambda)$, you got degree one. So this has to have degree strictly less than 1. And what is the only degree strictly less than 1? That's 0, so it needs to be a constant, okay? So this is the equation you get. Now if I put $x=\lambda$, what happens? Equals $r$. So this is what is codified here in this result, okay? So if $\lambda$ is a root, meaning $p(\lambda)$ is 0, then your remainder is 0. So if the remainder is 0, $(x - \lambda)$ becomes a factor of $p(x)$, right? $p(x)$ is $(x - \lambda)q(x)$, that's one way of proving it. If $(x - \lambda)$ is a factor of $p(x)$ then $r$ is zero. And what is $r$? It's the exact same as $p(\lambda)$. So $p(\lambda)$ is equal to zero and $\lambda$ is a root. So $\lambda$ is a root if and only if $(x - \lambda)$ is a factor, okay? So reasonably simple result. But we will use these kind of results over and over again, okay?

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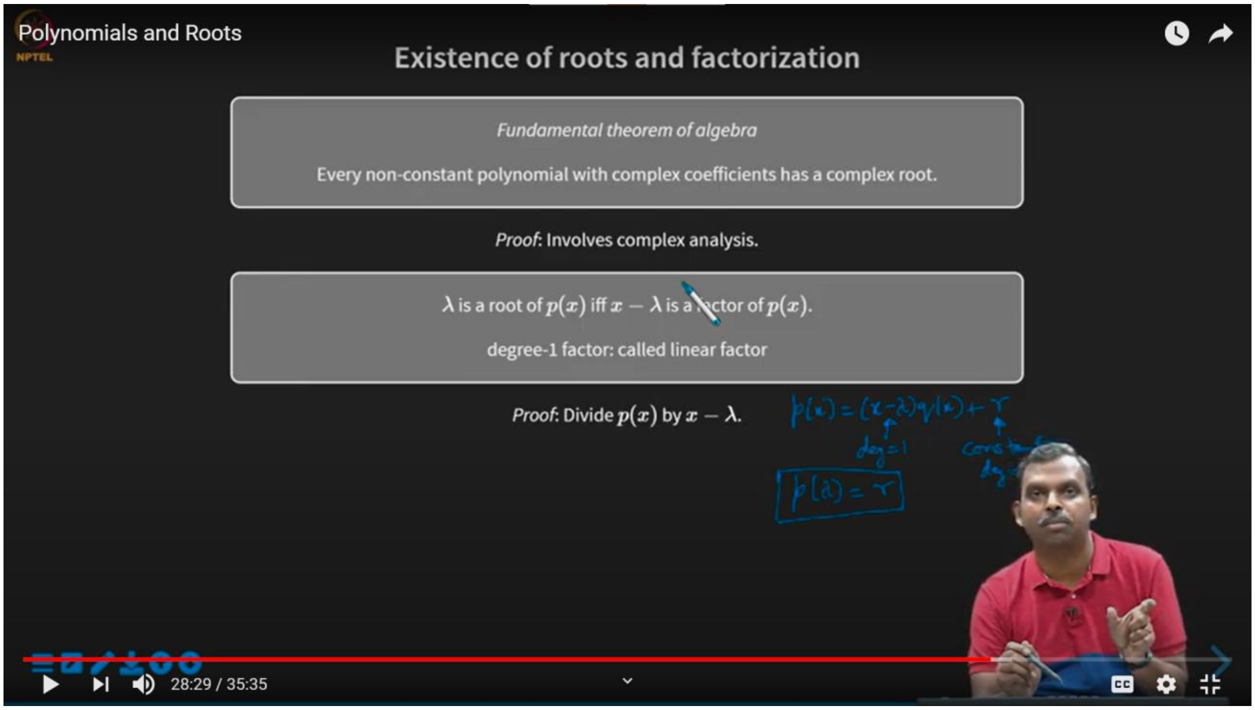


So once again, every complex polynomial, polynomial with complex coefficients has a complex root, we know that. That is the Fundamental Theorem of Algebra. Very nice. If there is a root, I can pick out a linear factor from the polynomial, okay? Notice every polynomial has a root. If there is a root, I can pick out a linear factor, okay? Now you can use this idea repeatedly and get very nice results of this form, okay? So if you have a polynomial of degree $n$ with complex coefficients, there are $n$ complex roots and there are no more, okay? There are at most, there are exactly $n$ roots in this case and you cannot have more, cannot have less, right? So in a polynomial with degree $n$ with complex coefficients, there will be $n$ complex roots and you can call them $\lambda_1, \lambda_2, \dots, \lambda_n$. In fact some of these may be repeated, okay? Remember that. When I say $n$ complex roots, I am allowing for repetitions, repetition is okay. But there will be $n$ of them, they will occur $n$ times, okay? So that way if you, when you have $n$ complex roots like that, you can also factor over and over again and you can write $p(x)$ as $p\_n(x - \lambda_1)\dots(x - \lambda_n)$, these two will be equal. Any $p(x)$ of degree $n$ can be factored into linear terms of this type and this is always true for complex coefficients, okay?

So notice how this argument worked. It's very important, you can use the Fundamental Theorem and the degree one factor result repeatedly. What do I mean by repeatedly, okay? So first you take this polynomial. there should be one root, right? You call that $\lambda_1$. Now $(x - \lambda_1)$ is a factor. So you pull out $p(x)$ as $(x - \lambda_1)$ times some quotient. Now on that quotient, you use the Fundamental Theorem again. As long as it has degree, as long as it's non-constant you keep on using it, you will always keep pulling out linear factors and you end up with a linear factorization, okay?

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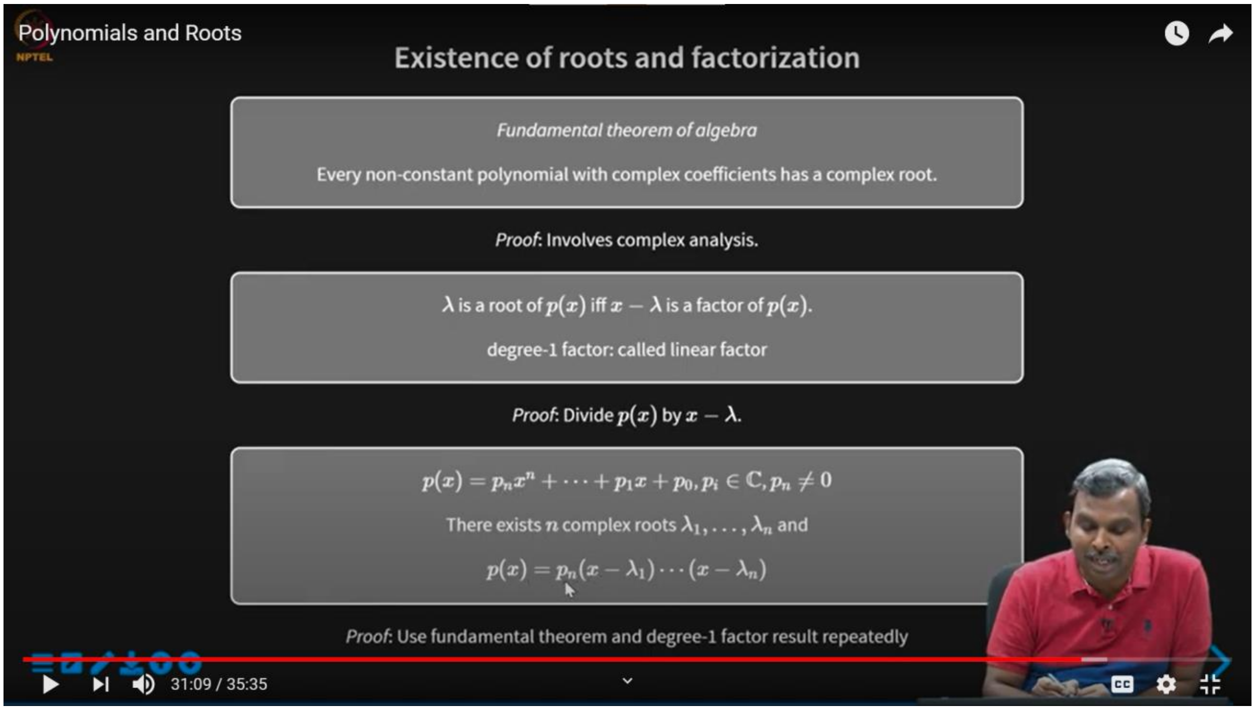


Now a couple of other results in general I want to point out. Forget about complex field. If you have a polynomial of degree $n$ in any field, it can have at most $n$ roots, okay? It cannot have more than $n$ roots, all right? So this is a very popular result. Once again one can prove it, actually it's not very hard, given the tools that you have, you can even prove it. But I am not going to do that, write that down, but that's generally true. Polynomial in an arbitrary field can have at most $n$ roots in that field, okay? So that's true. But notice what happens in complex . Complex you have exactly $n$, so such fields are special they are called some things and people study these kind of fields, so that's why the complex field is very, very interesting and important, okay? So that's a quick take on complex polynomials. We will use these kind of results over and over again. And notice how I put this $p_n$ here, okay? So $p_n$, how did I put this $p_n$? If you multiply all these guys, the coefficient of $x^n$ will be 1, right? And I know the coefficient of $x^n$ in $p$ is $p_n$, so I simply attach the $p_n$ to get the correct form. So this has to be an exact equality once I put $p_n$ there, right, So think about why that is true... So that is also a subtle little thing I have done here to get this exact equality, okay?

So that is, this is very important for us. We will use this as we study more ideas in operators and all that, okay? So this is existence.

Now what about the real field, okay? So if you have real coefficients, then what happens? So in general, you can have a real polynomial which has no real root, okay? It is possible. Like $x^2 + 1$. $x^2 + 1$ has no real roots. In fact you can have higher degree polynomial. Simply raise $(x^2 + 1)$ to higher powers, you will have polynomials with no real roots, it is possible, okay? However if you have a complex root or any root for that matter, then the complex conjugate, $\bar{\lambda}$ is, the complex conjugate, $\bar{\lambda}$ is also a root, okay? So let me write down what $\bar{\lambda}$ is. This is the complex conjugate. You know what complex conjugate is. $a + ib$, the conjugate of that is $a - ib$, okay? If $\lambda$ is a root, $\bar{\lambda}$ is also a root, think about how you will prove it. You have real polynomial, everything is real. If $p(\lambda)$ is 0, simply take conjugate, okay? $p(\lambda)$ whole conjugate is equal to 0. Now when you go in, every $p_n$ is real, so nothing will happen to it. The conjugate will go only into $x$, so you will get $p(\bar{\lambda})$ also equal to 0, okay? So that is the way you prove this result.

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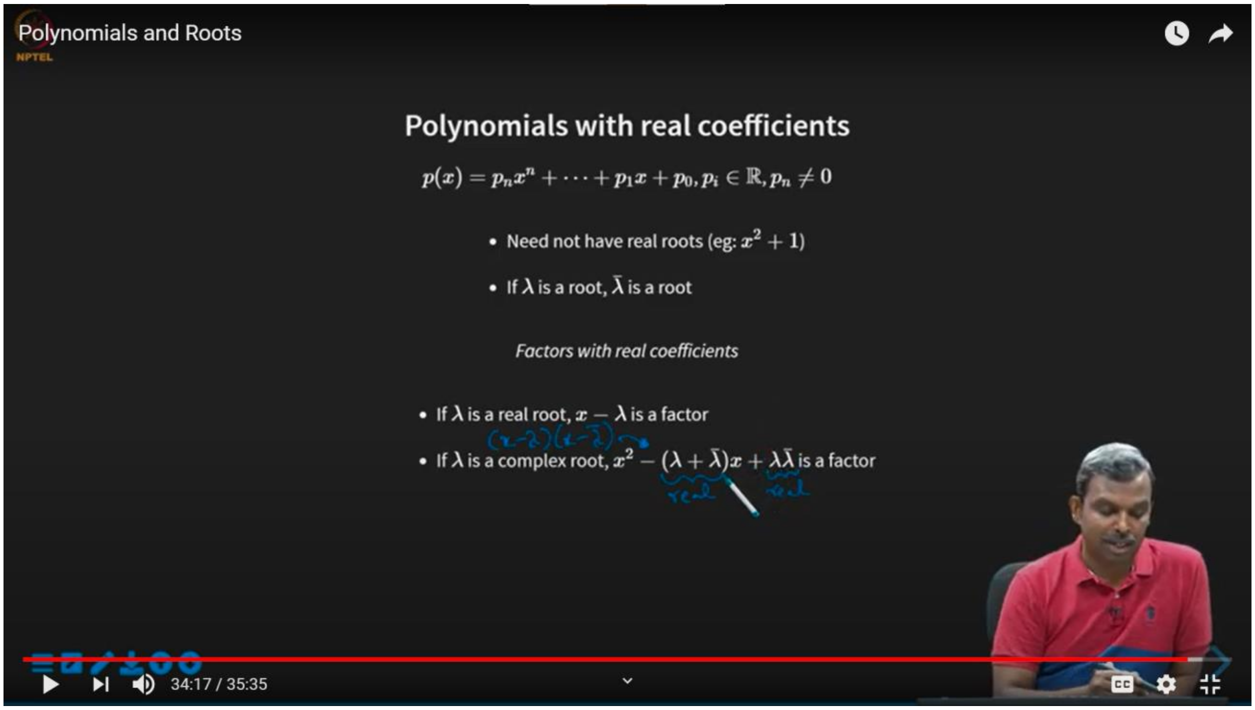


So what about factors with real coefficients? Supposing I want, I have a real polynomial and I want factors with real coefficients, right? See, if real numbers are inside complex numbers, so I can always factor them into complex roots and get linear factors, it's possible, right? Now on the other hand, if I want factors with real coefficients, can I do it, okay? So it turns out it's possible. If $\lambda$ is a real root, $(x - \lambda)$ is a factor. That we saw before. It turns out if $\lambda$ is a complex root, $x^2$ minus this guy is a factor. How did I get this? It's not very hard. So if $\lambda$ is a complex root, $(x - \lambda)$

is a factor and $(x - \bar{\lambda})$ is also a factor, right? Right? There are two factors because it is complex. $\lambda$ is a root, $\bar{\lambda}$ is also a root. So $(x - \lambda)(x - \bar{\lambda})$ is a root. Now when I multiply these two things, I will get this $(x^2 - (\lambda + \bar{\lambda})x + \lambda * \bar{\lambda})$. And what's interesting about this? This guy is real, this guy is also real, okay? So if you have a polynomial with real coefficients, you see that it may have only complex roots or may have real roots also. If it has a real root, then you have a linear factor. Linear real factor. If it has a complex root, you have a quadratic linear factor, okay? This is quadratic, sorry, quadratic real factor, okay? So not quadratic linear factor. so you have a quadratic real factor, okay? So any real polynomial, polynomial with real coefficients, you can ultimately factor it into quadratic factors and linear factors, okay? So the exact form of it I will leave alone. So this is how it will factor, okay?

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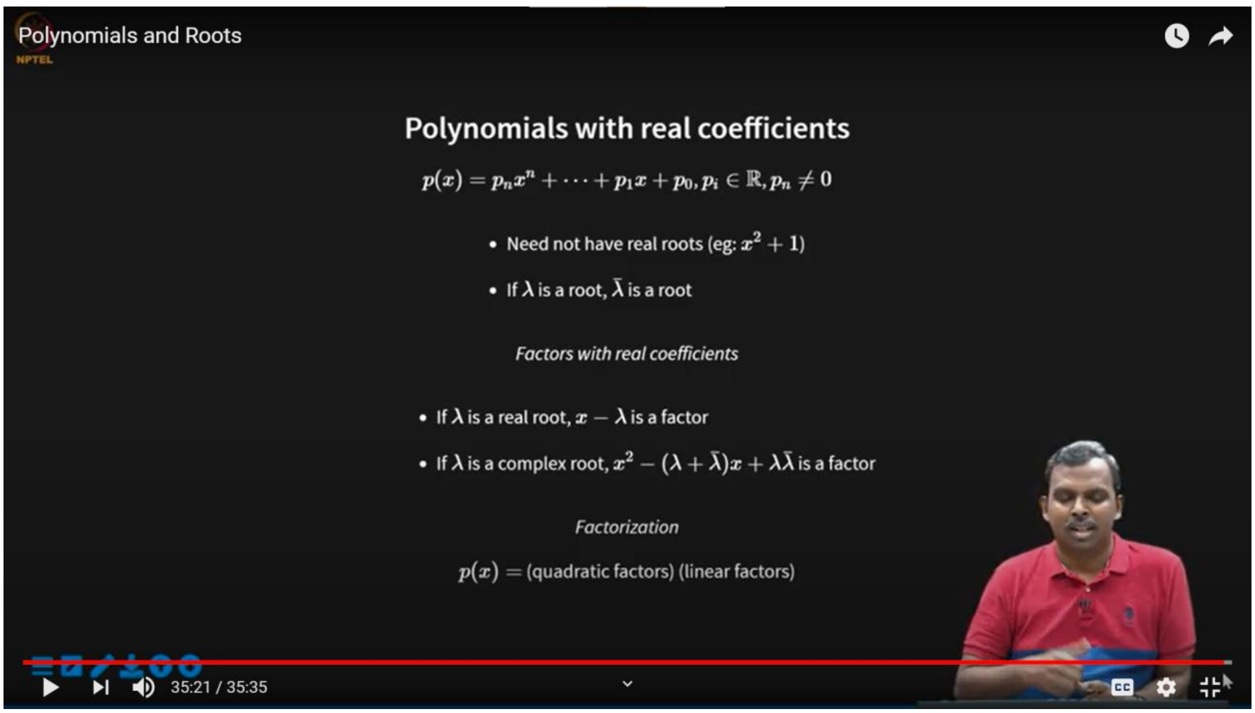


So once again what are the important things? Polynomials have roots, complex polynomials, polynomial with complex coefficients always have roots. And every time you have a root, you can factor out a linear polynomial. So for, over the complex field, the polynomials factor into linear factors. With real, you may not have real roots, but you always have complex roots. And when you have a complex root, you also have a complex conjugate root when you have real coefficients. So you can always take a real polynomial and write it as quadratic factors times and linear factors, okay? So this form is always possible. I also stated a couple of general results which are important to us. One is that, you know, a polynomial with, over any field of degree $n$ has at most $n$ roots, okay? It cannot have more than $n$ roots. That's, that's the result that one can prove. I'm not going to prove it. There are also other associated results about zero polynomial and all that which are slightly more theoretically interesting. I will let you read it up from the book if you are interested, okay? So this is a short lecture on polynomials and this is what we will need when we study eigenvalues next, okay? Thank you.